GEOMETRIC ASPECTS OF MINTY-BROWDER MONOTONICITY

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Introduction

The Minty-Browder monotonicity notion was an important achievement of nonlinear analysis. It throws a new light on many known results and becomes a good instrument in various application oriented new investigations [40].

Convex functionals in Hilbert spaces have as gradients monotone operators. This result was extended to Riemannian manifolds by C. Udrişte [34], [33]. He proved that the gradient of a (geodesic) convex functional [31] on a Riemannian manifold is monotone. T. Rapcsák [31] gives a more explicit form of this result. Hovewer, there are nongradient type monotone vector fields which (as far as we know) has not been studied yet.

Recent developments of convex optimization problems on Riemannian manifolds [31] focus the attention to nongradient type monotone vector fields, since in the Hilbert case in the analysis and solution of complementary problems, variational inequalities, and evolution equations it is commonly assumed that the defining map is monotone (and not necessarily of gradient type).

The main purpose of our thesis is to present global results on the geometrical aspects of the Minty-Browder monotonicity. We consider that this notion has a strong geometrical meaning, which has not been analysed yet.

The thesis is structured on chapters, sections and subsections. The first chapter is devoted to the euclidean case. It contains three sections. The first two deals with first order characterizations of monotone maps through the scalar derivative, a new notion introduced by us. In the two dimensional case the connection of scalar derivatives with the complex derivative and the Kasner circle is also encountered.

The third section analyses the connection of scalar derivative with conformity. It is proved that globally scalar differentiability of a map (which can be identified with a vector field) implies the conformity of the one parameter transformation group generated by this map (vector field). Locally, the connection is given using the conformal derivative, a new notion introduced by us. In this section the well-known connection between monotone operators and nonexpansive semigroups [40] is also presented, from a more geometrical point of view.

The second chapter is devoted to the Riemannian case. It contains six sections. In the first section the notion of geodesic monotone vector fields is introduced [34], which generalize the monotone operators. First order characterization of geodesic monotone vector fields are given, using the geodesic scalar derivatives, a new notion introduced by us. Different computational formulae are presented for this derivative. The constant curvature case is analysed separately, since it has important peculiarities. We present the connection of geodesic monotone vector fields with the first variation of lengths of geodesics and the existence of closed geodesics. For Hadamard manifolds (complete, simply connected Riemannian manifolds of nonpositive sectional curvature) a class of geodesic monotone vector fields is given which generalize the position vector fields [26]. It is proved that there are monotone vector fields on Hadamard manifolds, which are not gradient fields.

In the second section we analyse the connection of geodesic monotone vector fields with one parameter expansive transformation groups. We introduce the Killing monotone vector fields, which are the vector fields generated by such transformation groups. We proved that the notion of Killing monotone vector fields and geodesic monotone vector fields coincide. For brevity's sake they are called monotone vector fields. These results are generalizations of the results of section 1.3.2. We also analyse the connection of the Lie derivative with monotonicity and (geodesic) scalar derivatives. It is proved that a vector field is conformal if and only if it is (geodesic) scalar differentiable, generalizing the corresponding result of section 1.3.1.

The third and fourth section deals with complementary vector fields of maps on Hadamard manifolds. We define the notion of complementary vector fields and λ-complementary vector fields of maps on Hadamard manifolds. It is proved that for λ increasing the λ-complementary vector fields of projection maps to closed (geodesic) convex sets [31] are monotone. It is also proved that the complementary vector fields of nonexpansive maps are monotone. As a particular case composition of projection maps to closed convex sets are considered. These results are generalizations of some results of [38].

In the fifth section different types of generalized monotone vector fields are introduced. It is showed that for gradient fields, these generalized monotonicity properties correspond to generalized convexity properties of the underlying functions. These results extend some results of [11].

In the sixth section we prove that the zeros of a monotone vector field, defined in a convex open subset of an Hadamard manifold, form a convex set.

Chapter 1

Monotone operators in Euclidean spaces

In this chapter we restrict our attention to euclidean spaces. Most of the results presented here are true also for Hilbert spaces. However, the euclidean case is enough for illustrating the main ideas.

1.1 Scalar derivatives of operators in Euclidean spaces

The behaviour of the scalar product $\langle f(x) - f(y), x - y \rangle$ (with $f : \mathbf{R}^n \to \mathbf{R}^n$ and $\langle \, . \, , \, . \, \rangle$ the usual scalar product in $\mathbf{R}^n$) when x and y run over $\mathbf{R}^n$ is a good tool in characterizing important properties of f. If f is bounded, then this product converges to 0 for $x \to y$. Therefore it cannot be used in obtaining a local characterization. Hence it is natural to consider at y limits of the expressions of form $\langle f(x) - f(y), x - y \rangle / \langle x - y, x - y \rangle$ for $x \to y$. Thus we arrive by a natural way to a notion which we call scalar derivative. It is in general a multivalued mapping from $\mathbf{R}^n$ to $\mathbf{R}$ even if f is linear.

In order to avoid the difficulties in considering multifunctions we shall only consider so called upper and lower scalar derivatives, which are extensions to vectorial functions of the Dini derivatives.

We consider mostly the case when lower and upper scalar derivatives coincide. This restriction is a very strong one. In subsection 1.1.3 it is shown that for $n = 2$ the existence of a single-valued scalar derivative is strongly related to the complex differentiability. In subsection 1.1.4 we consider various examples and counter examples. Lower and upper scalar derivatives can be used in characterizing the monotone operators in the way this is done in subsection 1.1.5.

Convex functionals have as gradients monotone operators. Hence the scalar derivative can also be used to characterize convexity-like notions. Thus Proposition 2.1 and 2.2 in [11] together with the results in subsection 1.1.5 give some characterizations of convex and strictly convex functionals.

We have defined the notion of scalar derivative having in mind Minty's monotonicity notion [15]. Most of the results of this paper hold also in Hilbert spaces. We restrict ourselves to $\mathbf{R}^n$ since this space is sufficient to illustrate our basic ideas.

1.1.1 Some basic results concerning skew-symmetric operators

Definition 1.1.1.1 *Consider the operator $f : \mathbf{R}^n \to \mathbf{R}^n$. It will be called increasing (decreasing) if for any x and y in $\mathbf{R}^n$ one has*

$$\langle f(x) - f(y), x - y \rangle \geq 0 \ \ (\leq 0).$$

If

$$\langle f(x) - f(y), x - y \rangle > 0 \ \ (< 0)$$

whenever $x \neq y$, then f will be called strictly increasing (strictly decreasing).

Definition 1.1.1.2 *The linear operator $A : \mathbf{R}^n \to \mathbf{R}^n$ will be said skew-symmetric if for any x and y in $\mathbf{R}^n$ the relation $\langle Ax, y \rangle + \langle Ay, x \rangle = 0$ it holds.*

Theorem 1.1.1.3 *If $A : \mathbf{R}^n \to \mathbf{R}^n$ is linear, then the following statements are equivalent:*

1) A is skew-symmetric.

2) $\langle Ax - Ay, x - y \rangle = 0$ for any $x, y \in \mathbf{R}^n$.

3) Taking an arbitrary orthonormal basis in $\mathbf{R}^n$, A can be represented by a matrix $A = (a_{ij})_{i,j=1,\dots,n}$ such that $a_{ij} = -a_{ji}$ $\forall i, j \in \{1, 2, \dots, n\}$.

Proof. 1) $\Rightarrow$ 2). Take x and y arbitrarily in $\mathbf{R}^n$. By the definition of the skew-symmetric operator A we have $\langle Ax, y \rangle + \langle Ay, x \rangle = 0$. Put $y = x$. Then $\langle Ax, x \rangle = 0$ for arbitrary x in $\mathbf{R}^n$. Whence we have also $\langle Ax - Ay, x - y \rangle = 0$ by the linearity of A. The implication 2) $\Rightarrow$ 1) can be shown similarly.

The equivalence 1) $\Leftrightarrow$ 3) is obvious.

Remarks.

1. There exist injective skew-symmetric operators. For instance the operators represented by the matrices

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{for } n = 2 \text{ and}$$

$$A = \begin{pmatrix} 0 & -1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \text{ for } n = 4$$

are injective.

2. If n is odd, then there is no injective skew-symmetric operator in $\mathbf{R}^n$. Indeed let A be the matrix corresponding to an skew-symmetric operator. Let the superscript T denote transposition. Then $A^T = -A$ and hence det$A$=$-$det$A$ this means that $\det A = 0$.

Theorem 1.1.1.4 *Consider the operator $F : \mathbf{R}^n \to \mathbf{R}^n$. The following assertions are equivalent:*

1) $\langle F(x) - F(y), x - y\rangle = 0$, $\forall x, y \in \mathbf{R}^n$.

2) F is an affine operator with skew-symmetric linear term.

Proof. Suppose that 1) holds. Put $f(x) = F(x) - F(0)$ for x in $\mathbf{R}^n$. Then $f(0) = 0$ and $\langle f(x) - f(y), x - y\rangle = 0$ $\forall x, y \in \mathbf{R}^n$. Let x be arbitrary in $\mathbf{R}^n$ and $y = 0$. Then $\langle f(x), x\rangle = 0$ $\forall x \in \mathbf{R}^n$. The above relation yields also

$$\langle f(x), x\rangle - \langle f(x), y\rangle - \langle f(y), x\rangle + \langle f(y), y\rangle = 0, \forall x, y \in \mathbf{R}^n$$

and hence

$$\langle f(x), y\rangle + \langle f(y), x\rangle = 0, \forall x, y \in \mathbf{R}^n.$$

Put $x = \lambda x_1 + \mu x_2$ with arbitrary x_1 and x_2 in $\mathbf{R}^n$. Then

$$\langle f(\lambda x_1 + \mu x_2), y\rangle = -\langle f(y), \lambda x_1 + \mu x_2\rangle = -\lambda\langle f(y), x_1\rangle - \mu\langle f(y), x_2\rangle =$$

$$\lambda\langle f(x_1), y\rangle + \mu\langle f(x_2), y\rangle,$$

wherefrom

$$\langle f(\lambda x_1 + \mu x_2) - \lambda f(x_1) - \mu f(x_2), y\rangle = 0$$

for any x_1, x_2 and y in $\mathbf{R}^n$ and any λ, μ in $\mathbf{R}$, wherefrom we have the linearity of f. Since $\langle f(x) - f(y), x - y\rangle = 0$, for any x, y in $\mathbf{R}^n$, f is also skew-symmetric. Thus $F(x) = f(x) + F(0)$ and hence it is indeed affine with skew-symmetric linear term.

The implication 2) $\Rightarrow$ 1) is obvious.

1.1.2 The scalar derivative and its fundamental properties.

Definition 1.1.2.1 *Consider the operator $f : \mathbf{R}^n \to \mathbf{R}^n$. If the limit*

$$\lim_{x \to x_0} \frac{\langle f(x) - f(x_0), x - x_0 \rangle}{\|x - x_0\|^2} =: f^{\#}(x_0) \in \mathbf{R}$$

exists, (here $\|x - x_0\|^2 = \langle x - x_0,\ x - x_0 \rangle$) then it is called the scalar derivative *of the operator f in x_0. In this case f is said to be* scalarly differentiable *at x_0. If $f^{\#}(x)$ exists for every x in $\mathbf{R}^n$, then f is said to be* scalarly differentiable *on $\mathbf{R}^n$, with the scalar derivative $f^{\#}$.*

It follows from this definition that both the set of operators scalarly differerentiable in x_0, and the set of operators scalarly differentiable on $\mathbf{R}^n$ form linear spaces.

Definition 1.1.2.2 *Consider the operator $f : \mathbf{R}^n \to \mathbf{R}^n$. The limit*

$$\underline{f}^{\#}(x_0) := \liminf_{x \to x_0} \frac{\langle f(x) - f(x_0),\ x - x_0 \rangle}{\|x - x_0\|^2}$$

is called the lower scalar derivative *of f at x_0. Taking* $\limsup$ *in place of* $\liminf$ *we can define the* upper scalar derivative $\overline{f}^{\#}(x_0)$ *of f at x_0 similarly.*

Theorem 1.1.2.3 *The linear operator $A : \mathbf{R}^n \to \mathbf{R}^n$ is scalarly differentiable on $\mathbf{R}^n$ if and only if it is of the form $A = B + cI_n$ with B an skew-symmetric linear operator, I_n the identity of $\mathbf{R}^n$, and c a real number.*

Proof. Let us suppose that A is scalarly differentiable at $x_0 \in \mathbf{R}^n$. Then

$$A^{\#}(x_0) = \lim_{x \to x_0} \frac{\langle Ax - Ax_0,\ x - x_0 \rangle}{\|x - x_0\|^2} = \lim_{h \to 0} \frac{\langle Ah,\ h \rangle}{\|h\|^2} = A^{\#}(0).$$

Take $h = \lambda x$ with $x \in \mathbf{R}^n$ and $\lambda > 0$. Then

$$A^{\#}(0) = \lim_{\lambda \downarrow 0} \frac{\langle A\lambda x,\ \lambda x \rangle}{\|\lambda x\|^2} = \frac{\langle Ax,\ x \rangle}{\|x\|^2}.$$

That is $\dfrac{\langle Ax,\ x \rangle}{\|x\|^2} = c = A^{\#}(0)$. Accordingly

$$\langle (A - cI_n)x,\ x \rangle = 0,\ \forall x \in \mathbf{R}^n.$$

This means that $B = A - cI_n$ is a skew symmetric linear operator and hence A has the representation given in the theorem. Obviously, every $A = B + cI_n$ with B a skew symmetric linear operator has the scalar derivative c at every point of $\mathbf{R}^n$.

Theorem 1.1.2.4 *Suppose that $f : \mathbf{R}^n \to \mathbf{R}^n$, $f = (f_1, \dots, f_n)$ is scalarly differentiable at x_0. Then for every $i \in \{1, \dots, n\}$ there exists the partial derivative*

$$\frac{\partial f_i(x_0)}{\partial x^i} \text{ and } \frac{\partial f_1(x_0)}{\partial x^1} = \dots = \frac{\partial f_n(x_0)}{\partial x^n} = f^{\#}(x_0).$$

Proof. If we consider $x = (x_0^1, \dots, x^i, \dots, x_0^n)$, $x_0 = (x_0^1, \dots, x_0^n)$, by letting $x \to x_0$, we obtain that

$$\frac{\partial f_i(x_0)}{\partial x^i} = f^{\#}(x_0).$$

Theorem 1.1.2.5 *Suppose that $f : \mathbf{R}^n \to \mathbf{R}^n$ is differentiable at x_0 and scalarly differentiable at x_0. Then we have for the differential $df(x_0)$ of f at x_0 the relation*

$$df(x_0) = B + f^{\#}(x_0) I_n,$$

with $B : \mathbf{R}^n \to \mathbf{R}^n$ linear and skew-symmetric.

Proof. Let $t \in \mathbf{R}^n$ be given. Then

$$f^{\#}(x_0) = \frac{1}{||t||^2} \lim_{\lambda \downarrow 0} \left\langle \frac{f(x_0 + \lambda t) - f(x_0)}{\lambda}, \ t \right\rangle = \frac{1}{||t||^2} \langle df(x_0)(t), t \rangle,$$

wherefrom $\langle (df(x_0) - f^{\#}(x_0) I_n)(t), t \rangle = 0$, $\forall t \in \mathbf{R}^n$, that is, $B = df(x_0) - f^{\#}(x_0) I_n$ is linear and skew-symmetric.

Remarks.

1) The theorem holds for the Gateaux differential $\delta f(x_0)$ in place of $df(x_0)$. The differentiability condition will be often used next and hence we state the theorem for this stronger condition.

2) If we denote by $f'(x_0)$ the Jacobi matrix of f at x_0 in some coordinate representation and the symbols B and I_n stand for matrices of the corresponding operators, then our relation becomes

$$f'(x_0) = B + f^{\#}(x_0) I_n.$$

Theorem 1.1.2.6 *Suppose that $f : \mathbf{R}^n \to \mathbf{R}^n$, $f = (f_1, \dots, f_n)$ is differentiable at x_0. Then the following statements are equivalent:*

1) f is scalarly differentiable at x_0;

2) *a)* $\dfrac{\partial f_1(x_0)}{\partial x^1} = \dots = \dfrac{\partial f_n(x_0)}{\partial x^n}$;

b) $\dfrac{\partial f_i(x_0)}{\partial x^j} = -\dfrac{\partial f_j(x_0)}{\partial x^i}$ $\forall i, j \in \{1, \dots, n\}, i \neq j$.

The condition 2) will be called the Cauchy-Riemann relations *at x_0.*

Proof. 1) $\Rightarrow$ 2). By Remark 2 after Theorem 1.1.2.5 one has

$$f'(x_0) = B + f^{\#}(x_0) I_n,$$

where B is a skew symmetric matrix and $f'(x_0)$ is the Jacobi matrix of f at x_0. Since from the above relation

$$B = \begin{pmatrix} \dfrac{\partial f_1(x_0)}{\partial x^1} - f^{\#}(x_0) & \dfrac{\partial f_1(x_0)}{\partial x^2} & \dots & \dfrac{\partial f_1(x_0)}{\partial x^n} \\ \dfrac{\partial f_2(x_0)}{\partial x^1} & \dfrac{\partial f_2(x_0)}{\partial x_2} - f^{\#}(x_0) & \dots & \dfrac{\partial f_2(x_0)}{\partial x^n} \\ \dots & \dots & \dots & \dots \\ \dfrac{\partial f_n(x_0)}{\partial x^1} & \dfrac{\partial f_n(x_0)}{\partial x^2} & \dots & \dfrac{\partial f_n(x_0)}{\partial x^n} - f^{\#}(x_0) \end{pmatrix}$$

and since B is a skew symmetric matrix, we must have the relations at 2) of the theorem.

2) $\Rightarrow$ 1). Consider the Taylor expansions of $f_1, \dots, f_n$ around x_0:

$$f_1(x) = f_1(x_0) + \sum_{i=1}^{n} \frac{\partial f_1(x_0)}{\partial x^i}(x^i - x_0^i) + u_1(x)\|x - x_0\|$$
$$\dots$$
$$f_n(x) = f_n(x_0) + \sum_{i=1}^{n} \frac{\partial f_n(x_0)}{\partial x^i}(x^i - x_0^i) + u_n(x)\|x - x_0\|,$$

where $\lim\limits_{x \to x_0} u_i(x) = 0 \ \forall i \in \{1, \dots, n\}$. Usage of the above formulas gives

$$\frac{\langle f(x) - f(x_0), x - x_0 \rangle}{\|x - x_0\|^2} = \frac{1}{\|x - x_0\|^2} \Bigg[\sum_{i,j=1}^{n} \frac{\partial f_i(x_0)}{\partial x^j}(x^i - x_0^i)(x^j - x_0^j)$$
$$+ \sum_{i=1}^{n} u_i(x)\|x - x_0\|(x^i - x_0^i) \Bigg].$$

By the relations a) and b) one obtains

$$\frac{\langle f(x) - f(x_0), x - x_0 \rangle}{\|x - x_0\|^2} = \frac{1}{\|x - x_0\|^2} \Bigg[\frac{\partial f_1(x_0)}{\partial x^1}\|x - x_0\|^2 +$$
$$+ \sum_{i=1}^{n} u_i(x)\|x - x_0\|(x^i - x_0^i) \Bigg] = \frac{\partial f_1(x_0)}{\partial x^1} + \sum_{i=1}^{n} \frac{u_i(x)(x^i - x_0^i)}{\|x - x_0\|},$$

wherefrom, since $-1 \leq \dfrac{x^i - x_0^i}{\|x - x_0\|} \leq 1$ and

$\lim\limits_{x \to x_0} u_i(x) = 0$, it follows that

$$f^{\#}(x_0) = \lim_{x \to x_0} \frac{\langle f(x) - f(x_0), x - x_0 \rangle}{\|x - x_0\|^2} = \frac{\partial f_1(x_0)}{\partial x^1} = \dots = \frac{\partial f_n(x_0)}{\partial x^n}.$$

1.1.3 Case n=2. The relation of the scalar derivative with the complex derivative

We shall identify in this chapter the complex numbers with points in $\mathbf{R}^2$. The scalar product of these numbers will mean the scalar product of the vectors representing them in $\mathbf{R}^2$.

Theorem 1.1.3.1 *Let $f : \mathbf{C} \to \mathbf{C}$ be a complex function. The following statements are equivalent:*

1) f is differentiable at z_0 as a complex function.

2) f is differentiable at z_0 as a mapping $f : \mathbf{R}^2 \to \mathbf{R}^2$ and is scalarly differentiable at this point.

Proof. Follows directly from Theorem 1.1.2.6.

The differentiability condition of f at z_0 in 2) is essential. In examples 2) and 3) of subsection 1.1.4 we construct two discontinuous functions at 0, which are scalarly differentiable at this point.

Remarks.

1) Let G be an open subset of $\mathbf{C}$. Then f is holomorphic on G if and only if is differentiable as a vector function and scalarly differentiable on G. As it is well known the set of holomorphic functions on $\mathbf{C}$ is closed with respect to the compositions of functions.

2) The above remark justifies the following generalization of a holomorphic function.

Definition 1.1.3.2 *Let G be open in $\mathbf{R}^n$. The function $f : \mathbf{R}^n \to \mathbf{R}^n$ will be said* $\mathbf{R}$-holomorphic *on G if and only if it is differentiable and scalarly differentiable on G. The set of $\mathbf{R}$-holomorphic functions on G will be denoted by $\mathcal{H}(G)$.*

Theorem 1.1.3.3 *For the complex function $f : \mathbf{C} \to \mathbf{C}$ the following statements are equivalent:*

1) f is differentiable at $z_0 \in C$ as a complex function.

2) f and if are scalarly differentiable at z_0.

Proof. Let us denote $f = u + iv$, $z = x + iy$, $z_0 = x_0 + iy_0$. Then

$$\frac{f(z) - f(z_0)}{z - z_0} = \frac{u(z) - u(z_0) + i(v(z) - v(z_0))}{x - x_0 + i(y - y_0)} =$$

$$= \frac{[u(z) - u(z_0) + i(v(z) - v(z_0)][x - x_0 - i(y - y_0)]}{(x - x_0)^2 + (y - y_0)^2} =$$

$$= \frac{[u(z)-u(z_0)](x-x_0)+[v(z)-v(z_0)](y-y_0)}{(x-x_0)^2+(y-y_0)^2}+$$

$$+i\frac{[v(z)-v(z_0)](x-x_0)-[u(z)-u(z_0)](y-y_0)}{(x-x_0)^2+(y-y_0)^2}=$$

$$= \frac{\langle f(z)-f(z_0),\, z-z_0\rangle}{|z-z_0|^2} - i\frac{\langle (if)(z)-(if)(z_0),\, z-z_0\rangle}{|z-z_0|^2}.$$

From the obtained relation it follows that $\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}$ exists, that is, f is differentiable at z_0 as a complex function if and only if the limits

$$\lim_{z\to z_0}\frac{\langle f(z)-f(z_0),\, z-z_0\rangle}{|z-z_0|^2} \text{ and } \lim_{z\to z_0}\frac{\langle (if)(z)-(if)(z_0),\, z-z_0\rangle}{|z-z_0|^2}.$$

exist

Remark. The function f is holomorphic on the open set $G \subset C$ if and only if f and if are scalarly differentiable on G .

Corollary 1.1.3.4 *If $f : \mathbf{C} \to \mathbf{C}$ is differentiable at z_0 as a complex function then f and if are scalarly differentiable at z_0 and the relation*

$$f'(z_0) = f^{\#}(z_0) - i(if)^{\#}(z_0).$$

holds. This relation is equivalent with the relations:

$$Ref'(z_0) = f^{\#}(z_0)$$

$$Imf'(z_0) = -(if)^{\#}(z_0).$$

Proof. It is indeed contained in the proof of Theorem 1.1.3.3.

1.1.4 Miscellanea concerning scalar differentiability

Examples and counterexamples.

1) Let $n \in \mathbf{N}$, $n > 2$. Then the set $\mathcal{H}(\mathbf{R}^n)$ of the holomorphic functions on $\mathbf{R}^n$ is not closed under compositions of functions.

Indeed consider $A : \mathbf{R}^n \to \mathbf{R}^n$ represented by the matrix $A = (a_{ij})_{i,j=1,\ldots,n}$, where

$$a_{ij} = \begin{cases} 1 \text{ if } & i < j \\ 0 \text{ if } & i = j \\ -1 \text{ if } & i > j. \end{cases}$$

Obviously, A is a skew symmetric operator. Hence A is holomorphic on $\mathbf{R}^n$. Consider $A^2 = A \circ A$ and assume that it is holomorphic. Then by Theorem 2.1 it must be of the form $A^2 = B + cI_n$ with B a skew symmetric linear operator, c a real number and I_n the identical map. Let us denote the matrix representing A^2 by $D = (d_{ij})_{i,j=1,\dots,n}$, then $d_{12} + d_{21} = 0$, that is

$$(a_{11}a_{12} + \ldots + a_{1n}a_{n2}) + (a_{21}a_{11} + \ldots + a_{2n}a_{n1}) = 0$$

From the definition of A it follows that $2(n-2) = 0$ and hence $n = 2$, contradicting the hypothesis on n.
From this example and the results in chapter 3 it follows the assertion:

Theorem 1.1.4.1 *The set of scalarly differentiable linear functions in $\mathbf{R}^n$ is closed under composition if and only if $n \leq 2$.*

From the definition of the scalar derivative follows easily the assertion:

Lemma 1.1.4.2 *Let $0 = (0, \dots, 0) \in \mathbf{R}^n$ and let $f : \mathbf{R}^n \to \mathbf{R}^n$ be a function having the properties:*

a) $f(0) = 0$

b) $\langle f(x), x \rangle = 0$, $\forall x \in \mathbf{R}^n$.

Then f is scalarly differentiable at 0 and $f^{\#}(0) = 0$.

Usage of this lemma allows us to construct the following two examples of discontinuous functions at 0, which are scalarly differentiable at this point.

2) $f : \mathbf{R}^2 \to \mathbf{R}^2$ with

$$f(x,y) = \begin{cases} \left(\dfrac{y}{x^2+y^2}, \dfrac{-x}{x^2+y^2} \right) & if \;\; x^2+y^2 \neq 0 \\ (0,0) \quad if \;\; x = y = 0. \end{cases}$$

3) $f : \mathbf{R}^n \to \mathbf{R}^n$ $(n \geq 2)$, with

$$f(x) = \begin{cases} \dfrac{1}{\|x\|^2} Ax \;\; if \;\; x \neq 0, \\ 0 \quad if \;\; x = 0, \end{cases}$$

where A is a nonzero, linear, skew-symmetric operator.

In fact, 3) generalizes 2). Both functions satisfy the conditions of the above lemma and hence they are scalarly differentiable at 0. Let us show that f in 3) is not continuous at 0. Since $A \neq 0$, there exists some t in $\mathbf{R}^n$ with $At \neq 0$. Put $x = \lambda t$, $\lambda > 0$. Then the relation

$$\lim_{\lambda \downarrow 0} \frac{A(\lambda t)}{\lambda^2 \|t\|^2} = \lim_{\lambda \downarrow 0} \frac{1}{\lambda} \frac{At}{\|t\|^2} \neq 0$$

shows that f is not continuous at 0.

4) Example of a function $f:\mathbf{R}^2\to\mathbf{R}^2$ which is continuous at 0, scalarly differentiable at this point, but not differentiable as a vector function.

Consider f given by

$$f(x,y)=\begin{cases}\left(\dfrac{xy^2}{x^2+y^2},\ \dfrac{-x^2y}{x^2+y^2}\right) & if\ \ x^2+y^2\neq 0\\[2ex] (0,0) \quad if\ \ x=y=0\end{cases}$$

Then f fulfils the conditions of the lemma, hence it is scalarly differentiable at 0.

The continuity of the two components of f is a standard exercise in calculus.

If f is differentiable at 0, then its components f_1 and f_2 are differentiable real valued functions. Since $\dfrac{\partial f_1(0,0)}{\partial x}=\dfrac{\partial f_1(0,0)}{\partial y}=0$, then $df_1(0,0)=0$. Hence we cannot have

$$\lim_{\substack{x\to 0\\ y\to 0}}\frac{|f_1(x,y)-f_1(0,0)-df_1(0,0)(x,y)|}{\sqrt{x^2+y^2}}\neq 0.$$

Taking for instance $x=y>0$ the above limit will be $\frac{1}{2\sqrt{2}}$.

5) Example of a function $f:\mathbf{R}^2\to\mathbf{R}^2$ which is continuous at 0, possesses partial derivatives at 0, does not satisfy the Cauchy-Riemann conditions at 0, but it is scalarly differentiable at 0.

Take

$$\begin{cases}\left(\dfrac{-y^3}{x^2+y^2},\ \dfrac{xy^2}{x^2+y^2}\right) & if\ \ x^2+y^2\neq 0,\\[2ex] (0,0) \quad if\ \ x=y=0.\end{cases}$$

The continuity of f at 0 can be verified passing to polar coordinates. By direct verifications:

$$\frac{\partial f_1(0,0)}{\partial x}=\frac{\partial f_2(0,0)}{\partial x}=\frac{\partial f_2(0,0)}{\partial y}=0,$$

$$\frac{\partial f_1(0,0)}{\partial y}=-1,$$

that is, the Cauchy-Riemann conditions does not hold at 0.

The scalar differentiability of f at 0 follows from the fact, that it satisfies the conditions of the lemma.

6) Example of a function $f : \mathbf{R}^2 \to \mathbf{R}^2$ which is continuous at 0, satisfies the Cauchy-Riemann conditions at 0, but is not scalarly differentiable at this point.

Take

$$\begin{cases} \left(\dfrac{x^2 y}{x^2+y^2}, \ \dfrac{x^2 y}{x^2+y^2} \right) & if \ \ x^2+y^2 \neq 0, \\ \\ (0,0) \quad if \ \ x = y = 0. \end{cases}$$

As in the above examples, the components of f are continuous at 0. Further,

$$\frac{\partial f_1(0,0)}{\partial x} = \frac{\partial f_1(0,0)}{\partial y} = \frac{\partial f_2(0,0)}{\partial x} = \frac{\partial f_2(0,0)}{\partial y} = 0,$$

that is, the Cauchy-Riemann conditions are satisfied at 0.

Assume that f is scalarly differentiable at 0. Then the limit

$$\lim_{\substack{x\to 0\\ y \to 0}} \frac{f_1(x,y)x + f_2(x,y)y}{x^2+y^2} = \lim_{\substack{x\to 0\\ y \to 0}} \frac{x^2 y(x+y)}{(x^2+y^2)^2}.$$

must exist. Put $x = 0$ and $y \to 0$. Then this limit will be 0. Put $x = y \neq 0$, $x \to 0$. Then this limit will be $\frac{1}{2}$. That is, the limit does not exist.

7) Example of a nonlinear $\mathbf{R}$-holomorphic function $f : \mathbf{R}^n \to \mathbf{R}^n$ for arbitrary n $(n > 2)$.

By a direct verification it can be seen that

$$f(x^1, x^2, \ldots, x^n) = ((x^1)^2 - (x^2)^2 - \ldots - (x^n)^2, 2x^1x^2, \ldots, 2x^1x^n)$$

has scalar derivative $f^{\#}(x^1, x^2, \ldots, x^n) = 2x^1$ and satisfies the Cauchy-Riemann conditions at every point.

Remark. In [1] is proved that there are no nonlinear functions of other type as that in 7) which satisfies the Cauchy-Riemann equations at every point (i.e. there are no nonlinear $\mathbf{R}$-holomorphic functions of other type). Since $\mathcal{H}(\mathbf{R}^n)$ for $n > 2$ is not closed with respect to the composition (see example 1 and Theorem 1.1.4.1), we cannot derive in this way other holomorphic functions.

1.1.5 Characterization of monotonicity by scalar derivatives.

By using the notion of upper (lower) scalar derivative we obtain the following assertion:

Theorem 1.1.5.1 *Let G be an open convex set in $\mathbf{R}^n$. Then the following statements are equivalent:*

1) $f : G \to \mathbf{R}^n$ is an increasing (decreasing) function;
2) $\underline{f}^{\#}(x) \geq 0$ $(\overline{f}^{\#}(x) \leq 0)$ for each x in G.

Proof. The implication 1) $\Rightarrow$ 2) is obvious.

2) $\Rightarrow$ 1). Take $\epsilon > 0$ arbitrarily and put $g = f + \epsilon I_n$. Then $\underline{g}^{\#}(x) = \underline{f}^{\#}(x) + \epsilon > 0$, $\forall x \in G$.

Take a, b in G, $a \neq b$. For x in the line segment $[a, b]$ determined by a and b, one has by hypothesis:

$$\liminf_{y \to x} \frac{\langle g(y) - g(x),\, y - x\rangle}{\|y - x\|^2} > 0.$$

and hence there exists $\delta(x) > 0$ such that for any y in $I_x =]x - \delta(x)(b-a), x + \delta(x)(b-a)[\subset G$, $\langle g(y) - g(x),\, y - x\rangle > 0$ holds as far as $y \neq x$. Obviously, $[a, b] \subset \bigcup_{x \in [a,b]} I_x$, that is $\{I_x : x \in [a, b]\}$ is an open cover of the compact set $[a, b]$. Hence $[a, b] \subset I_{y_1} \cup I_{y_2} \cup \ldots \cup I_{y_{m-1}}$ for an appropriate set $y_1, \ldots, y_{m-1}$ of points in $]a, b[$. We can suppose that $y_1, \ldots, y_{m-1}$ are ordered from a to b. Hence $a = y_0 \in I_{y_1}$, $b = y_m \in I_{y_{m-1}}$. We can also consider that no interval I_{y_i} is contained in any other. Take $\xi_i \in I_{y_{i-1}} \cap I_{y_i} \cap]y_{i-1}, y_i[$. Then by the construction of these intervals

$$\langle g(\xi_i) - g(y_{i-1}),\, \xi_i - y_{i-1}\rangle > 0$$

$$\langle g(y_i) - g(\xi_i),\, y_i - \xi_i\rangle > 0$$

and since ξ_i is in $]y_{i-1}, y_i[$,

$$y_i - \xi_i = \alpha(y_i - y_{i-1}),$$

$$\xi_i - y_{i-1} = \beta(y_i - y_{i-1}),$$

for appropriate positive α and β. Hence

$$\langle g(\xi_i) - g(y_{i-1}),\, y_i - y_{i-1}\rangle > 0,$$

$$\langle g(y_i) - g(\xi_i,\, y_i - y_{i-1}\rangle,$$

wherefrom

$$\langle g(y_i) - g(y_{i-1}),\, y_i - y_{i-1}\rangle > 0$$

But $y_i - y_{i-1} = \lambda_i(b - a)$ for some positive λ_i, and then we must have also

$$\langle g(y_i) - g(y_{i-1}),\, b - a\rangle > 0.$$

By summing the above relations from $i = 1$ to $i = m$, we obtain

$$\langle g(b) - g(a),\, b - a\rangle > 0.$$

Rewriting this relation using the definition of g we have

$$\langle f(b) - f(a),\, b - a\rangle + \epsilon \|b - a\|^2 > 0.$$

By letting $\epsilon \to 0$ we conclude that

$$\langle f(b) - f(a),\, b - a\rangle \geq 0.$$

The case $\overline{f}^{\#}(x) \leq 0$, $\forall x \in G$ can be handled similarly.

Theorem 1.1.5.2 *Let G be an open convex set in $\mathbf{R}^n$ and suppose that $f : G \to \mathbf{R}^n$ satisfies*

$$\underline{f}^{\#}(x) > 0 \;\; (\overline{f}^{\#}(x) < 0),\, \forall x \in G.$$

Then f is strictly increasing (strictly decreasing) on G.

The proof of this theorem is in fact contained in the proof of Theorem 1.1.5.1.

Corollary 1.1.5.3 *Let $f : \mathbf{R}^n \to \mathbf{R}^n$ be given. The following statements are equivalent:*

1) $f^{\#}(x) = 0$, $\forall x \in \mathbf{R}^n$.

2) f is an affine function with skew-symmetric linear part.

Proof. The implication 2) $\Rightarrow$ 1) is trivial.

To show that 1) $\Rightarrow$ 2) we apply Theorem 1.1.5.1 to conclude that

$$\langle f(y) - f(x),\, y - x\rangle = 0$$

$\forall x,\, y \in \mathbf{R}^n$, and then by usage of Theorem 1.1.1.4 we conclude assertion 2).

Theorem 1.1.5.4 *Let G be convex and open in $\mathbf{R}^n$ and let $f : G \to \mathbf{R}^n$ be a Gateaux differentiable function on G. Then the following statements are equivalent:*

1) f is increasing (decreasing) on G;

2) The Gateaux differential of f is positive (negative) semidefinite in every point of G.

Proof.

1) $\Rightarrow$ 2).

Suppose that $\langle f(y) - f(x),\, y - x\rangle \geq 0$ $\forall x,\, y \in \mathbf{R}^n$. Take $y = x + \lambda t$ with $\lambda \in \mathbf{R}$, $\lambda > 0$ and $t \in \mathbf{R}^n$ arbitrarily. Then

$$\left\langle \frac{f(x + \lambda t) - f(x)}{\lambda},\;\; t \right\rangle \geq 0$$

and letting $\lambda \to 0$ we obtain for the Gateaux differential $\delta f(x)$ that $\langle \delta f(x)t,\, t\rangle \geq 0$.

2) $\Rightarrow$ 1).

Suppose that the Gateaux differential is positive semidefinite at each point of G. Take a, b in G, $a \neq b$ and $x \in [a, b]$. Then $\langle \delta f(x)t,\, t\rangle \geq 0$ for every $t \in \mathbf{R}^n$, that is,

$$\lim_{\lambda \downarrow 0} \left\langle \frac{f(x+\lambda t) - f(x)}{\lambda}, \quad t \right\rangle \geq 0 \, \forall t \in \mathbf{R}^n.$$

Take $t = b - a$ and $y = x + \lambda t$, then

$$\lim_{\substack{y \to x \\ y \in ab}} \frac{\langle f(y) - f(x), y - x\rangle}{\|y - x\|^2} \geq 0$$

where ab denotes the line determined by a and b. By an appropriate adaptation of the method in the proof of Theorem 1.1.5.1 we can conclude

$$\langle f(b) - f(a),\, b - a\rangle \geq 0.$$

Theorem 1.1.5.5 *Let G be a convex open subset of $\mathbf{R}^n$. If $f : G \to \mathbf{R}^n$ is Gateaux differentiable on G and the Gateaux differential is in each x of G positive definite (negative definite), then f is strictly increasing (strictly decreasing) on G.*

Proof. Suppose that the Gateaux differential is positive definite. Let a, b in G, $a \neq b$. Let $x \in [a, b]$. Then $\langle \delta f(x)t,\, t\rangle > 0$ for every $t \in \mathbf{R}^n \setminus \{0\}$, that is,

$$\lim_{\lambda \downarrow 0} \left\langle \frac{f(x+\lambda t) - f(x)}{\lambda}, \quad t \right\rangle > 0 \ \forall t \in \mathbf{R}^n \setminus \{0\}.$$

Take $t = b - a$ and $y = x + \lambda t$. Then

$$\lim_{\substack{y \to x \\ y \in ab}} \frac{\langle f(y) - f(x), y - x\rangle}{\|y - x\|^2} > 0$$

where ab denotes the line determined by a and b. A reasoning similar with that in the proof of Theorem 1.1.5.1 yields that

$$\langle f(b) - f(a),\, b - a\rangle > 0.$$

1.2 Computational formulae for the scalar derivative

In this section we shall consider relations of the scalar derivatives with the directional derivatives and with the basic notions of the spectral theory. In the two dimensional case we exhibit a geometric connection with the Kasner circle. The obtained formulae will be used for determining the monotonicity domains of operators on $\mathbf{R}^n$.

1.2.1 Scalar derivatives and directional derivatives

Let $f : \mathbf{R}^n \to \mathbf{R}^n$ be given and $x,\ h \in \mathbf{R}^n$. If the limit

$$f'(x; h) = \lim_{t \downarrow 0} \frac{1}{t}(f(x + th) - f(x))$$

exists with t in $\mathbf{R}$, then it is called the directional derivative of f at x in the direction h. To have a geometrical sense we suppose that $h \neq 0$, but $f'(x; 0) = 0$ can be taken obviously for each x.

The operator f is called locally Lipschitz at x, if there exist a neighbourhood V of x and a positive real number L such that for any y and z in V the inequality

$$\|f(y) - f(z)\| \leq L\|y - z\|.$$

holds.
Using these notions we have the following

Theorem 1.2.1.1 *If the operator f is locally Lipschitz at x and the directional derivative $f'(x; h)$ exists for each h, then*

$$\underline{f}^{\#}(x) = \inf_{\|h\|=1} \langle f'(x; h),\ h\rangle$$

and

$$\overline{f}^{\#}(x) = \sup_{\|h\|=1} \langle f'(x; h),\ h\rangle$$

Proof. We have from the definitions of the lower scalar derivative and of the directional derivative that

$$\underline{f}^{\#}(x) = \liminf_{g \to 0} \frac{\langle f(x + g) - f(x),\ g\rangle}{\|g\|^2} \leq \lim_{t \downarrow 0} \left\langle \frac{f(x + th) - f(x)}{t},\ \ h \right\rangle =$$

$$= \langle f'(x; h),\ h\rangle,\ \forall h,\ \|h\| = 1. \tag{1.2.1.1}$$

Suppose that $l = \underline{f}^{\#}(x)$ and consider the sequence (h_m) with $h_m \neq 0$ and $\lim_{m \to \infty} h_m = 0$ so as to have

$$\lim_{m \to \infty} \frac{\langle f(x + h_m) - f(x),\ h_m\rangle}{\|h_m\|^2} = l$$

Put $s_m = h_m/\|h_m\|$, $m \in \mathbf{N}$. Then by the compactness of the unit sphere in $\mathbf{R}^n$, there exists a subsequence (s_{m_k}) of s_m such that $\lim_{k \to \infty} s_{m_k} = s$, $\|s\| = 1$. Since

$$\lim_{k \to \infty} \frac{\langle f(x + h_{m_k}) - f(x),\ h_{m_k}\rangle}{\|h_{m_k}\|^2} = l,$$

we have

$$l = \lim_{k\to\infty} \left\langle \frac{f(x + \|h_{m_k}\|s) - f(x)}{\|h_{m_k}\|}, \quad s_{m_k} \right\rangle +$$

$$+ \lim_{k\to\infty} \frac{\langle f(x + \|h_{m_k}\|s_{m_k}) - f(x + \|h_{m_k}\|s), \, h_{m_k}\rangle}{\|h_{m_k}\|^2},$$

wherefrom

$$l = \langle f'(x; s), \, s\rangle + \lim_{k\to\infty} \frac{\langle f(y_{m_k}) - f(z_{m_k}, \, h_{m_k}\rangle}{\|h_{m_k}\|^2} \tag{1.2.1.2}$$

with $y_{m_k} = x + \|h_{m_k}\|s_{m_k}$, $z_{m_k} = x + \|h_{m_k}\|s$.

We have

$$\lim_{k\to\infty} y_{m_k} = \lim_{k\to\infty} z_{m_k} = x \tag{1.2.1.3}$$

and

$$\|y_{m_k} - z_{m_k}\| = \|h_{m_k}\| \, \|s_{m_k} - s\|. \tag{1.2.1.4}$$

Since f is locally Lipschitz at x, there exist a neighbourhood V of x and a real number $L > 0$ such that for any y and z in V

$$\|f(y) - f(z)\| \le L\|y - z\|. \tag{1.2.1.5}$$

From relations (1.2.1.3) there exist a k_0 such that for every $k \ge k_0$ one has $y_{m_k}, \, z_{m_k} \in V$. Hence by (1.2.1.4) and (1.2.1.5) it follows that

$$\|f(y_{m_k}) - f(z_{m_k})\| \le L\|h_{m_k}\| \, \|s_{m_k} - s\|, \ \forall k \ge k_0,$$

wherefrom

$$\frac{|\langle f(y_{m_k}) - f(z_{m_k}), \, h_{m_k}\rangle|}{\|h_{m_k}\|^2} \le \frac{\|f(y_{m_k}) - f(z_{m_k})\|}{\|h_{m_k}\|} \le L\|s_{m_k} - s\|,$$

and then

$$\lim_{k\to\infty} \frac{\langle f(y_{m_k}) - f(z_{m_k}), \, h_{m_k}\rangle}{\|h_{m_k}\|^2} = 0.$$

Using now (1.2.1.1), (1.2.1.2) and the obtained relation, we conclude that

$$\underline{f}^{\#}(x) = \inf_{\|h\|=1} \langle f'(x; h), \, h\rangle.$$

The second relation in the theorem can be deduced in a similar way.

Theorem 1.2.1.2 *If the operator f is Frechét differentiable at x, with the differential $df(x)$, then*

$$\underline{f}^{\#}(x) = \inf_{\|h\|=1} \langle df(x)(h), \, h\rangle,$$

$$\overline{f}^{\#}(x) = \sup_{\|h\|=1} \langle df(x)(h), \, h\rangle.$$

Proof.

We have by definitions

$$\underline{f}^{\#}(x) = \liminf_{g\to 0}\frac{\langle f(x+g)-f(x),\, g\rangle}{\|g\|^2} \le \lim_{t\downarrow 0}\left\langle \frac{f(x+th)-f(x)}{t},\ h \right\rangle =$$

$$= \langle df(x)(h),\, h\rangle$$

for each h, $\|h\| = 1$. Hence

$$\underline{f}^{\#}(x) \le \inf_{\|h\|=1}\langle df(x)(h),\, h\rangle. \tag{1.2.1.6}$$

Conversely, for the expression

$$\frac{f(x+g)-f(x)-df(x)(g)}{\|g\|} = w(x,g)$$

we have $\lim\limits_{g\to 0} w(x,g) = 0$, hence

$$\underline{f}^{\#}(x) = \liminf_{g\to 0}\frac{\langle f(x+g)-f(x),\, g\rangle}{\|g\|^2} =$$

$$\liminf_{g\to 0}\left\langle\ w(x,g) + df(x)\left(\frac{g}{\|g\|}\right),\ \frac{g}{\|g\|}\right\rangle =$$

$$\lim_{g\to 0}\left\langle\ w(x,g),\ \frac{g}{\|g\|}\right\rangle + \liminf_{g\to 0}\left\langle\ df(x)\left(\frac{g}{\|g\|}\right),\ \frac{g}{\|g\|}\right\rangle.$$

We used in the last relation the fact that the first limit exist since

$$|\langle w(x,g),\, \frac{g}{\|g\|}\rangle| \le \|w(x,g)\| \to 0 \ \text{ for } \ g\to 0.$$

Hence

$$\underline{f}^{\#}(x) = \liminf_{g\to 0}\left\langle\ df(x)\left(\frac{g}{\|g\|}\right),\ \frac{g}{\|g\|}\right\rangle \ge \inf_{\|h\|=1}\langle df(x)(h),\, h\rangle.$$

Compare this relation with (1.2.1.6) to conclude that

$$\underline{f}^{\#}(x) = \inf_{\|h\|}\langle df(x)(h),\, h\rangle.$$

A similar way yields the proof of the second relation of the theorem.

Remark. Let us consider the function $f : \mathbf{R}^2 \to \mathbf{R}^2$ given by

$$f(x,y) = \begin{cases} \left(\dfrac{xy}{(x^2+y^2)^{\frac{1}{2}}},\ \dfrac{xy}{(x^2+y^2)^{\frac{1}{2}}}\right), & x^2+y^2 \ne 0 \\ (0,0) & ,\ x=y=0. \end{cases}$$

Then f is locally Lipschitz at every $(x,\, y)$ and has directional derivatives in each direction. Hence Theorem 1.2.1.1 applies, but not Theorem 1.2.1.2, since f is not differentiable at $(0,\, 0)$.

Using Theorem 1.2.1.1 we get that

$$\underline{f}^{\#}(0,\, 0) = -(2)^{-1/2},\, \overline{f}^{\#}(0,\, 0) = (2)^{-1/2}.$$

(Thanks are due to prof. J. Kolumbán who suggested us the above example.)

Corollary 1.2.1.3 *Let the operator f be Frechét differentiable at x. Then*

(a) $\underline{f}^{\#}(x) \geq 0$, ($\overline{f}^{\#}(x) \leq 0$) if and only if $df(x)$ is positive semidefinite (negative semidefinite).

(b) $\underline{f}^{\#}(x) > 0$, ($\overline{f}^{\#}(x) < 0$) if and only if $df(x)$ is positive definite (negative definite).

(c) f is scalarly differentiable at x and $f^{\#}(x) = 0$ if and only if $df(x)$ is a skew symmetric linear operator.

Proof. The assertions (a) and (b) are obvious from Theorem 1.2.1.2. For (c) compare this theorem with Theorem 1.1.1.3.

For $A : \mathbf{R}^n \to \mathbf{R}^n$ a linear operator, we denote by A_s the operator $(A + A^*)/2$, where A^* is the adjoint of A. Let $\sigma(A_s)$ be the spectrum of A_s. With this notation we have the

Lemma 1.2.1.4 *If $A : \mathbf{R}^n \to \mathbf{R}^n$ is a linear operator, then*

$$\min_{\|h\|=1} \langle Ah,\, h\rangle = \min_{\|h\|=1} \sigma(A_s), \quad \max_{\|h\|=1} \langle Ah,\, h\rangle = \max_{\|h\|=1} \sigma(A_s).$$

Proof.

$$\min_{\|h\|=1} \langle Ah,\, h\rangle = \min_{\|h\|=1} \left\langle \frac{A + A^*}{2} h, \quad h \right\rangle = \min_{\|h\|=1} \langle A_s h,\, h\rangle.$$

Since A_s is symmetric by the well-known relation we have $\min\limits_{\|h\|=1} \langle A_s h,\, h\rangle = \min\limits_{\|h\|=1} \sigma(A_s)$

Theorem 1.2.1.5 *Let f be Frechét differentiable at x with the differential $df(x)$. Then we have*

$$\underline{f}^{\#}(x) = \min \sigma(df(x)_s); \quad \overline{f}^{\#}(x) = \max \sigma(df(x)_s).$$

Proof. The assertion follows directly from Theorem 1.2.1.2 by using Lemma 1.2.1.4.

Corollary 1.2.1.6 *Let $f : \mathbf{R}^n \to \mathbf{R}^n$ be differentiable at the convex open set G in $\mathbf{R}^n$. Consider the following assertions:*

(a1) For each x in G all the eigenvalues of $df(x)_s$ are nonnegative (nonpositive).

(a2) For each x in G all the eigenvalues of $df(x)_s$ are positive (negative).

(b1) f is increasing (decreasing) on G.

(b2) f is strictly increasing (strictly decreasing) on G.

Then $(a1) \Leftrightarrow (b1)$ and $(a2) \to (b2)$.

Proof. Compare Theorem 1.2.1.5 with Theorems 1.1.5.1 and 1.1.5.2.

1.2.2 Applications

A. The two-dimensional case.

We shall use in this section the terminology and some facts from complex analysis in the form they are exposed in [8] and [4].

We recall that a function $f = u + iv : \mathbf{C} \to \mathbf{C}$ is called $\mathbf{R}$-*differentiable* at z in $\mathbf{C}$ if f considered as operator from $\mathbf{R}^2$ to $\mathbf{R}^2$ by the identification of $\mathbf{C}$ with $\mathbf{R}^2$ is Frechét differentiable at z.

Theorem 1.2.2.1 *If $f = u + iv : \mathbf{C} \to \mathbf{C}$ is $\mathbf{R}$-differentiable at $z \in \mathbf{C}$ then*

$$\underline{f}^{\#}(z) = \mathrm{Re}\frac{\partial f}{\partial z} - \left| \frac{\partial f}{\partial \overline{z}} \right|,$$

$$\overline{f}^{\#}(z) = \mathrm{Re}\frac{\partial f}{\partial z} + \left| \frac{\partial f}{\partial \overline{z}} \right|,$$

with $\dfrac{\partial f}{\partial z} = \dfrac{1}{2}\left(\dfrac{\partial f}{\partial x} - i\dfrac{\partial f}{\partial y} \right)$ *and* $\dfrac{\partial f}{\partial \overline{z}} = \dfrac{1}{2}\left(\dfrac{\partial f}{\partial x} + i\dfrac{\partial f}{\partial y} \right)$ *taken in z.*

Proof. Let $z = x + iy$. Then the Frechét differential $df(z)$ of f in z can be identified with the matrix

$$\begin{bmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\[2ex] \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{bmatrix}$$

taken in (x, y). We have then that

$$df(z)_s = \frac{df(z) + df(z)^*}{2} = \begin{bmatrix} \dfrac{\partial u}{\partial x} & \dfrac{1}{2}\left(\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \right) \\[2ex] \dfrac{1}{2}\left(\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \right) & \dfrac{\partial v}{\partial y} \end{bmatrix}$$

By solving the characteristic equation of this matrix we get the eigenvalues:

$$\lambda_{1,2} = \frac{1}{2}\left\{ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \pm \left[\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right)^2 + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^2 \right]^{1/2} \right\}$$

and hence from Theorem 1.2.1.5

$$\underline{f}^{\#}(x) = \frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) - \frac{1}{2}\left[\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right)^2 + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^2 \right]^{1/2}$$

$$\overline{f}^{\#}(x) = \frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + \frac{1}{2}\left[\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right)^2 + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^2 \right]^{1/2}$$

Now taking into account the definitions of $\dfrac{\partial f}{\partial z}$ and $\dfrac{\partial f}{\partial \overline{z}}$ we conclude the proof.

Theorem 1.2.2.1 together with Corollary 1.2.1.6 yield

Corollary 1.2.2.2 *Let $f : \mathbf{C} \to \mathbf{C}$ be an $\mathbf{R}$-differentiable function on the convex open set G in $\mathbf{C}$. Then the following assertions are equivalent:*

(a) f is increasing (decreasing) on G;
(b) Re$\dfrac{\partial f}{\partial z} \geq \left| \dfrac{\partial f}{\partial \overline{z}} \right| \left(\mathrm{Re}\dfrac{\partial f}{\partial z} \leq - \left| \dfrac{\partial f}{\partial \overline{z}} \right| \right)$ for every z in G.

Remark. Let $f : \mathbf{C} \to \mathbf{C}$ be given and suppose that f is $\mathbf{R}$-differentiable at z. Then we have the following expression for the (complex) directional derivative in the direction $h \neq 0$ (see [8] Proposition 2.91 or [4] p. 34):

$$f'(z;\, h) = \frac{\partial f}{\partial z} + \frac{\partial f}{\partial \overline{z}} e^{-2i\theta}$$

with $\theta = \arg h$. That is, since $\dfrac{\partial f}{\partial z}$ and $\dfrac{\partial f}{\partial \overline{z}}$ do not depend on h (they are taken as above in z), the directional derivatives describe a circle, the so-called Kasner's circle, when h varies $\dfrac{\partial f}{\partial z}$ is the centre, $\left| \dfrac{\partial f}{\partial \overline{z}} \right|$ is the radius of the circle. Hence $\underline{f}^{\#}(z)$ is the minimal oriented distance of the Kasner circle from the imaginary axis, $\overline{f}^{\#}(z)$ is the maximal oriented distance of this circle from the imaginary axis. Thus Corollary 1.2.2.2 has the following equivalent form:

Corollary 1.2.2.3 *Let $f : \mathbf{C} \to \mathbf{C}$ be an $\mathbf{R}$ differentiable function on the convex open set G of $\mathbf{C}$. Then f is increasing (decreasing) on G if and only if the Kasner circle is contained for every z in G in the closed right half-plane (closed left half-plane) of the plane $\mathbf{C}$.*

1.2.2.4 Examples

(1) Let $f : \mathbf{R}^2 \to \mathbf{R}^2$ be given by $f(x, y) = (xy, x+y)$. Let us identify $\mathbf{R}^2$ with $\mathbf{C}$ and consider f to be a complex function. Then

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y} \right) = \frac{1}{2}[y + 1 + i(1-x)]$$

(with $z = x + iy$) and thus $\mathrm{Re}\dfrac{\partial f}{\partial z} = (y+1)/2$.

$$\frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} \right) = \frac{1}{2}[y - 1 + i(x+1)]$$

and hence $\left| \dfrac{\partial f}{\partial \overline{z}} \right| = \dfrac{1}{2}[(x+1)^2 + (y-1)^2]^{1/2}$.

We have from Theorem 1.2.2.1 that

$$\underline{f}^{\#}(x, y) = \frac{y+1}{2} - \frac{1}{2}[(x+1)^2 + (y-1)^2]^{1/2}$$

$$\overline{f}^{\#}(x, y) = \frac{y+1}{2} + \frac{1}{2}[(x+1)^2 + (y-1)^2]^{1/2}$$

wherefrom we get that $\underline{f}^{\#}(x, y) > 0$ is equivalent with $y > \left(\dfrac{x+1}{2} \right)^2$. That is, f is strictly increasing in the domain above the parabola $y = \left(\dfrac{x+1}{2} \right)^2$ (see Theorem 1.1.5.1).

The condition $\overline{f}^{\#}(x, y) \leq 0$ yields a contradiction, hence f is nowhere decreasing (see Theorem 1.1.5.1).

(2) Let $f : \mathbf{R}^2 \to \mathbf{R}^2$ be given by $f(x, y) = (\sin x - y, x + \sin y)$. The symmetrization of the Jacoby matrix of this operator is then

$$(df(x,y))_s = \begin{bmatrix} \cos x & 0 \\ 0 & \cos y \end{bmatrix}.$$

The eigenvalues of this matrix are $\cos x$ and $\cos y$ and hence we have according to Theorem 1.2.1.5 that

$$\underline{f}^{\#}(x, y) = \min\{\cos x, \cos y\},$$

$$\overline{f}^{\#}(x, y) = \max\{\cos x, \cos y\}.$$

We have $\underline{f}^{\#}(x, y) > 0$ and f is thus strictly increasing for $(x, y) \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2} \right) \times \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2} \right)$. Similarly, $\overline{f}^{\#}(x, y) < 0$ and f is strictly decreasing for $(x, y) \in \left(\dfrac{\pi}{2}, \dfrac{3\pi}{2} \right) \times \left(\dfrac{\pi}{2}, \dfrac{3\pi}{2} \right)$. Similar assertions hold for

2π multiple translations of the above domains in the direction of the x axis and the y axis.

B. The case $n > 2$.

1.2.2.5. Examples

1) Let $f : \mathbf{R}^n \to \mathbf{R}^n$ be given by

$$f(x^1, x^2, \dots, x^n) = ((x^1)^2 + \dots + (x^n)^2, -2x^1x^2, \dots, -2x^1x^n).$$

By the symmetrization of the Jacoby matrix we get

$$(df(x))_s = \begin{bmatrix} 2x^1 & 0 & \dots & 0 \\ 0 & -2x^1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & -2x^1 \end{bmatrix},$$

whose eigenvalues are $\lambda_1 = 2x^1$, $\lambda_2 = \dots = \lambda_n = -2x^1$. By Theorem 1.2.1.2 we have

$$\underline{f}^{\#}(x) = -2|x^1| \quad \overline{f}^{\#}(x) = 2|x^1|.$$

Hence $\underline{f}^{\#}(x) \geq 0$ or $\overline{f}^{\#}(x) \leq 0$ if and only $x^1 = 0$. Thus f is nowhere strictly decreasing or strictly increasing.

2) Consider the function $f : \mathbf{R}^3 \to \mathbf{R}^3$ given by

$$f(x, y, z) = (x^2, y + z, z).$$

Then

$$df(x, y, z)_s = \begin{bmatrix} 2x & 0 & 0 \\ 0 & 1 & 1/2 \\ 0 & 1/2 & 1 \end{bmatrix}.$$

It can be checked that $(df(x, y, z)_s$ is strictly positive definite for $x > 0$. From Theorem 1.1.5.5 it follows that f is strictly increasing for $x > 0$.

1.3 Monotonicity, scalar differentiability and conformity

In this section we are going to relate the existence of the scalar derivative with the notion of conformity. In the global case the scalar differentiability of a map (which can be identified with a vector field) on a convex open domain coincide with the conformity of the one parameter transformation group generated by this map (vector field). In the local case the correspondence is given using the conformal derivative, a new notion introduced by us.

We shall also present a more geometrical interpretation of the well-known correspondence between nonexpansive (expansive) maps and decreasing (increasing) ones [40], which generalize the Lie correspondence between skew-symmetric maps and isometries.

1.3.1 The coefficient of conformity and the conformal derivative

Definition 1.3.1.1 *Let $f : \mathbf{R}^n \to \mathbf{R}^n$ a map and $p \in \mathbf{R}^n$. If the limit*

$$f^c(p) = \lim_{q \to p} \frac{\|f(q) - f(p)\|}{\|q - p\|},$$

exists then f will be called conformally differentiable in p *and $f^c(p)$ the* conformal derivative of f in p. *If f is conformally differentiable in each point of a subset U of $\mathbf{R}^n$ then we shall say that f is* conformally differentiable on U.

Lemma 1.3.1.2 *Let $f : \mathbf{R}^n \to \mathbf{R}^n$ be a differentiable function in $p_0 \in \mathbf{R}^n$. Then f is conformally diferentiable in p_0 if and only if $\|df_{p_0}(v)\| = \lambda_0 \|v\|$ for all vectors v where λ_0 is some nonnegative constant. If $f : \mathbf{R}^n \to \mathbf{R}^n$ is differentiable on $\mathbf{R}^n$, then f is conformally differentiable on $\mathbf{R}^n$ if and only if $\|df_p(v)\| = \lambda(p)\|v\|$ for all $p \in \mathbf{R}^n$ and for all vectors v, where $\lambda(p)$ is some nonnegative real valued function of p which does not depend on v. In this case $f^c(p) = \lambda(p)$.*

Proof:
" $\Longrightarrow$ " Let $q = p + tv$, so that $t > 0$, $t \to 0$ and v an arbitrary but fixed vector. Then $q \to p$, so

$$f^c(p) = \lim_{t \to 0} \frac{\|f(p + tv) - f(p)\|}{|t|\|v\|} = \frac{1}{\|v\|}\|df_p(v)\|,$$

from where we obtain the required equality with $\lambda(p) = f^c(p)$.
" $\Longleftarrow$ " We have

$$\frac{\|f(p+v) - f(p)\|}{\|v\|} = \sqrt{\frac{\|f(p+v) - f(p)\|^2}{\|v\|^2}} = \sqrt{\frac{\|df_p(v) + \omega(p,v)\|v\|\,\|^2}{\|v\|^2}}$$

$$= \sqrt{\frac{\langle df_p(v) + \omega(p,v)\|v\|, df_p(v) + \omega(p,v)\|v\|\rangle}{\|v\|^2}}$$

$$\begin{aligned} &= \sqrt{\frac{\|df_p(v)\|^2}{\|v\|^2} + 2\frac{\langle df_p(v), \omega(p,v)\rangle}{\|h\|} + \|\omega(p,v)\|^2} \\ &= \sqrt{\lambda(p)^2 + 2\frac{\langle df_p(v), \omega(p,v)\rangle}{\|h\|} + \|\omega(p,v)\|^2}, \end{aligned} \tag{1.3.1.1}$$

where

$$\omega(p,v) \longrightarrow 0, \text{ whenever } v \to 0. \tag{1.3.1.2}$$

On the other hand

$$\frac{|\langle df_p(v), \omega(p,v)\rangle|}{\|v\|} \leq \frac{\|df_p(v)\|\|\omega(p,v)\|}{\|v\|} = \lambda(p)\|\omega(p,v)\|.$$

So

$$\frac{\langle df_p(v), \omega(p,v)\rangle}{\|v\|} \longrightarrow 0, \text{ whenever } v \to 0. \tag{1.3.1.3}$$

Relations (1.3.1.1),(1.3.1.2) and (1.3.1.3) imply that the limit

$$\lim_{v\to 0} \frac{\|f(p+v) - f(p)\|}{\|v\|}$$

exists and is equal to $\lambda(p)$. So f is conformally differentiable at p and $f^c(p) = \lambda(p)$.

Theorem 1.3.1.3 *Let $f : U \to \mathbf{R}^n$ be a smooth map, where $U \subset \mathbf{R}^n$ is an open set. Then f is conformal if and only if it is conformally differentiable on U and $f^c(p) \neq 0$, for all $p \in U$. In this case the coefficient of conformity $\lambda(p)$ is equal to $f^c(p)$ at each p from U.*

The proof is straightforward consequence of Lemma 1.3.1.2.

Theorem 1.3.1.4 *Let $f : U \to \mathbf{R}^n$ be a smooth map where $U \subset \mathbf{R}^n$ is an open set. Then f is conformal if and only if*

$$df_p^* \circ df_p = \lambda(p)^2 I,$$

for all p and some real valued positive function $\lambda(p)$, where df_p^* denotes the adjoint operator of df_p and I the identity operator.

This theorem is an easy consequence of Lemma 1.3.1.2.

1.3.2 Monotone vector fields and expansive maps

The following theorem is a well known result from the theory of Lie groups, which states that the Lie algebra of $O(n)$ is the set of skew-symmetric matrices. However since we shall use the same idea in the foregoing that the proof of this theorem and since this is an easy proof of a classical result we shall state it and prove it.

Theorem 1.3.2.1 *Let v be a vector field on $\mathbf{R}^n$ and $\psi(\varepsilon, p)$ the one parameter transformation group generated by v. Then v is skew-symmetric as a function from $\mathbf{R}^n$ to $\mathbf{R}^n$ if and only if $\psi(\varepsilon, p)$ is an isometry for every ε fixed.*

Proof:
Theorem 1.1.1.3 implies that

$$\langle v(\psi(\varepsilon, q)) - v(\psi(\varepsilon, p)), \psi(\varepsilon, q) - \psi(\varepsilon, p)\rangle = 0$$

for all $p, q \in \mathbf{R}^n$ and $\varepsilon \in \mathbf{R}$. Hence

$$\frac{1}{2}\frac{d}{d\varepsilon}\|\psi(\varepsilon, q) - \psi(\varepsilon, p)\|^2 = 0,$$

since

$$\|\psi(\varepsilon,q)-\psi(\varepsilon,p)\|^2 = \langle \psi(\varepsilon,q)-\psi(\varepsilon,p), \psi(\varepsilon,q)-\psi(\varepsilon,p)\rangle$$

and

$$\frac{d}{d\varepsilon}\psi(\varepsilon,p) = v(\psi(\varepsilon,p)).$$

Thus we have

$$\|\psi(\varepsilon,q)-\psi(\varepsilon,p)\| = \text{constant}$$

for p, q fixed. If we put $\varepsilon = 0$ in this relation we obtain

$$\|\psi(\varepsilon,q)-\psi(\varepsilon,p)\| = \|q-p\|$$

for all p and q, since $\psi(0,p) = p$. The converse can be proved similarly.

Theorem 1.3.2.2 *Let v be a vector field on $\mathbf{R}^n$ and $\psi(\varepsilon,p)$ ($\varepsilon > 0$) the one parameter transformation group generated by v. Then v is increasing (decreasing) as a function from $\mathbf{R}^n$ to $\mathbf{R}^n$ if and only if $\|\psi(\varepsilon,p)-\psi(\varepsilon,q)\|$ is increasing (decreasing) as a real function of ε for all p, q fixed.*

Proof:
"$\Longrightarrow$"

$$\langle v(\psi(\varepsilon,q)) - v(\psi(\varepsilon,p)), \psi(\varepsilon,q)-\psi(\varepsilon,p)\rangle \geq 0$$

$\forall p, q \in \mathbf{R}^n$ and $\forall \varepsilon \in \mathbf{R}^n$. Hence as before

$$\frac{1}{2}\frac{d}{d\varepsilon}\|\psi(\varepsilon,q)-\psi(\varepsilon,p)\|^2 \geq 0,$$

from where it follows that

$$\|\psi(\varepsilon,q)-\psi(\varepsilon,p)\|$$

is increasing
"$\Longleftarrow$" If

$$\|\psi(\varepsilon,p)-\psi(\varepsilon,q)\|$$

is increasing so is

$$\|\psi(\varepsilon,p)-\psi(\varepsilon,q)\|^2,$$

hence

$$\frac{d}{d\varepsilon}\|\psi(\varepsilon,p)-\psi(\varepsilon,q)\|^2 \geq 0$$

which is equivalent to

$$\langle v(\psi(\varepsilon,q)) - v(\psi(\varepsilon,p)), \psi(\varepsilon,q) - \psi(\varepsilon,p)\rangle \geq 0$$

$\forall p,q \in \mathbf{R}^n$ and $\forall \varepsilon \in \mathbf{R}^n$. For $\varepsilon = 0$ we have that

$$\langle v(q) - v(p), q - p\rangle \geq 0$$

for all p and q. The case v decreasing can be treated similarly.

The above theorem implies the following result, found in [40]:

Theorem 1.3.2.3 *Let v be a vector field on $\mathbf{R}^n$ and $\psi(\varepsilon,p)$ the one parameter transformation group generated by v through p. Then v is increasing (decreasing) as a function from $\mathbf{R}^n$ to $\mathbf{R}^n$ if and only if $\psi(\varepsilon,p)$ is an expansive (nonexpansive) function of p for all $\varepsilon > 0$ fixed.*

Proof:
If v is increasing then it follows from Theorem 1.3.2.2 that

$$\|\psi(\varepsilon,p) - \psi(\varepsilon,q)\|$$

is increasing for all p, q fixed. Particularly $\varepsilon > 0$ yields

$$\|\psi(\varepsilon,p) - \psi(\varepsilon,q)\| \geq \|p - q\|$$

so $\psi(\varepsilon,p)$ is an expansive function of p.
Conversely if $\psi(\varepsilon,p)$ is expansive for all $\varepsilon > 0$ fixed then we have that

$$\|\psi(\delta-\varepsilon,\psi(\varepsilon,p)) - \psi(\delta-\varepsilon,\psi(\varepsilon,q))\| \geq \|\psi(\varepsilon,p) - \psi(\varepsilon,q)\|$$

for all $\delta > \varepsilon > 0$ and all p, q from $\mathbf{R}^n$. But

$$\psi(\delta-\varepsilon,\psi(\varepsilon,p)) = \psi(\delta,p),$$

since $\psi(\varepsilon,p)$ is an one parameter transformation group, so

$$\|\psi(\delta,p) - \psi(\delta,q)\| \geq \|\psi(\varepsilon,p) - \psi(\varepsilon,q)\|$$

for all $\delta > \varepsilon > 0$. So

$$\|\psi(\varepsilon,p) - \psi(\varepsilon,q)\|$$

is increasing as a real function of ε for all p, q fixed. Hence by Theorem 1.3.2.2 v is increasing as a function from $\mathbf{R}^n$ to $\mathbf{R}^n$.

1.3.3 The connection between the coefficient of conformity and the scalar derivative

Definition 1.3.3.1 *A vector field on* $\mathbf{R}^n$ *is* scalarly differentiable *if it is scalarly differentiable as a function from* $\mathbf{R}^n$ *to* $\mathbf{R}^n$.

Theorem 1.3.3.2 *Let* v *be a scalarly differentiable vector field on* $\mathbf{R}^n$ *and* $\psi^\varepsilon(p) = \psi(\varepsilon, p)$ $(\varepsilon > 0)$ *the one parameter conformal transformation group generated by* v.
Then we have

$$\|d\psi_p^\varepsilon(h)\| = \exp(\int_0^\varepsilon v^\#(\psi(\delta,p))d\delta).\|h\| \tag{1.3.3.1}$$

for all h*, or equivalently*

$$\psi^c(\varepsilon,p) = \exp(\int_0^\varepsilon v^\#(\psi(\delta,p))d\delta.$$

Proof:
From the chain rule we have that

$$\frac{d}{d\varepsilon}d\psi_p^\varepsilon = d\frac{d}{d\varepsilon}\psi(\varepsilon,p) = d[v\circ\psi^\varepsilon]_p = dv_{\psi(\varepsilon,p)}\circ d\psi_p^\varepsilon.$$

We have that

$$v^\#(\psi(\varepsilon,p)) = \lim_{t\to 0}\frac{\langle v(\psi(\varepsilon,p)+th) - v(\psi(\varepsilon,p)), th\rangle}{\|th\|^2}$$

$$= \frac{1}{\|h\|^2}\lim_{t\to 0}\left\langle \frac{v(\psi(\varepsilon,p)+th) - v(\psi(\varepsilon,p))}{t}, \quad h \right\rangle = \frac{\langle dv_{\psi(\varepsilon,p)}(h), h\rangle}{\|h\|^2}$$

for all h.
If we put in this chain of relations $d\psi_p^\varepsilon(h)$ instead of h we obtain:

$$v^\#(\psi(\varepsilon,p)) =$$

$$\frac{\langle (dv_{\psi(\varepsilon,p)}\circ d\psi_p^\varepsilon)(h), d\psi_p^\varepsilon(h)\rangle}{\|d\psi_p(h)\|^2} = \frac{\left\langle \frac{d}{d\varepsilon}d\psi_p^\varepsilon(h), \quad d\psi_p^\varepsilon(h) \right\rangle}{\|d\psi_p(h)\|^2} =$$

$$\frac{1}{2}\frac{\frac{d}{d\varepsilon}\|d\psi_p^\varepsilon(h)\|^2}{\|d\psi_p^\varepsilon(h)\|^2}.$$

So

$$\frac{d}{d\varepsilon}\ln\|d\psi_p^\varepsilon(h)\| = v^\#(\psi(\varepsilon,p)). \tag{1.3.3.2}$$

Integrating (1.3.3.2) from 0 to ε we obtain

$$\ln\|d\psi_p^\varepsilon(h)\| - \ln\|h\| = \int_0^\varepsilon v^\#(\psi(\delta,p))d\delta,$$

and from here the required relation. Here we have used that $d\psi_p^0$ is the identical map. This is an easy consequence of the Taylor expansion of ψ and it is proved for example in [25].

Chapter 2

Monotone vector fields on Riemannian manifolds

In this chapter we generalize the notion of Minty-Browder monotonicity to Riemannian manifolds. For global examples of monotone vector fileds we often consider Hadamard manifolds (complete, simply connected Riemannian manifolds of nonpositive sectional curvature). However these examples can be extended locally for positive curvature manifolds, using the comparison theorems. To avoid the technical difficulties we consider just the Hadamard manifolds, where these vector fields can be defined globally.

2.1 Geodesic monotone vector fields

Although many results of convex analysis and optimization theory were generalized to Riemannian manifolds, there are few attempts to generalize the Minty-Browder monotonicity notion.
C. Udrişte proves [33] that a function f defined on an open and (geodesic) convex subset [31] of a Riemannian manifold is (geodesic) convex [31], if and only if its gradient is geodesic monotone. This result is expressed, in terms of the differential of f, and is stated in [34] too. T. Rapcsák in [31] reformulates the result of Udrişte, giving explicitly the inequality which express the geodesic monotonicity of the gradient of a convex function. However, as far as we know the nongradient type geodesic monotone vector fields has not been studied yet. Connected to the notion of geodesic monotonicity we shall introduce the strictly geodesic monotone, virtually geodesic monotone and trivially geodesic monotone vector fields.
We shall prove that a vector field X on a Riemannian manifold is geodesic monotone (strictly geodesic monotone), if and only if the first variation of the length of every geodesic arc γ with infinitesimal variation the restriction of X to γ is nonnegative (positive).
Analysing the existence of geodesic monotone vector fields, we shall prove that there is no strictly geodesic monotone vector field on a Riemannian manifold which contains a closed geodesic. Similarly, if every geodesic of a Rieman-

nian manifold is closed then there is no virtually monotone vector field on the manifold. As a consequence, since every compact and complete Riemannian manifold contains a closed geodesic [12], there are no strictly monotone vector fields on compact and complete Riemannian manifolds. For the existence of strictly geodesic monotone vector fields, we consider Hadamard manifolds (simply connected, complete Riemannian manifolds with nonpositive sectional curvature), proving that for f strictly monotone, the f-position vector fields of such manifolds are strictly geodesic monotone, where the f-position vector fields generalize the notion of position vector fields.
We can generalize the notion of scalar derivative (see section 1.1, 1.2), introduced by us, for characterizing such vector fields. The results are generalizations of some results of section 1.1 and 1.2.
The case of constant sectional curvature Riemannian manifolds presents some interesting pecularities, therefore we shall consider it separately. For illustrating the ideas it is enough to consider just the cases of S^n, and H^n, where S^n and H^n are the n-dimensional unit sphere and hyperbolical space respectively. Of course similar results holds for every constant curvature manifold.

2.1.1 Geodesic monotone vector fields and convex functionals

Definition 2.1.1.1 *Let M be a Riemannian manifold.*

(i) A subset K of M is called (geodesic) convex [31] *if for any two points of M there is a geodesic arc contained in K joining these points.*

(ii) Let K be a convex subset of M. A function $f : K \to \mathbf{R}$ is called (geodesic) convex [31] (strictly (geodesic) convex [31]) , *if $f \circ \gamma : [0, l] \to \mathbf{R}$ is convex (strictly convex) for every unit speed geodesic arc $\gamma : [0, l] \to M$ contained in K.*

If N is an arbitrary manifold, we shall denote by $\mathrm{Sec}(TN)$ the family of sections of the tangent bundle TN of N. Using this notation we have the following definition:

Definition 2.1.1.2 *Let $(M,\, g)$ be a Riemannian manifold, $K \subset M$ a geodesic convex open set and $X \in Sec(TK)$ a vector field on K.*
X is called geodesic monotone *if for every $x, y \in K$ and every unit speed geodesic arc $\gamma : [0, l] \to M$ joining x and y ($\gamma(0) = x$, $\gamma(l) = y$) and contained in K we have that*

$$g(X_x,\, \dot{\gamma}(0)) \leq g(X_y,\, \dot{\gamma}(l)),$$

where $\dot{\gamma}$ denotes the tangent vector of γ with respect to the arclength.
X is called strictly geodesic monotone *if for every distinct x and y we have*

$$g(X_x,\, \dot{\gamma}(0)) \; < \; g(X_y,\, \dot{\gamma}(l)).$$

X is called virtually geodesic monotone *if it is geodesic monotone and there are x and y such that*

$$g(X_x,\, \dot\gamma(0)) \,<\, g(X_y,\, \dot\gamma(l)).$$

X is called trivially geodesic monotone *if for every x, y we have that*

$$g(X_x,\, \dot\gamma(0)) = g(X_y, \dot\gamma(l)).$$

Since the length of the tangent vector of an arbitrary parametrized geodesic is constant, the relations of definition 2.1.1.2 can be given for any parametrization of γ. It is also easy to see that X is geodesic monotone (strictly geodesic monotone), if and only if for every arbitrarily parametrized geodesic γ the function

$$\upsilon : \tau \mapsto g(X_{\gamma(\tau)},\, \gamma'(\tau))$$

is monotone (strictly monotone), where $\gamma'(\tau)$ is the tangent vector of γ, with respect to its parameter τ. Similarly X is trivially monotone, if and only if υ is constant.

The following example makes the connection between geodesic monotone vector fields and monotone operators of an Euclidean space, showing that with few modifications the formers are generalizations of the laters:

Example 2.1.1.3 *Let E be an Euclidean space, $G \subset E$ an open and convex set and $h : G \to E$ a monotone operator. Then the vector field $X \in Sec(TG)$; $x \mapsto h(x)_x$, where $h(x)_x$ is the tangent vector in 0 of the curve $t \mapsto x + th(x)$, is geodesic monotone.*

The following theorem is a modified version of Udrişte's result [34], [33].

Theorem 2.1.1.4 *Let M be a Riemannian manifold and K an open and convex subset of M. A function $f : K \to \mathbf{R}$ is convex (strictly convex), if and only if its gradient* grad f *is geodesic monotone (strictly geodesic monotone). (Udrişte, 1976)*

C. Udrişte gives the inequality which expreses the geodesic monotonicity in terms of df, the differential of f.
T. Rapcsák [31] states the inequality in explicit form, using the gradient of f. However, neither of them speaks (in general) about monotone vector fields.

2.1.2 Geodesic monotone vector fields and the first variation of the length of a geodesic

In this subsection we make a connection between the geodesic monotone vector fields and the first variation of the lengths of geodesics. First of all we recall some preliminary definitions and results:

Definition 2.1.2.1 *Let $(M,\, g)$ be a Riemannian manifold and $\xi : [\lambda, \nu] \to M$ a piecewise smooth continuous curve with breaking points $\lambda < t_1 < \ldots < t_{n-1} < \nu$. Let $t_0 = \lambda$ and $t_n = \nu$. A* piecewise smooth deformation *of the curve ξ is a continuous map*

$$\delta :\, [\lambda, \nu] \times [-\epsilon, \epsilon] \to M;\ \ \epsilon > 0,$$

which is smooth on the rectangles $[t_{i-1}, t_i] \times [-\epsilon, \epsilon]$; $i = \overline{1, n}$ and $\delta(t, 0) = \xi(t)$; $\forall t \in [\lambda, \nu]$.
Let s be the parameter which varies in $[-\epsilon, \epsilon]$. The vector field V along ξ defined by

$$V(t) = \frac{d\delta}{ds}|_{s=0}.$$

is called the infinitesimal deformation *induced by δ. For a fixed $s \in [-\epsilon, \epsilon]$, the curve $\xi_s(t) = \delta(t, s)$ is called the* deformed curve*; its arclength is a smooth function*

$$L(s) = \int_\lambda^\nu \|\xi_s'(t)\| dt.$$

We call the first variation *of the arclength of ξ corresponding to the deformation δ the number*

$$L'(0) = \frac{dL}{ds}|_{s=0}.$$

Definition 2.1.2.2 *Let $\xi : [\lambda, \nu] \to M$ be a piecewise smooth curve of a Riemannian manifold $(M,\, g)$ and*

$$\lambda < t_1 < \ldots < t_{n-1} < \nu$$

its breaking points. Then the vectors

$$\Theta\xi'(t_i) = \xi'(t_i^+) - \xi'(t_i^-) \in T_{\xi(t_i)}M;\quad i = \overline{1, n-1}$$

are called the breakings *of the tangent vector.*

Then we have the following proposition:

Proposition 2.1.2.3 *Let $\xi : [\lambda, \nu] \to M$ be a piecewise smooth curve of a Riemannian manifold, parametrized by arclength $\lambda < t_1 < \ldots < t_{n-1} < \nu$ its breaking points, δ a piecewise smooth deformation of ξ and $V : [\lambda, \nu] \to TM$ the infinitesimal deformation induced by δ. Then the first variation of the length of ξ is*

$$L'(0) = -\int_\lambda^\nu g(\ddot{\xi}(t),\, V(t))dt - \sum_{i=1}^{n-1} g(\Theta\dot{\xi}(t_i),\, V(t_i)) + g(\dot{\xi},\, V)|_\lambda^\nu. \qquad (2.1.2.1)$$

The next theorem is the main result of this subsection, connecting the notion of geodesic monotonicity with the first variations of the lengths of geodesics:

Theorem 2.1.2.4 *Let (M, g) be a Riemannian manifold, K an open and geodesic convex subset of M and X a smooth vector field on K. Then, X is geodesic monotone (strictly geodesic monotone), if and only if for all unit speed geodesic arcs $\gamma : [0, l] \to M$ contained in K and all deformation δ inducing the infinitesimal variation V equal to the restriction of X to γ, the first variation of the length of γ is nonnegative (positive).*

Proof. Let x, y be two arbitrary, distinct points of M and $\gamma : [0, l] \to M$ a unit speed geodesic arc contained in K and joining x, y, such that $\gamma(0) = x$ and $\gamma(l) = y$. Let δ be a deformation of γ inducing the infinitesimal variation V equal to the restriction of X to γ. Since γ is a geodesic, the first and second terms in the variation formula (2.1.2.1) vanish. Hence we have for the first variation of the length of γ:

$$L'(0) = g(\dot{\gamma}, V)|_0^l.$$

Since V is the restriction of X to γ, we have that

$$L'(0) = g(\dot{\gamma}(l), X_y) - g(\dot{\gamma}(0), X_x). \tag{2.1.2.2}$$

Using the definition of geodesic monotonicity (strict geodesic monotonicity), (2.1.2.2) proves the theorem.

2.1.3 Closed geodesics and monotone vector fields

In this subsection we analyse the connection between the existence of closed geodesics and geodesic monotone vector fields. We start with the following theorem:

Theorem 2.1.3.1 *Let (M, g) be a Riemannian manifold which contains a closed geodesic γ. Then there are no strictly geodesic monotone vector fields on M.*

Proof. Suppose that there is a strictly monotone vector field X on M. Let τ be the parameter of γ and $x = \gamma(\tau_1)$, $y = \gamma(\tau_2)$ two arbitrary points of the curve. Then we have

$$g(X_x, \gamma'(\tau_1)) < g(X_y, \gamma'(\tau_2)). \tag{2.1.3.1}$$

and

$$g(X_y, \gamma'(\tau_2)) < g(X_x, \gamma'(\tau_1)), \tag{2.1.3.2}$$

since $g(X_{\gamma'(\tau)}, \gamma'(\tau))$ is strictly monotone and γ is closed, respectively. But inequalities (2.1.3.1) and (2.1.3.2) imply a contradiction. Hence there are no strictly geodesic monotone vector fields on M.

Using the same argument as in the proof of Theorem 2.1.3.1 we obtain the following result:

Theorem 2.1.3.2 *If every geodesic of a Riemannian manifold is closed then there are no virtually monotone vector fields on the manifold.*

Since every compact and complete Riemannian manifold contains a closed geodesic [12], we have the following theorem:

Theorem 2.1.3.3 *There are no strictly geodesic monotone vector fields on compact and complete Riemannian manifolds.*

2.1.4 The geodesic monotonicity of position vector fields

In this subsection we define the notion of f-position vector fields of a Riemannian manifold M and analyse its geodesic monotonicity under certain topological and metrical restrictions imposed to M.
Recall that a complete, simply connected Riemannian manifold, of nonpositive sectional curvature is called an *Hadamard manifold*.
By the Gauss lemma ([26] p.127) it can be seen easily that the following definition generalize the notion of position vector fields ([26] p.178) on Hadamard manifolds. (The definition is given just for local position vector fields. However by the theorem of Hadamard ([26] p.278) these vector fields can be defined globally on Hadamard manifolds)

Definition 2.1.4.1 *Let M be an Hadamard manifold and $o \in M$. If $x \neq o$, denote by l_x the distance of x from o, and by γ_x the unit speed radial geodesic joining o with x; $\gamma_x(0) = o$.*
Let $f : [0, \infty[\to [0, \infty[$. Then the vector field $P^{f,o}$, defined by

$$P_x^{f,o} = \begin{cases} f(l_x)\dot{\gamma}_x(l_x), & if \;\; x \neq o, \\ 0, & if \;\; x = o. \end{cases}$$

is called the f-position vector field *of M at o.*

In particular, the position vector field at o of an Hadamard manifold is equal to the id-position vector field of M at o, where id denotes the identity map of $[0, \infty[$.
The local f-position vector field at a point of an arbitrary Riemannian manifold (in a normal neighbourhood) can be defined similarly.
In the following definition indices $i = \overline{1,3}$ are considered modulo 3:

Definition 2.1.4.2 *A* geodesic triangle *T in a Riemannian manifold M is a set formed by three segments of minimizing unit speed geodesics (called* sides *of the triangle)*

$$\gamma_i : [0, l_i] \to M; \quad i = \overline{1,3},$$

in such a way that $\gamma_i(l_i) = \gamma_{i+1}(0)$; $i = \overline{1,3}$. The endpoints of the geodesic segments are called vertices *of T. The angle*

$$\angle(-\dot{\gamma}_i(l_i), \dot{\gamma}_{i+1}(0)); \quad i = \overline{1,3}$$

is called the (interior) angle *of the corresponding vertex.*

Then we have the following lemma, proved in [5] p. 259.

Lemma 2.1.4.3 *Let M be an Hadamard manifold of sectional curvature κ. Let a, b and c be three points of M. Then (by the Hadamard theorem* [26], p. 278*) such points determine a unique geodesic triangle T in M with vertices a, b, c. Let α, β and γ be the angles of the vertices a, b, c, respectively, and let A, B, C be the lengths of the sides opposite the vertices a, b, c, respectively. Then*

(i) $A^2 + B^2 - 2AB\cos\gamma \leq C^2$ $(< C^2$, *if* $\kappa < 0)$

(ii) $\alpha + \beta + \gamma \leq \pi$ $(< \pi$, *if* $\kappa < 0)$.

Corollary 2.1.4.4 *Let M be an Hadamard manifold and a, b and c three points of M. Then (by the Hadamard theorem* [26], p. 278*) such points determine a unique geodesic triangle T in M with vertices a, b, c. Let α, β and γ be the angles of the vertices a, b, c, respectively, and let A, B, C be the lengths of the sides opposite the vertices a, b, c, respectively. If $f : [0,\infty[\to [0,\infty[$ is a strictly increasing function, then we have:*

$$f(C)\cos\beta + f(B)\cos\gamma > 0.$$

Proof. By (ii) of Lemma 2.1.4.3 we have that $\beta+\gamma < \pi$. Hence either $\beta < \pi/2$, or $\gamma < \pi/2$. Without losing generality we can suppose that $\beta \leq \gamma$. We consider two cases:

(1) $0 < \beta \leq \gamma < \dfrac{\pi}{2}$.

In this case the inequality of the lemma is trivial.

(2) $\gamma \geq \dfrac{\pi}{2}$ and $\beta < \dfrac{\pi}{2}$.

In this case we have $\cos\gamma < 0$. Hence by (i) of Lemma 2.1.4.3 we have that

$$B^2 < A^2 + B^2 - 2AB\cos\gamma \leq C^2.$$

Thus we get $0 < B < C$, and consequently $0 \leq f(0) < f(B) < f(C)$, since f is strictly increasing. Using (ii) of Lemma 2.1.4.3 and $\cos\beta > 0$ we obtain that

$$f(C)\cos\beta + f(B)\cos\gamma > f(B)(\cos\beta + \cos\gamma) > 0.$$

Theorem 2.1.4.5 *Let M be an Hadamard manifold and $f : [0,\infty[\to [0,\infty[$ a strictly increasing function. Then for all $o \in M$ the f-position vector field at o is strictly geodesic monotone.*

Proof. Let $x \neq y \in M$.

1. If x, y belongs to the same geodesical ray (with respect to o), then we trivially have that

$$g(P_x^{f,o}, \dot{\gamma}(0)) < g(P_x^{f,o}, \dot{\gamma}(l)),$$

where $\gamma : [0,l] \to M$ is the unit speed geodesic arc joining x and y, such that $\gamma(0) = x$ and $\gamma(l) = y$.

2. If x, y do not belong to the same geodesical ray the above inequality can be easily deduced by using Corollary 2.1.4.4.

In particular the position vector field at o is strictly geodesic monotone. However, the monotonicity of this vector field is an easy consequence of Theorem 2.1.2.4, Gauss lemma ([26] p.127), Corollary 3. [26] p.128, and the convexity of the distance function from a point in an Hadamard manifold.
We remark that, if f is bounded above, then P^f is a smooth, strictly geodesic monotone and bounded vector field. (For example we can take $f = \arctan|_{[0,\infty[}$.)

Lemma 2.1.4.6 *Let $\phi : [0,\infty[\to [0,\infty[$ be a differentiable function with $\phi(0) = 0$. Then there exists a differentiable function $\psi : [0,\infty[\to [0,\infty[$, with $\psi(0) = \frac{d\phi}{dt}(0)$, $\phi(t) = t\psi(t)$, $t \in [0,1]$.*

Proof. It suffices to define, for fixed t

$$\psi(t) = \int_0^1 \frac{d\phi(ts)}{d(ts)} ds$$

and, after changing the variables, observe that

$$t\psi(t) = \int_0^t \frac{d\phi(ts)}{d(ts)} d(ts) = \phi(t).$$

Using Lemma 2.1.4.6 to the function f of Theorem 2.1.4.5, we have that there is a smooth function $h : [0,\infty[\to [0,\infty[$, with $h(0) = \frac{df}{dt}(0)$ such that $f(t) = th(t)$. Hence, using definition 2.1.4.1, we have the following proposition:

Proposition 2.1.4.7 *Let M be an Hadamard manifold and $o \in M$. Let $f : [0,\infty[\to [0,\infty[$ be a smooth function with $f(0) = 0$ and $h : [0,\infty[\to [0,\infty[$ the smooth function defined by*

$$\begin{cases} h(t) = \dfrac{f(t)}{t}, \ \textit{if } t \neq 0, \\ \\ h(0) = \dfrac{df}{dt}(0). \end{cases}$$

Then the f-position vector field $P^{f,o}$ of M at o assigns to each $x \in M$, the vector

$$P^{f,o}_x = h(l_x)P^o_x,$$

where P^o is the position vector field at o and l_x is the distance of x from o.

Proposition 2.1.4.8 *We use the notations of Proposition 2.1.4.7. Let f be strictly monotone and $k : M \to [0, \infty[$ the smooth function defined by $k(x) = h(l_x)$. Then a necessary and sufficient condition for $P^{f,o}$ to be the gradient of a strictly convex function is*

$$dk \wedge P^{o^\flat} = 0, \tag{2.1.4.1}$$

where for a smooth vector field X on M, $X^\flat$ is the smooth 1-form defined by

$$X^\flat(Y) = g(X, Y),$$

for every smooth vector field Y on M.
If the integrability condition (2.1.4.1) is satisfied, then $P^{f,o}$ =grad μ, where μ is the strictly convex function defined by

$$\mu(x) = \int_o^x kP^{o^\flat}, \tag{2.1.4.2}$$

where the integral is taken along any curve joining o with x (for example we can take it along γ_x, the radial geodesic joining o with x).

Proof. Suppose that $P^{f,o}$ =grad μ, where $\mu : M \to \mathbf{R}$. By Theorem 2.1.1.4 μ must be strictly convex, since $P^{f,o}$ is strictly geodesic monotone. Then $P^{f,o^\flat}(Y) = g(\text{grad } \mu, Y)$. Hence $P^{f,o^\flat} = d\mu$. Let $\tilde{q} : T_o(M) \to \mathbf{R}$; $\tilde{q}(v) = (v, v)$ and $q = \tilde{q} \circ \exp_o^{-1}$. Since $2P$ =grad q (by Corollary 3. [26] p.128), a similar argument proves that $2P^{o^\flat} = dq$. Hence $P^{o^\flat}$ is exact. By Proposition 2.1.4.7 we have that

$$P^{f,o^\flat} = kP^{o^\flat}. \tag{2.1.4.3}$$

Since $P^{f,o^\flat}$ is exact, it is closed. Hence (2.1.4.3) implies

$$0 = dP^{f,o^\flat} = dk \wedge P + kdP^{o^\flat}.$$

But $dP^{o^\flat} = 0$, since $P^{o^\flat}$ is exact. Hence we have that

$$dk \wedge P = 0.$$

Since M is simply connected its first cohomology group $H^1(M, \mathbf{R}) = 0$. Thus if the integrability condition (2.1.4.1) is satisfied (i.e $P^{f,o^\flat}$ is closed) then $P^{f,o^\flat}$ is exact and

$$P^{f,o^\flat} = d\mu,$$

where

$$\mu(x) = \int_o^x P^{f,o^\flat} = \int_o^x kP^{o^\flat}. \tag{2.1.4.4}$$

The first equality of (2.1.4.4) was obtained by using formula (6.50) p.130 of [16]. Hence we have that P^f =grad μ, with μ given by (2.1.4.2).

Hence, if the integrability condition $dk \wedge P = 0$ is not satisfied, $P^{f,o}$ cannot be obtained as a gradient of a strictly convex function. This shows that, there are strictly geodesic monotone vector fields on an Hadamard manifold which cannot be obtained as gradients of strictly convex functions.

2.1.5 Geodesic scalar derivative

In this subsection we shall generalize the notion of scalar derivative, and some of our results concerning the characterization of monotone operators through the scalar derivative, included in sections 1.1 and 1.2 (see also [21], [22]).
Let us start by defining the geodesic scalar derivative of a vector field:

Definition 2.1.5.1 *Let $(M,\, g)$ be a Riemannian manifold, $K \subset M$ a convex open set and $X \in Sec(TK)$ a vector field on K. Then the* lower geodesic scalar derivative *(*upper geodesic scalar derivative*) of X is the function*

$$\underline{X}^{\#} :\, K \to \mathbf{R};\;\; \underline{X}^{\#}(x) = \liminf_{\substack{t\downarrow 0\\ \gamma\in\Gamma}} \frac{g(X_{\gamma(t)},\, \dot\gamma(t)) - g(X_x,\, \dot\gamma(0))}{t}$$

$$(\overline{X}^{\#} :\, K \to \mathbf{R};\;\; \overline{X}^{\#}(x) = \limsup_{\substack{t\downarrow 0\\ \gamma\in\Gamma}} \frac{g(X_{\gamma(t)},\, \dot\gamma(t)) - g(X_x,\, \dot\gamma(0))}{t}).$$

where Γ denotes the family of unit speed geodesic arcs $\gamma :\, [0,l] \to M$ starting from x (i.e $\gamma(0) = x$), and contained in K.
If $x_0 \in K$ and $\underline{X}^{\#}(x_0) = \overline{X}^{\#}(x_0) =: X^{\#}(x_0)$, then X is called geodesic scalar differentiable in x_0 *and $X^{\#}(x_0)$ is called the* geodesic scalar derivative of X in x_0*. If X is geodesic scalar differentiable in every $x \in K$ then X is called* geodesic scalar differentiable*, and the function $X^{\#} :\, x \mapsto X^{\#}(x)$ is called the* geodesic scalar derivative *of X.*

The following theorem is a local characterization of geodesic monotone vector fields.

Theorem 2.1.5.2 *Let $(M,\, g)$ be a Riemannian manifold, $K \subset M$ a convex open set and $X \in Sec(TK)$ a vector field on K. Then we have the following assertions:*

(i) X $(-X)$ is geodesic monotone if and only if $\underline{X}^{\#}(x) \geq 0$ $(\overline{X}^{\#}(x) \leq 0)$ for all $x \in K$.

(ii) X is trivially monotone if and only if X is scalar differentiable and $X^{\#}(x) = 0$ for all $x \in K$.

Proof.

(i) First we suppose that $\underline{X}^{\#}(x) \geq 0$ for every $x \in K$ and prove that X is geodesic monotone. Let a and b be two arbitrary distinct points of K and $\gamma : [0, l] \to M$ a unit speed geodesic arc joining a and b ($\gamma(0) = a$, $\gamma(l) = b$) contained in K. Let $x = \gamma(t)$ be an arbitrary point of the geodesic arc.
Let $\epsilon > 0$ be an arbitrary but fixed positive number. Since $\underline{X}^{\#}(x) \geq 0$, there is a $\delta(t) > 0$ such that for every $s \in I_t =]t - \delta(t), t + \delta(t)[$ we have that

$$\frac{g(X_{\gamma(s)}, \dot{\gamma}(s)) - g(X_x, \dot{\gamma}(t))}{s - t} > -\frac{\epsilon}{l}. \tag{2.1.5.1}$$

(The geodesic arc can be continued to an open one contained in K and $\delta(t)$ chosen sufficiently small so that $\gamma(s)$ can be defined.) But $\{I_t : t \in [0, l]\}$ is an open covering of the compact set $[0, l]$. Hence $[0, l] \subset I_{t_1} \cup I_{t_2} \cup \ldots \cup I_{t_{m-1}}$ for some positive integer m and some points $t_1 < t_2 < \ldots < t_{m-1}$ of $[0, l]$. This yields $0 =: t_0 \in I_{t_1}$ and $l =: t_m \in I_{t_{m-1}}$. Obviously we can choose the intervals $\{I_i : i = \overline{1, m-1}\}$ so that no interval is contained in another. Let $\xi_i \in I_{t_{i-1}} \cap I_{t_i} \cap]t_{i-1}, t_i[$ for $i = \overline{1, m}$. Then, using (2.1.5.1) we have that

$$g(X_{\gamma(\xi_i)}, \dot{\gamma}(\xi_i)) - g(X_{\gamma(t_{i-1})}, \dot{\gamma}(t_{i-1})) > -\frac{\epsilon}{l}(\xi_i - t_{i-1}), \tag{2.1.5.2}$$

and

$$g(X_{\gamma(t_i)}, \dot{\gamma}(t_i)) - g(X_{\gamma(\xi_i)}, \dot{\gamma}(\xi_i)) > -\frac{\epsilon}{l}(t_i - \xi_i), \tag{2.1.5.3}$$

for $i = \overline{1, m}$. Summing the inequalities (2.1.5.2) and (2.1.5.3) we obtain

$$g(X_{\gamma(t_i)}, \dot{\gamma}(t_i)) - g(X_{\gamma(t_{i-1})}, \dot{\gamma}(t_{i-1})) > -\frac{\epsilon}{l}(t_i - t_{i-1}), \tag{2.1.5.4}$$

for $i = \overline{1, m}$. Summing the inequalities (2.1.5.4) for $i = \overline{1, m}$ we get

$$g(X_{\gamma(l)}, \dot{\gamma}(l)) - g(X_{\gamma(0)}, \dot{\gamma}(0)) > -\epsilon.$$

Since ϵ is an arbitrary positive number we have that

$$g(X_{\gamma(l)}, \dot{\gamma}(l)) - g(X_{\gamma(0)}, \dot{\gamma}(0)) \geq 0.$$

Hence for any two distinct points a and b of K and an arbitrary geodesic arc joining a and b contained in K, we have that

$$g(X_b, \dot{\gamma}(l)) \geq g(X_a, \dot{\gamma}(0)).$$

Thus X is geodesic monotone.
Now suppose that X is geodesic monotone. Then by the definitions of geodesic monotonicity and geodesic scalar derivative, we have that $\underline{X}^{\#}(x) \geq 0$ for all $x \in K$. The statement of (i) for $-X$ is obtained by using the identity

$$\underline{(-X)}^{\#} = -\overline{X}^{\#}.$$

(ii) We have that X is trivially geodesic monotone if and only if X and $-X$ is geodesic monotone. Hence X is trivially geodesic monotone, if and only if $0 \leq \underline{X}^{\#}(x) \leq \overline{X}^{\#}(x) \leq 0$, for all $x \in K$. Hence $\underline{X}^{\#}(x) = \overline{X}^{\#}(x) = 0$. Thus X is trivially geodesic monotone if and only if it is scalar differentiable and $X^{\#}(x) = 0$ for every $x \in K$.
Similarly to Theorem 2.1.5.2 we can prove the following theorem:

Theorem 2.1.5.3 *Let (M, g) be a Riemannian manifold, $K \subset M$ a convex open set and $X \in Sec(TK)$ a vector field on K. If $\underline{X}^{\#}(x) > 0$ for every $x \in K$, then X is strictly geodesic monotone.*

Remark 2.1.5.4 *Theorems 2.1.5.2 and 2.1.5.3 hold for uncontinuous vector fields too.*

For the smooth case we have the following Corollary of Theorems 2.1.5.2 and 2.1.5.3:

Corollary 2.1.5.5 *Let (M, g) be a Riemannian manifold, ∇ the Levi-Civitá connection of M, $K \subset M$ a convex set and $X \in Sec(TK)$ a smooth vector field on K. For $x \in K$ define $\phi(x) : T_x(M) \to T_x(M)$ by $\phi(x)(v) = \nabla_v X$ for $v \in T_x(M)$, where $T_x(M)$ is the tangent space of M in x. Then we have the following assertions:*

(i) $\underline{X}^{\#}(x) = \inf\limits_{\substack{v \in T_x(M) \\ \|v\|=1}} g(\phi(x)(v), v)$

(ii) X is geodesic monotone if and only if $\phi(x)$ is positive semidefinite for every x in K

(iii) X is trivially monotone if and only if $\phi(x)$ is skew-symmetric for every $x \in K$

(iv) If $\phi(x)$ is positive definite for every $x \in K$, then X is strictly geodesic monotone.

Proof. Let x be an arbitrary point of K. Then we have

$$\underline{X}^{\#}(x) = \inf_{\gamma \in \Gamma} \lim_{t \downarrow 0} \frac{g(X_{\gamma(t)}, \dot{\gamma}(t)) - g(X_x, \dot{\gamma}(0))}{t}. \tag{2.1.5.5}$$

Equation (2.1.5.5) can be rewritten as

$$\underline{X}^{\#}(x) = \inf_{\gamma \in \Gamma} \frac{d}{dt}|_{t=+0} g(X_{\gamma(t)}, \dot{\gamma}(t)). \tag{2.1.5.6}$$

Using that ∇ is the Levi-Civitá connection of M and γ is geodesic, (2.1.5.6) becomes

$$\underline{X}^{\#}(x) = \inf_{\gamma \in \Gamma} g(\nabla_{\dot{\gamma}(0)} X|_x, \dot{\gamma}(0)). \tag{2.1.5.7}$$

Since for every $v \in T_x(M)$ there is $\gamma \in \Gamma$ such that $\dot{\gamma}(0) = v$, from (2.1.5.7) we obtain that

$$\underline{X}^{\#}(x) = \inf_{\substack{v \in T_x(M) \\ \|v\|=1}} g(\phi(x)(v), v), \tag{2.1.5.8}$$

where $\| \, . \, \|$ is the norm generated by the metrical tensor of M. Using (i) From (2.1.5.8) and theorems 2.1.5.2 and 2.1.5.3 we obtain the assertions (ii), (iii) and (iv) of the corollary.

Using Corollary 2.1.5.5 we obtain:

Theorem 2.1.5.6 *Let (M, g) be a Riemannian manifold, $K \subset M$ a convex open set and $X \in Sec(TK)$ a smooth vector field on K. Then the following assertions are equivalent:*

(i) X is trivially geodesic monotone,

(ii) X is a Killing vector field.

Proof. By Corollary 2.1.5.5 (iii) we have that X is trivially geodesic monotone if and only if A_Y is skew-symmetric, where A_Y is the endomorphism of smooth vector fields defined by $A_Y(X) = \nabla_Y X$. But this is exactly a necessary and sufficient condition for X to be a Killing vector field.

2.1.6 Geodesic monotone vector fields on S^n

Denote by $\langle \, . \, , \, . \, \rangle$ the canonical scalar product of $\mathbf{R}^{n+1}$. Consider S^n, the standard unit sphere, to be a Riemannian manifold, with the metric induced from $\mathbf{R}^{n+1}$ (we shall denote its metric by $\langle \, . \, , \, . \, \rangle$ too). We start this subsection by remarking that there are no virtually monotone vector fields on the whole sphere. This is an obvious consequence of Theorem 2.1.3.2, since every geodesic of the sphere is closed. Hence we can seek for virtually monotone vector fields just on convex proper subsets of the sphere. Denote by $\overset{g}{\underset{y \to x}{}}$ the convergence of y to x along geodesics. Then we have the following theorem:

Theorem 2.1.6.1 *Let K be an open and convex subset of S^n, X a vector field on K and x an arbitrary point of K. Then we have the following formula for the lower geodesic scalar derivative of X in x:*

$$\underline{X}^{\#}(x) = \liminf_{\substack{g \\ y \to x}} \frac{\langle X_y - X_x, \, y - x \rangle}{\|y - x\|^2}.$$

Proof. Every unit speed geodesic of the n-dimensional unit sphere S^n starting from $x \in S^n$ $(\gamma(0) = x)$ has equation:

$$\gamma(t) = (\cos t)x + (\sin t)e, \tag{2.1.6.1}$$

where $e = \dot{\gamma}(0) \in T_x(S^n)$ is the tangent unit vector of γ in the starting point. We want to find the equation of the unit speed geodesic, denoted with the

same letter γ, starting from $x \in S^n$ and passing through $y \in S^n$. Obviously this is equivalent to finding the tangent unit vector e of γ in the staring point. This can be calculated in the following way:
First we find the parameter l corresponding to y and after that we can compute the unit tangent vector of γ: $e = \dot{\gamma}(0)$ using (2.1.6.1). Having in mind that x and y are unit vectors of $T_0(\mathbf{R}^{n+1})$, the tangent space of $\mathbf{R}^{n+1}$ in the origin, we have that

$$\cos l = \langle x,\, y\rangle. \tag{2.1.6.2}$$

Thus we get

$$e = \pm\frac{y - \langle x,\, y\rangle x}{\sqrt{1-\langle x,\, y\rangle^2}}. \tag{2.1.6.3}$$

From (2.1.6.1), (2.1.6.2) and (2.1.6.3) we have that

$$\dot{\gamma}(l) = \mp\sqrt{1-\langle x,\, y\rangle^2}\, x \pm \langle x,\, y\rangle \frac{y - \langle x,\, y\rangle x}{\sqrt{1-\langle x,\, y\rangle^2}}. \tag{2.1.6.4}$$

If Γ has the same meaning as in the previous chapter, then we have

$$\underline{X}^{\#}(x) = \liminf_{\substack{l\downarrow 0\\ \gamma\in\Gamma}} \frac{\langle X_{\gamma(l)},\, \dot{\gamma}(l)\rangle - \langle X_{\gamma(0)},\, \dot{\gamma}(0)\rangle}{l}. \tag{2.1.6.5}$$

Since $\lim\limits_{l\to 0}(\sin l)/l = 1$ (2.1.6.5) becomes

$$\underline{X}^{\#}(x) = \liminf_{\substack{l\downarrow 0\\ \gamma\in\Gamma}} \frac{\langle X_{\gamma(l)},\, \dot{\gamma}(l)\rangle - \langle X_{\gamma(0)},\, \dot{\gamma}(0)\rangle}{\sin l}. \tag{2.1.6.6}$$

Inserting $\gamma(0) = x$, $\gamma(l) = y$, (2.1.6.4), (2.1.6.3), $\langle X_x,\, x\rangle = \langle X_y,\, y\rangle = 0$ and $\sin l = \pm\sqrt{1-\langle x,\, y\rangle^2}$ into (2.1.6.6) we obtain

$$\underline{X}^{\#}(x) = \liminf_{y \overset{g}{\to} x} \frac{\langle X_y - X_x,\, y - x\rangle}{1-\langle x,\, y\rangle^2}. \tag{2.1.6.7}$$

Since $1 - \langle x,\, y\rangle^2 = (1-\langle x,\, y\rangle)(1+\langle x,\, y\rangle)$ and $1+\langle x,\, y\rangle \to 2$, when $y \overset{g}{\to} x$, (2.1.6.7) becomes

$$\underline{X}^{\#}(x) = \liminf_{y \overset{g}{\to} x} \frac{\langle X_y - X_x,\, y - x\rangle}{2-2\langle x,\, y\rangle}. \tag{2.1.6.8}$$

But since $\|x\| = \|y\| = 1$ we have that $2 - 2\langle x,\, y\rangle = \|y - x\|^2$, which inserted in (2.1.6.8) gives the required equality of the theorem.

For the smooth case we have the following corollary of Theorem 2.1.6.1:

Corollary 2.1.6.2 *If $X = f|_K$, where $f : \mathbf{R}^{n+1} \to \mathbf{R}^{n+1}$ is a smooth operator satisfying $\langle f(x),\, x\rangle = 0$ when $\|x\| = 1$, then*

$$\underline{X}^{\#}(x) = \inf_{\substack{\|h\|=1,\\ \langle x,\, h\rangle = 0}} \langle df(x)(h),\, h\rangle,$$

where $df(x)$ is the Frechét differential of f in x.

Proof. Suppose that $y \to x$ along the arbitrary but fixed geodesic γ starting from x ($\gamma(0) = x$). This means that $y = \gamma(t)$, where $t \downarrow 0$. Hence using $X = f|_K$ we have that

$$\frac{\langle X_y - X_x,\, y - x\rangle}{\|y - x\|^2} =$$

$$\left\langle \frac{f(\gamma(t)) - f(\gamma(0))}{t} : \left\| \frac{\gamma(t) - \gamma(0)}{t} \right\|,\ \frac{\gamma(t) - \gamma(0)}{t} : \left\| \frac{\gamma(t) - \gamma(0)}{t} \right\| \right\rangle \tag{2.1.6.9}$$

If $y \to x$ along γ i.e $t \downarrow 0$ then using $(d/dt)|_{t=0} f(\gamma(t)) = df(x)(\dot\gamma(0))$, and that γ is of unit speed we obtain from (2.1.6.9):

$$\lim_{\substack{y \to x \\ y \in Im\gamma}} \frac{\langle X_y - X_x,\, y - x\rangle}{\|y - x\|^2} = \langle df(x)(\dot\gamma(0)),\, \dot\gamma(0)\rangle, \tag{2.1.6.10}$$

where we have denoted by Imγ the image of γ. Equation (2.1.6.10) holds for every γ, hence by definition of $\underline{X}^{\#}(x)$ we get

$$\underline{X}^{\#}(x) = \inf_{\gamma \in \Gamma} \langle df(x)(\dot\gamma(0)),\, \dot\gamma(0)\rangle. \tag{2.1.6.11}$$

Since for every $h \in T_x(S^n)$, $\|h\| = 1$ there is a $\gamma \in \Gamma$ such that $\dot\gamma(0) = h$, (2.1.6.11) becomes the required equality of the corollary.

Remarks 2.1.6.3 *We shall use the same notations as above. Then we have that*
1. Similarly to Theorem 2.1.6.1 it can be proved that X is geodesic monotone (strictly geodesic monotone for $K \neq S^n$) if and only if

$$\langle X_y - X_x,\, y - x\rangle \geq 0 \quad (\langle X_y - X_x,\, y - x\rangle > 0),$$

for arbitrary two distinct points x and y of K.

If X is smooth, using Theorem 2.1.5.2 and Corollary 2.1.6.2 we have that X is geodesic monotone if and only if

$$\langle df(x)(h),\, h\rangle \geq 0,$$

for all $x \in K$ and all $h \in \mathbf{R}^{n+1} - \{0\}$ with $\langle x,\, h\rangle = 0$.

Similarly using Theorem 2.1.5.3 and Corollary 2.1.6.2 if X is smooth and

$$\langle df(x)(h),\, h\rangle > 0,$$

for all $x \in K \neq S^n$ and all $h \in \mathbf{R}^{n+1} - \{0\}$ with $\langle x,\, h\rangle = 0$, then X is strictly geodesic monotone.

2. If $f = A$, where A is a skew symmetric operator, then it follows easily that X is trivially geodesic monotone.

In the following theorem we shall give a method for computing the scalar derivative of a vector field on the sphere.

Theorem 2.1.6.4 *Let X be a vector field on the sphere, such that $X = f|_{S^n}$, where*
$f : \mathbf{R}^{n+1} \to \mathbf{R}^{n+1}$ is a smooth operator satisfying $\langle f(x), x\rangle = 0$, for all $x \in \mathbf{R}^{n+1}$ with $\|x\| = 1$. Let $x \in S^n$ and let $B(x)$ be an orthogonal matrix having in the first column the components of x relative to the canonical basis of $\mathbf{R}^{n+1}$ and

$$C(x) = B^T(x)J_sf(x)B(x),$$

where T denotes transposition and

$$J_sf(x) = \frac{Jf(x) + Jf(x)^T}{2}$$

is the symmetrizant of the Jacobi matrix $Jf(x)$ of f in x. Let $C^-(x)$ be the $n \times n$ matrix obtained from $C(x)$ by deleting the first row and first column. Then $\underline{X}^\#(x)$ is the smallest eigenvalue of $C^-(x)$.

Before proving Theorem 2.1.6.4 we shall state a corollary of it.

Corollary 2.1.6.5 *If $K \subset S^n$ is open and convex and X is a vector field on K such that $C^-(x)$ is positive definite for all $x \in K$ then X is strictly geodesic monotone.*

The Proof of Corollary 2.1.6.5 is an easy consequence of Theorems 2.1.5.3 and 2.1.6.4.

Proof of Theorem 2.1.6.4. Corollary 2.1.6.2 implies that

$$\underline{X}^\#(x) = \inf_{\substack{\|h\|=1 \\ \langle x,\, h\rangle=0}} \langle J_sf(x)h,\, h\rangle.$$

Using the conditional extreme value theorem we obtain

$$\underline{X}^\#(x) = \inf\{\lambda : \ J_sf(x) = \lambda h + \mu x, \ \text{for some } \mu \in \mathbf{R}, \ \text{and } h \in \mathbf{R}^{n+1} \\ \text{with } \|h\| = 1 \text{ and } \langle x,\, h\rangle = 0\}. \tag{2.1.6.12}$$

Consider an ortonormalt basis of $\mathbf{R}^{n+1}$ with first vector x. Then in the new basis we have: $h = (0, h_2 \dots, h_{n+1})$ and the matrix of $d_sf(x) = (df(x) + df(x)^T)/2$ is $C(x)$. If $C(x) = (c_{ij}(x))_{1\le i,j\le n+1}$, then equation $J_sf(x)h = \lambda h + \mu x$ in the new base is equivalent to

$$\mu = c_{12}h_2 + \ldots + c_{1,n+1}h_{n+1} \text{ and} \tag{2.1.6.13}$$

$$C^-(x)h^- = \lambda h^-, \tag{2.1.6.14}$$

where $h^- = (h_2, \dots, h_{n+1})$. From (2.1.6.12) and (2.1.6.14) follows immediately the statement of the theorem.

Of course similar results can be obtained for positive contant sectional curvature manifolds.

Example 2.1.7.6 Let $f : \mathbf{R}^3 \to \mathbf{R}^3$, $f(x) = (x_1^2 - 1,\, x_1x_2,\, x_1x_3)$, for $x = (x_1,\, x_2,\, x_3)$. Since $\langle f(z),\, z\rangle = 0$ for all $z \in S^2$ there is a smooth vector field X on S^2 such that $X = f|_{S^2}$. Hence $X_z = (z_1^2 - 1,\, z_1z_2,\, z_1z_3)$, for arbitrary point $z = (z_1,\, z_2,\, z_3)$ of S^2. In spherical coordinates ψ, θ we have

$$\begin{cases} z_1 = \cos\psi\cos\theta, \\ z_2 = \cos\psi\sin\theta, \\ z_3 = \sin\psi. \end{cases}$$

We can choose $B(z)$ to be

$$\begin{pmatrix} \cos\psi\cos\theta & -\sin\theta & \sin\psi\cos\theta \\ \cos\psi\sin\theta & \cos\theta & \sin\psi\sin\theta \\ \sin\psi & 0 & -\cos\psi \end{pmatrix}. \tag{2.1.6.15}$$

The Jacobi matrix of f in z is equal to

$$Jf(z) = \begin{pmatrix} 2z_1 & 0 & 0 \\ z_2 & z_1 & 0 \\ z_3 & 0 & z_1 \end{pmatrix}.$$

Hence

$$J_sf(z) = \frac{1}{2}\begin{pmatrix} 4z_1 & z_2 & z_3 \\ z_2 & 2z_1 & 0 \\ z_3 & 0 & 2z_1 \end{pmatrix}.$$

In spherical coordinates this can be written as

$$J_sf(z) = \frac{1}{2}\begin{pmatrix} 4\cos\psi\cos\theta & \cos\psi\sin\theta & \sin\psi \\ \cos\psi\sin\theta & 2\cos\psi\cos\theta & 0 \\ \sin\psi & 0 & 2\cos\psi\cos\theta \end{pmatrix}. \tag{2.1.6.16}$$

Using (2.1.6.15) and (2.1.6.16) we can calculate $C(z) = B^T(z)J_sf(z)B(z)$ in spherical coordinates. Deleting the first row and first column of $C(z)$ we obtain

$$C^-(z) = \begin{pmatrix} \cos\psi\cos\theta & 0 \\ 0 & \cos\psi\cos\theta \end{pmatrix}. \tag{2.1.6.17}$$

Equation (2.1.6.17) can be written in cartesian coordinates as

$$C^-(z) = \begin{pmatrix} z_1 & 0 \\ 0 & z_1 \end{pmatrix}. \tag{2.1.6.18}$$

Applying Theorem 2.1.6.4 to (2.1.6.18) we obtain $\underline{X}^\#(z) = z_1$. Using Theorem 2.1.5.3 the vector field X is strictly geodesic monotone on the hemisphere

$$\begin{cases} z_1^2 + z_2^2 + z_3^2 = 1, \\ z_1 > 0 \end{cases}$$

2.1.7 Geodesic monotone vector fields on H^n

Consider the following model for the n dimensional hyperbolic space of constant sectional curvature $K = -1$:

$$\mathbf{H}^n = \{\xi = (\xi_1, \dots, \xi_n, \xi_{n+1}) \in \mathbf{R}^{n+1} : \ \xi_{n+1} > 0 \ and \ \{\xi, \xi\} = -1\},$$

where for $\xi = (\xi^1, \dots, \xi^{n+1})$, $\eta = (\eta^1, \dots, \eta^{n+1}) \in \mathbf{R}^{n+1}$, $\{\xi, \eta\} = \xi^1\eta^1 + \dots + \xi^n\eta^n - \xi^{n+1}\eta^{n+1}$. The metric of $\mathbf{H}^n$ is induced from the Lorentz metric $\{.,.\}$ of $\mathbf{R}^{n+1}$ and it will be denoted by the same symbol. Then an arbitrary geodesic γ of H^n starting from x ($\gamma(0) = x$) will have the equation

$$\gamma(t) = (\cosh t)x + (\sinh t)e, \tag{2.1.7.1}$$

where $e = \dot{\gamma}(0) \in T_x(H^n)$ is the tangent unit vector of γ in the starting point. Also for arbitrary vector field X on H^n we have

$$\{X_x, x\} = 0, \tag{2.1.7.2}$$

for all $x \in H^n$. If the geodesic γ is passing through $y \in H^n$ corresponding to parameter l then (2.1.7.1) yields

$$y = \cosh l \, x + \sinh l \, e. \tag{2.1.7.3}$$

Multiplying (2.1.7.3) by x, using the definition of H^n and (2.1.7.2) we obtain

$$\cosh l = -\{y, x\}. \tag{2.1.7.4}$$

We also have

$$\cosh^2 t - \sinh^2 t = 1, \tag{2.1.7.5}$$

and

$$\lim_{l \to 0} \frac{\sinh l}{l} = 1. \tag{2.1.7.6}$$

Using (2.1.7.1), (2.1.7.2), (2.1.7.3), (2.1.7.4), (2.1.7.5), (2.1.7.6) and the definition of $\underline{X}^{\#}(x)$ we can prove, similarly to the previous subsection, the following theorems:

Theorem 2.1.7.1 *Let K be an open and convex subset of H^n, X a vector field on K and x an arbitrary point of K. Then we have the following formula for the lower geodesic scalar derivative of X in x:*

$$\underline{X}^{\#}(x) = \liminf_{\substack{g \\ y \to x}} \frac{\{X_y - X_x, \, y - x\}}{\{y - x, \, y - x\}},$$

where the limit is taken just for those y wich satisfy the relation $\{y - x, y - x\} \neq 0$.

Corollary 2.1.7.2 *Using the notations of Theorem 2.1.7.1 if $X = f|_K$, where $f: \mathbf{R}^{n+1} \to \mathbf{R}^{n+1}$ is a smooth operator satisfying $\{f(x), x\} = 0$ when $\{x, x\} = -1$, then*

$$\underline{X}^{\#}(x) = \inf_{\substack{\{h,\, h\}=1, \\ \{x,\, h\}=0}} \{df(x)(h),\, h\},$$

where $df(x)$ is the Frechét differential of f in x.

Remarks 2.1.7.3 *We shall use the same notations as above. Then we have that*
1. Similarly to Theorem 2.1.7.1 it can be proved that X is geodesic monotone (strictly geodesic monotone) if and only if

$$\{X_y - X_x,\, y - x\} \geq 0 \;\; (\{X_y - X_x,\, y - x\} > 0),$$

for arbitrary two distinct points x and y of K.

If X is smooth, using Theorem 2.1.5.2 and Corollary 2.1.7.2 we have that X is geodesic monotone if and only if

$$\{df(x)(h),\, h\} \geq 0,$$

for all $x \in K$ and all $h \in \mathbf{R}^{n+1} - \{0\}$ with $\{x, h\} = 0$ and $\{h, h\} \neq 0$.

Similarly using Theorem 2.1.5.3 and Corollary 2.1.7.2 if X is smooth and

$$\{df(x)(h),\, h\} > 0,$$

for all $x \in K$ and all $h \in \mathbf{R}^{n+1} - \{0\}$ with $\{x, h\} = 0$ and $\{h, h\} \neq 0$, then X is strictly geodesic monotone.

2. If $f = A$, where $A \in o(n, 1)$ then it follows easily that X is trivially geodesic monotone.

We also have an example similar to example 2.1.7.6 of the previous subsection.

Example 2.1.8.4 Let

$$f: \mathbf{R}^3 \to \mathbf{R}^3,$$

$$f(x) = (-x_1 x_3,\, -x_2 x_3,\, x_3^2 - 1),$$

for $x = (x_1, x_2, x_3)$. Since

$$\{f(z),\, z\} = 0,$$

for all $z \in \mathbf{H}^2$ there is a smooth vector field X on H^2 such that

$$X = f|_{\mathbf{H}^2}.$$

Hence

$$X_z = (-z_1 z_2,\ -z_2 z_3,\ z_3^2 - 1),$$

for arbitrary point $z = (z_1,\ z_2,\ z_3)$ of $\mathbf{H}^2$. By an easy computation

$$\{df(z)(h),\ h\} = z_3,$$

for all $z \in \mathbf{H}^2$ and $h \in \mathbf{R}^3$ with $\{h,\ h\} = 1$, $\{h,\ z\} = 0$. Hence using Corollary 2.1.7.2 we have that

$$\underline{X}^{\#}(z) = z_3,$$

for all $z \in \mathbf{H}^2$. But $z_3 > 0$, for all $z = (z_1,\ z_2,\ z_3) \in \mathbf{H}^2$. Thus using Theorem 2.1.5.3 X is a strictly geodesic monotone vector field on the whole $\mathbf{H}^2$.

2.2 Killing monotone vector fields

The theory of one parameter transformation groups is a useful tool for many application oriented investigations. Many of these results are connected with the qualitative theory of differential equations (see for example [25]), very important in theoretical physics. In this paper we connect the topic of one parameter transformation groups with results of nonlinear analysis on Riemannian manifolds, concerning geodesic monotonicity.
We relate the geodesic monotone vector fields to expansive maps. (A map $\phi : M \to M$ is called expansive if its tangential maps are expansive in every point of M.) This is done through the Killing monotone vector fields, a notion introduced by us. The Killing monotone vector fields are vector fields on a Riemannian manifold $(M,\ g)$, generated by expansive one parameter transformation groups, where we mean by expansive one parameter transformation groups smooth one parameter transformation groups ϕ_t over Riemannian manifolds, with ϕ_t expansive for $t > 0$ (from this follows easily that ϕ_t is nonexpansive for $t < 0$). The Killing strictly monotone vector fields can be introduced similarly.
We prove that a vector field X on M is Killing monotone (Killing strictly monotone) if and only if the Lie derivative $\mathcal{L}_X g$, of the metrical tensor g with respect to X is positively semidefinite (positively definite) in every point of M. The positive semidefiniteness (positive definiteness) of $\mathcal{L}_X g$ is proved to be equivalent with the positive semidefiniteness (positive definiteness) of the endomorphism A_X, defined by $A_X U = \nabla_U X$, with respect to g, where ∇ is the Levi-Civitá connection of M. This result has two corollaries, regarding the monotonicity of vector fields on Riemannian submanifolds.

Finally, we prove that a vector field is Killing monotone if and only if it is geodesic monotone. From these follows that a necessary and sufficient condition for X to be geodesic monotone is the positive semidefiniteness of $\mathcal{L}_X g$ in every point. We also prove that a necessary condition for X to be strictly geodesic monotone is the positive definiteness of $\mathcal{L}_X g$ in every point of M. We express the lower geodesic scalar derivative of X in terms of the Lie derivative, and we prove that X is geodesic scalar differentiable if and only if it is conformal.

2.2.1 Expansive one parameter transformation groups

Let $(M,\, g)$ be a Riemannian manifold.

Definition 2.2.1.1 *A smooth map $\phi : M \to M$ is called* expansive *(*nonexpansive*), if for all $x \in M$ the tangent map of ϕ; $T\phi(x) : T_xM \to T_{\phi(x)}M$ is expansive (nonexpansive) i.e.*

$$\|T\phi(x)U\| \geq \|U\| \quad (\|T\phi(x)U\| \leq \|U\|) \tag{2.2.1.1}$$

for all $U \in T_xM$, where $\|\,.\,\|$ is the norm generated by the metric g.
ϕ is called strictly expansive *(*strictly nonexpansive*) if in* (2.2.1.1) *we have strict inequalities for $U \neq 0$.*

Definition 2.2.1.2 *A one parameter smooth transformation group $\phi_t : M \to M$ is called* expansive *(*nonexpansive*) if ϕ_t is expansive (nonexpansive) for all $t > 0$. The* strictly expansive *(*strictly nonexpansive*)* one parameter transformation groups *can be defined similarly.*

Definition 2.2.1.3 *Let $\phi_t : M \to M$ be an expansive (nonexpansive) one parameter transformation group and $X : M \to TM$ the vector field generating ϕ_t i.e.*

$$X(z) = \frac{d}{dt}|_{t=0}\phi_t(z),$$

for all $z \in M$.
Then the vector field X is called an infinitesimal expansion *(*nonexpansion*). The* infinitesimal strict expansions *(*infinitesimal strict nonexpansions*) can be defined similarly. The infinitesimal expansions (infinitesimal strict expansions) will also be called* Killing monotone vector fields *(*Killing strictly monotone vector fields*).*

Remark 2.2.1.4 *The Killing vector fields are trivially Killing monotone vector fields. Hence the Killing monotone vector fields are generalizations of the Killing vector fields.*

The following theorem gives a characterization of the Killing monotone vector fields using the Lie derivative.

Theorem 2.2.1.5 *$X : M \to TM$ is a Killing monotone vector field (Killing strictly monotone vector field) if and only if (if) the Lie derivative of the metrical tensor $\mathcal{L}_X g$ is positive semidefinite (positive definite) in every point of M.*

Proof. Suppose that X is Killing monotone. Denote by $\mathcal{T}M$ the set of all smooth vector fields on M. Let $U,\, V \in \mathcal{T}M$ be two arbitrary vector fields, $z \in M$ an arbitrary point and ϕ_t the one parameter transformation group generated by X. Then we have that

$$(\mathcal{L}_X g)(U,\, V) = X(g(U,\, V)) - g([X,\, U],\, V) - g(U,\, [X,\, V]). \tag{2.2.1.2}$$

If we put $V = U$ in (2.2.1.2), then we obtain

$$(\mathcal{L}_X g)(U, U) = X(g(U, U)) - 2g([X, U], U). \tag{2.2.1.3}$$

On the other hand

$$X(g(U, U))_z = \lim_{\substack{t\to 0\\ t>0}} \frac{g(U(\phi_t(z)), U(\phi_t(z))) - g(U(z), U(z))}{t}. \tag{2.2.1.4}$$

Since X is Killing monotone we have that

$$g(U(\phi_t(z)), U(\phi_t(z))) \geq g(T\phi_t^{-1}U(\phi_t(z)), T\phi_t^{-1}U(\phi_t(z))). \tag{2.2.1.5}$$

Inserting (2.2.1.5) in (2.2.1.4) we get

$$X(g(U, U))_z \geq \lim_{\substack{t\to 0\\ t>0}} (g(\frac{T\phi_t^{-1}U(\phi_t(z)) - U(z)}{t}, T\phi_t^{-1}U(\phi_t(z)))$$

$$+g(U(z), \frac{T\phi_t^{-1}U(\phi_t(z)) - U(z)}{t})). \tag{2.2.1.6}$$

But

$$\lim_{\substack{t\to 0\\ t>0}} \frac{T\phi_t^{-1}U(\phi_t(z)) - U(z)}{t} = [X, U]_z. \tag{2.2.1.7}$$

Inserting (2.2.1.7) in (2.2.1.6) we obtain

$$X(g(U, U))_z \geq 2g([X, U], U)_z. \tag{2.2.1.8}$$

From (2.2.1.3) and (2.2.1.8) it follows that $\mathcal{L}_X g$ is positive semidefinite at every $z \in M$.

Conversely, suppose that $\mathcal{L}_X g$ is positive semidefinite (positive definite) at every $z \in M$. We have to prove that $g(T\phi_t U, T\phi_t U) \geq g(U, U)$ ($>$) for all $t > 0$ and $U \in \mathcal{T}M$ ($U \in \mathcal{T}M \setminus \{0\}$). But this is trivial since

$$\frac{d}{dt} g(T\phi_t U, T\phi_t U)|_{t=0} = (\mathcal{L}_X g)(U, U).$$

Let $X \in \mathcal{T}M$ and $A_X : \mathcal{T}M \to \mathcal{T}M$ the endomorphism defined by

$$A_X U = \nabla_U X,$$

where ∇ is the Levi-Civitá connection of M. Then we have the following lemma:

Lemma 2.2.1.6

$$\mathcal{L}_X g(U, U) = 2g(A_X U, U).$$

Proof. Since ∇ is the Levi-Civitá connection of M we have that

$$X(g(U, U)) = 2g(\nabla_X U, U). \tag{2.2.1.9}$$

Inserting (2.2.1.9) in (2.2.1.3), bearing in mind that ∇ is torsion free and using the definition of A_X we obtain the required relation.

By Theorem 2.2.1.5 and Lemma 2.2.1.6 we have the following theorem:

Theorem 2.2.1.7 *$X \in \mathcal{T}M$ is Killing monotone (Killing strictly monotone) if and only if (if) A_X is positive semidefinite (positive definite) relative to the metrical tensor g.*

Using Theorem 2.2.1.7 and Corollary 2.1.5.5 it is easy to prove the following

Theorem 2.2.1.8 *Let $K \in M$ be an open and convex set and $X \in \mathcal{T}K$. Then we have the following two assertions:*

(i) X is Killing monotone if and only if it is geodesic monotone.

(ii) X is a Killing vector field if and only if it is trivially geodesic monotone.

Hence the notion of Killing monotone vector fields (Killing vector fields) coincides with those of geodesic monotone (trivially geodesic monotone) ones. We shall call for brevity's sake these vector fields *monotone* (*trivially monotone*). However, the relation between strictly geodesic monotone vector fields and Killing strictly monotone vector fields seems to be difficult and we cannot say anything about it yet.

We recall the following definition (see [13] vol. II p.53)

Definition 2.2.1.9 *Let M be a Riemannian manifold. A submanifold M of N is called* auto-parallel *if, for each vector $X \in T_xN$ and for each curve τ in N starting from x, the parallel displacement of X along τ (with respect to the affine connection of the ambient space M) yields a vector tangent to N.*

Regarding this notion we recall the following proposition (see [13] vol. II p.55 Proposition 8.2)

Proposition 2.2.1.10 *Let N be a submanifold of the Riemannian manifold M. Denote by ∇ the Levi-Civitá connection of M. Then the following two conditions are equivalent:*

(i) N is an auto-parallel submanifold of M

(ii) If X and Y are vector fields on N, then $\nabla_X Y$ is tangent to N at every point of N.

Theorems 2.2.1.7 and 2.2.1.8 (i) have the following corollary:

Corollary 2.2.1.11 *Let (M, g) be a Riemannian manifold, X a monotone vector field of M and N an integral submanifold of X. Then the restriction of X to N is a monotone vector field of N (with respect to the induced metric on N).*

Proof. Denote the restriction of X to N by the same letter. Let $\tilde{\nabla}$ the Levi-Civitá connection of N (with respect to the induced metric on N) and ∇ be the Levi-Civitá connection of M. Then, by the Gauss formula we have (see [13] vol. II p.15 (I))

$$\nabla_Y X = \tilde{\nabla}_Y X + \alpha(X, Y),$$

where α is the second fundamental form of N and $Y \in \mathcal{T}N$. (Y can be extended to M. The extension was denoted by the same letter.) Since $\alpha(X, Y)$ is normal to N we have:

$$g(\nabla_Y X, Y) = g(\tilde{\nabla}_Y X, Y).$$

Since $g(\nabla_Y X, Y) \geq 0$, we get $g(\tilde{\nabla}_Y X, Y) \geq 0$ for all $Y \in calTN$. By Theorem 2.2.1.7 and Theorem 2.2.1.8 (i) the restriction of X to N is a monotone vector field of N. ♣

Example Let $f : \mathbf{R}^3 \to \mathbf{R}^3$ $f(x, y, z) = (x^3 + x, y + z, z)$. Then the symmetrizant of the Jacobi matrix of f is

$$df(x, y, z)_s = \begin{bmatrix} x^2+1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} \\ & & \\ 0 & \frac{1}{2} & 1 \end{bmatrix}$$

It can be checked that $df(x, y, z)_s$ is positive definite for all (x, y, z). From Theorem 1.1.5.5 it follows that f is a monotone operator. Hence the vector field X, where X_p is the tangent vector in $t = 0$ of the curve $t \mapsto p + tf(p)$, is monotone. Consider the cylinder surface S given by the equation $h(x, y, z) = 0$ $z > 0$, where $h(x, y, z) = y - z \log z$. By a straightforward computation it can be verified that

$$X_x \frac{\partial h}{\partial x} + X_y \frac{\partial h}{\partial y} + X_z \frac{\partial h}{\partial z} = 0,$$

for all $(x, y, z) \in S$. Hence S is an integral surface of X. By Corollary 2.2.1.11 the restriction of X to S is monotone on S with respect to the metric induced from the canonical metric of $\mathbf{R}^3$. ♣

We also have the following corollary of Theorem 2.2.1.7 and Theorem 2.2.1.8 (i) (which generalize Theorem 8.9 of [1] vol. II p. 59):

Corollary 2.2.1.12 *Let N be an auto-parallel submanifold of the Riemannian manifold (M, g). Let X be a monotone vector field of M. At each point of N decompose X into a vector tangent to N and a vector normal to N. Then the tangential component of X is a monotone vector field of N with respect to the induced metric on N.*

Proof. Let $\tilde{\nabla}$ the Levi-Civitá connection of N (with respect to the induced metric on N) and ∇ be the Levi-Civitá connection of M. At each point of N we decompose X as follows:

$$X = X' + X'',$$

where X' is tangent to N and X'' is normal to N. Let g denote also the induced Riemannian metric tensor of N. By Theorem 2.2.1.7 and Theorem 2.2.1.8, it suffices to prove that $g(\tilde{\nabla}_Y X', Y) \geq 0$ for all vector fields Y of N. By the same theorems we have $g(\nabla_Y X, Y) \geq 0$. By the Gauss formula (see [13] vol. II p.15 (I)) we have

$$\nabla_Y X' = \tilde{\nabla}_Y X' + \alpha(X', Y). \tag{2.2.1.10}$$

where α is the second fundamental form of N. Since $\alpha(X', Y)$ is normal to M (2.2.1.10) implies

$$0 \leq g(\nabla_Y X, Y) = g(\tilde{\nabla}_Y X', Y) + g(\nabla_Y X'', Y). \tag{2.2.1.11}$$

Since $g(X'', Y) = 0$ and ∇ is the Levi-Civitá connection of N we have

$$g(\nabla_Y X'', Y) = -g(X'', \nabla_Y Y) \tag{2.2.1.12}$$

Proposition 2.2.1.10 implies

$$g(X'', \nabla_Y Y) = 0. \tag{2.2.1.13}$$

Equations (2.2.1.12) and (2.2.1.13) yield

$$g(\nabla_Y X'', Y) = 0 \tag{2.2.1.14}$$

Inserting (2.2.1.14) into (2.2.1.11) we get $g(\tilde{\nabla}_Y X', Y) \geq 0$ for all vector fields Y of N. ♣

Example Let $M = \mathbf{R}^n$ and $\langle\, .\, ,\, .\, \rangle$ the canonical scalar product of $\mathbf{R}^n$. The auto-parallel submanifolds of M are the affine subspaces. Suppose that N is an affine subspace of M. Let $f : \mathbf{R}^n \to \mathbf{R}^n$ be a monotone operator. Then, f induces the monotone vector field X, where X_x is the tangent vector in $t = 0$ of the curve

$$t \mapsto x + tf(x).$$

In the points of N we can decompose X as follows:

$$X = Y + Z, \tag{2.2.1.15}$$

where Y is tangent to N and Z is normal to N. By Corollary 2.2.1.13 Y is a monotone vector field on N. We shall verify this assertion directly. We must prove that $\langle Y_x - Y_y,\, x - y\rangle \geq 0$ for all $x, y \in N$. Since Z is normal to N and N is an affine subspace of E we have

$$\langle Z_x - Z_y,\, x - y\rangle = 0. \tag{2.2.1.16}$$

Equations (2.2.1.15) and (2.2.1.16) imply

$$\langle Y_x - Y_y,\, x - y\rangle = \langle X_x - X_y,\, x - y\rangle. \tag{2.2.1.17}$$

But $\langle X_x - X_y,\, x - y\rangle \geq 0$, by the monotonicity of X. Hence (2.2.1.17) implies

$$\langle Y_x - Y_y,\, x - y\rangle \geq 0,$$

for any $x, y \in N$. ♣

From Theorem 2.2.1.7 and Corollary 2.1.5.5 (i) we get the following result:

Theorem 2.2.1.13 *Let $K \in M$ be an open and convex set and $X \in \mathcal{T}K$. Then X is Killing strictly monotone if $\underline{X}^{\#}(x) > 0$ for all $x \in K$.*

In the previous section we gave examples of strictly geodesic monotone vector fields on the 3-dimensional half sphere and on the 3-dimensional hyperbolical space. These examples were given by proving the positiveness of the lower scalar derivative. Hence by Theorem 2.2.1.14 these vector fields are Killing strictly monotone too.

2.2.2 Geodesic scalar derivatives and conformity

Using Lemma 2.2.1.6 and Corollary 2.1.5.5 (i), we easily get:

Theorem 2.2.2.1 *Let $X \in \mathcal{T}M$ be a smooth vector field on M and $x \in M$. Then the lower (upper) geodesic scalar derivative of X in x is given by the following formula:*

$$\underline{X}^{\#}(x) = \frac{1}{2}\inf_{g(h,h)=1}(\mathcal{L}_X g)_x(h,h) \;\; (\overline{X}^{\#}(x) = \frac{1}{2}\sup_{g(h,h)=1}(\mathcal{L}_X g)_x(h,h)).$$

Theorem 2.2.2.2 *Using the previous notations if $\mathcal{L}_X g$ is positive definite in every point of K then X is strictly geodesic monotone.*

Proof. Let x be an arbitrary point of K. Since

$$\underline{X}^{\#}(x) = \frac{1}{2}\inf_{g(h,h)=1}(\mathcal{L}_X g)_x(h,h)$$

and the unit ball in T_xM is compact, there is a $h_0 \in T_xM$ with $g(h_0, h_0) = 1$ such that

$$\underline{X}^{\#}(x) = \frac{1}{2}\mathcal{L}_X g_x(h_0, h_0) > 0.$$

Since x has been arbitrarily chosen Theorem 2.6 implies that X is strictly geodesic monotone. ♣

In the previous section we gave a class of strictly geodesic monotone vector fields, a class which generalizes the notion of position vector fields on an Hadamard manifold.
We recall the following definition:

Definition 2.2.2.3 *Let (M, g) be a Riemannian manifold and X a smooth vector field on X. Denote by $\mathcal{L}_X g$ the Lie derivative of g with respect to X. Then X is called* conformal *if there is a smooth map $\lambda : M \to \mathbf{R}$ such that*

$$\mathcal{L}_X g = \lambda g.$$

Theorem 2.2.2.4 *Let (M, g) be a Riemannian manifold and X a smooth vector field on X. Then X is geodesic scalar differentiable if and only if X is conformal.*

Proof. By Theorem 2.2.2.1 X is geodesic scalar differentiable if and only if there is a smooth function $\lambda : M \to \mathbf{R}$ such that

$$\mathcal{L}_X g(Y, Y) = \lambda g(Y, Y) \tag{2.2.2.1}$$

for all $Y \in \mathcal{T}M$. From (2.2.1.2) it follows that $\mathcal{L}_X g$ is symmetric. But a symmetric bilinear form is determined by its corresponding quadratic form [6]. Hence (2.2.2.1) implies

$$\mathcal{L}_X g = \lambda g.$$

In addition we have that $\lambda = 2X^{\#}$.

2.3 Projection maps on Hadamard manifolds

The projection maps onto closed convex sets of Hilbert spaces are an important tool for many application oriented investigations. These maps constituted (and still constitute) a great challenge for many matematicians. The introduction of Minty-Browder monotonicity notion threw a new light on this topic. E. H. Zarantonello's paper [38] is one of the most important in this subject. In [38] a series of nice monotonicity properties of a projection map π_C onto a closed convex subset C of a Hilbert space are proved. We extend one of them, the monotonicity of $I - \pi_C$, to Hadamard manifolds. We introduce a vector field related to a map $f : M \to M$, called the complementary vector field of f, which generalize the complementary map $I - g$ of a map g from an euclidean space onto itself. As a further generalization we define the λ-complementary vector field of f. We prove that if f is a projection map onto a closed convex set of M and λ increasing, then the λ-complementary vector field of f is monotone. In particular this holds for the complementary vector field too.

2.3.1 Some basic consequences of the Comparison Theorems

It is well known that $exp : M \times TM \to M$ is a smooth map. Hence by the definition of the position vector field we have the following proposition:

Proposition 2.3.1.1 *Let M be an Hadamard manifold and $\lambda : [0,\infty[\to [0,\infty[$ a function of class $\mathcal{C}^k$ with $\lambda(0)=0$. Then the map $P^\lambda : M\times M \to TM$ defined by*

$$P^\lambda(x,y) = P_y^{\lambda,x}$$

is of class $\mathcal{C}^k$, where $P^{\lambda,x}$ is the λ-position vector field at x.

In the following definition indices $i=\overline{1,n}$ are considered modulo n.

Definition 2.3.1.2 *A* geodesic n-sided polygon *in a Riemannian manifold M is a set formed by n segments of minimizing unit speed geodesics (called* sides *of the polygon)*

$$\gamma_i : [0,l_i] \to M; \quad i=\overline{1,n},$$

in such a way that $\gamma_i(l_i)=\gamma_{i+1}(0)$; $i=\overline{1,n}$. The endpoints of the geodesic segments are called vertices *of the polygon. The angle*

$$\angle(-\dot\gamma_i(l_i), \dot\gamma_{i+1}(0)); \quad i=\overline{1,n}$$

is called the (interior) angle *of the corresponding vertex.*

Let M be an Hadamard manifold. If a, b, c are three arbitrary points of M then ab will denote the geodesic distance of a from b and $abc_\triangle$ the geodesic triangle of vertices a, b, c (which by the theorem of Hadamard is uniquely defined). In general a geodesic polygon in M, of consecutive vertices $a_1, \dots, a_n$, will be denoted by $a_1\dots a_n$.

Corollary 2.3.1.3 *Let a, b, c, d be the consecutive vertices of a quadrilateral Q in an Hadamard manifold M and α, β, γ, δ the angles of the vertices a, b, c, d, respectively. Then*

$$\alpha+\beta+\gamma+\delta < 2\pi$$

Proof. Let α_1, α_2 be the angles of the vertex a in $adc_\triangle$ and $abc_\triangle$, respectively. Similarly, let γ_1 and γ_2 be the angles of the vertex c in $adc_\triangle$ and $abc_\triangle$, respectively (see fig. 2.1).
It is known that the angle formed by two half straight lines of a trieder is bounded by the sum of the other two angles formed by half straight lines. Hence

$$\alpha_1+\alpha_2 \ge \alpha, \tag{2.3.1.1}$$

and

$$\gamma_1+\gamma_2 \ge \gamma. \tag{2.3.1.2}$$

On the other hand by (ii) of Lemma 2.1.4.3 we have that

$$\alpha_1+\gamma_1+\delta \le \pi, \tag{2.3.1.3}$$

$$\alpha_2+\gamma_2+\beta \le \pi. \tag{2.3.1.4}$$

Summing inequalities (2.3.1.3), (2.3.1.4) and using (2.3.1.1), (2.3.1.2) we obtain

$$\alpha+\beta+\gamma+\delta \le 2\pi.$$

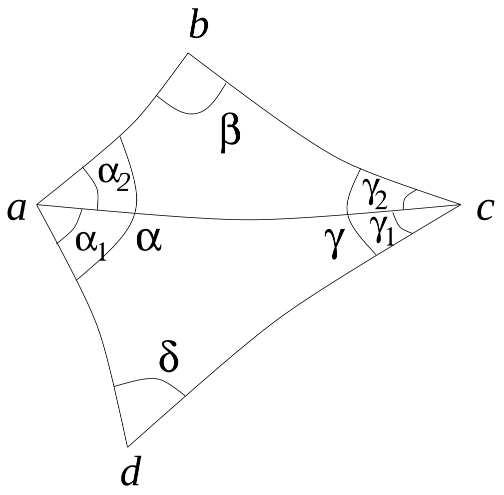

Figure 2.1:

Corollary 2.3.1.4 *Let $abc_\triangle$ be a geodesic triangle in an Hadamard manifold M with $A = bc$, $B = ca$, $C = ab$. Let m be the midpoint of the geodesic arc joining b to c, and L_A the length of the geodesic arc joining a to m. Then we have*

$$4L_A^2 \leq 2B^2 + 2C^2 - A^2.$$

Proof. Let ϕ be the angle between the geodesic arc joining m to a and the geodesic arc joining m to b (see fig. 2.2).

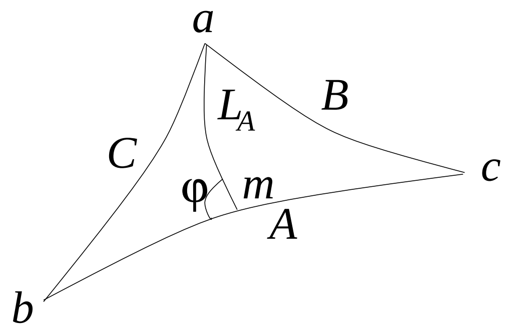

Figure 2.2:

Then using Lemma 2.1.4.3 (i) to $abm_\triangle$ and $acm_\triangle$, respectively, we obtain the inequalities

$$\frac{A^2}{4} + L_A^2 - AL_A \cos\phi \leq B^2, \tag{2.3.1.5}$$

and

$$\frac{A^2}{4} + L_A^2 + AL_A \cos\phi \leq C^2. \tag{2.3.1.6}$$

Summing inequalities (2.3.1.5), (2.3.1.6) and multiplying by 2 we obtain the required inequality of the corollary.

Lemma 2.3.1.5 *Let M be an Hadamard manifold. Let a, b, c and d be four points of M and let Q be a quadrilateral of consecutive vertices a, b, c and d. Let α, β, γ, δ be the angles of the vertices a, b, c, d, respectively. Suppose that α is nonacute and β is obtuse (nonacute). If $\lambda : [0, \infty[\to [0, \infty[$ is an increasing function, then we have the following assertions:*

(i) $cd > ab$ $(cd \geq ab)$.

(ii) $\lambda(da) \cos \delta + \lambda(bc) \cos \gamma \geq 0$

Proof.

(i) 1. $M = E_2$, where E_2 is the euclidean plane.

Since α is nonacute and β is obtuse (nonacute) the quadrilateral $abcd$ is convex. In this case $\alpha + \beta + \gamma + \delta = 2\pi$. Hence either $\delta \geq \pi - \alpha$ or $\gamma \geq \pi - \beta$. We can suppose without loss of the generality that $\delta \geq \pi - \alpha$. Hence the parallel to the straight line ab through d intersect the segment bc in p (see fig. 2.3).

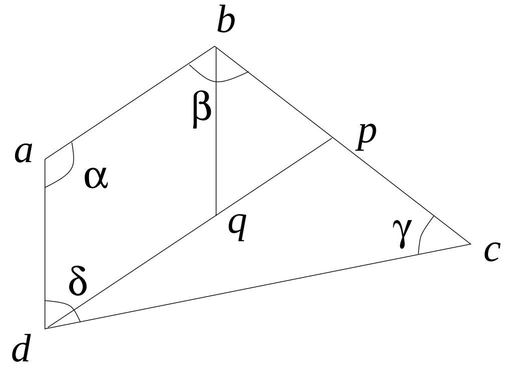

Figure 2.3:

If $p = c$ then $cd > ab$ $(cd \geq ab)$ trivially holds. Suppose that $p \neq c$. Then $\widehat{dpc} > \pi/2$ $(\widehat{dpc} \geq \pi/2)$. Thus, since the sum of the angles of $dpc_\triangle$ is π, we have that $\widehat{pcd} < \pi/2$ $(\widehat{pcd} \leq \pi/2)$. Hence $\widehat{dpc} > \widehat{pcd}$ $(\widehat{dpc} \geq \widehat{pcd})$. This implies that

$$dp < cd \ \ (dp \leq cd). \tag{2.3.1.7}$$

Since $\beta \geq \pi/2$, the parallel to the straight line ad through b intersect the segment dp in q (see fig. 2.3). Then $abqd$ is a parallelogram. Hence

$$ab = dq \leq dp. \tag{2.3.1.8}$$

Inequalities (2.3.1.7) and (2.3.1.8) imply that $ab < cd$ $(ab \leq cd)$.

2. In the general case denote by α_1, α_2 the angles of the vertex a in $adc_\triangle$, $abc_\triangle$ respectively (see fig. 2.4 a)).

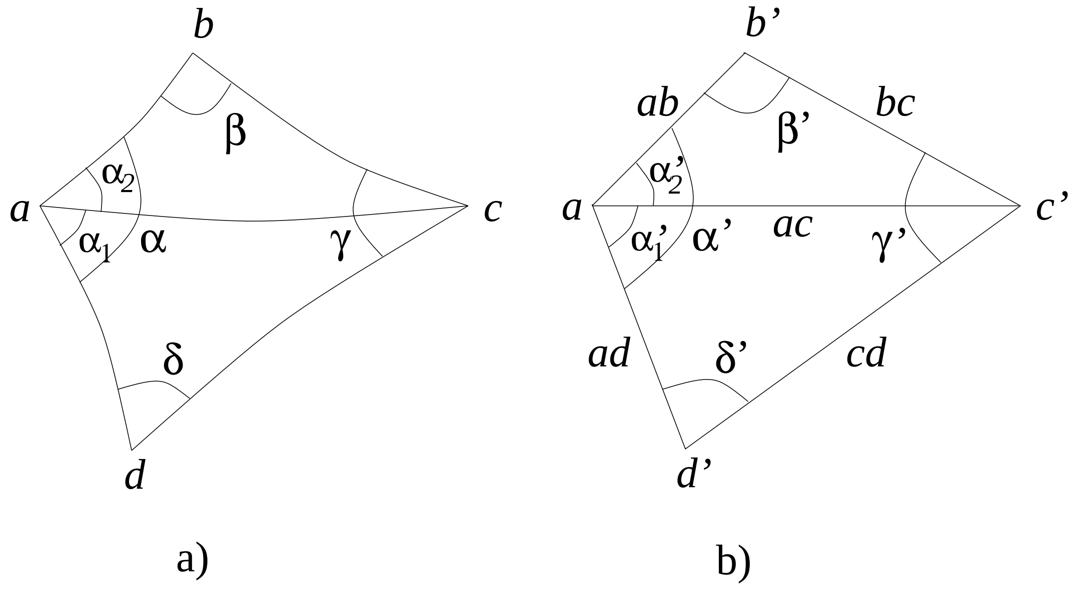

Figure 2.4:

It is known that the angle formed by two half straight lines of a trieder is bounded by the sum of the other two angles formed by half straight lines. Hence

$$\alpha_1 + \alpha_2 \geq \alpha \tag{2.3.1.9}$$

Since the sides of a geodesic triangle are geodesic arcs of minimal length, the length of a side is bounded by the sum of the lengths of the other two. Hence there is a triangle $ac'd'_\triangle$ in $T_a(M)$, such that $ad' = ad$, $ac' = ac$ and $c'd' = cd$. Similarly in the plane of $ac'd'_\triangle$ there is a point b', such that $ab' = ab$, $b'c' = bc$ and b', d' are in different half planes of the straight line ac'. Denote the angles of the quadrilateral $ab'c'd'$ corresponding to the vertices a, b', c' d' by α', β', γ', δ', respectively. We denote by α'_1, α'_2 the angles corresponding to the vertex a in $ac'd'_\triangle$, $ab'c'_\triangle$, respectively (see fig. 2.4 b)). Then by Lemma 2.1.4.3 (i) applied to the $ac'd'_\triangle$ and $ab'c'_\triangle$ we have that

$$\alpha'_1 \geq \alpha_1, \tag{2.3.1.10}$$

$$\alpha'_2 \geq \alpha_2, \tag{2.3.1.11}$$

and

$$\beta' \geq \beta. \tag{2.3.1.12}$$

Summing inequalities (2.3.1.10), (2.3.1.11) and using (2.3.1.12) we obtain that

$$\alpha' \geq \alpha \tag{2.3.1.13}$$

By inequalities (2.3.1.12) and (2.3.1.13) the angle α', β' of the quadrilateral $ab'c'd'$ is nonacute, obtuse (nonacute), respectively. Hence by the 1^{st} case we have that $c'd' \geq ab'$ ($c'd' > ab'$). But $c'd' = cd$ and $ab' = ab$. Hence $cd > ab$ ($cd \geq ab$).

(ii) By Corollary 2.3.1.3 we have that $\gamma + \delta \leq \pi$. Hence either $\gamma \leq \pi/2$, or $\delta \leq \pi/2$. Without losing the generality we can suppose that $\gamma \leq \delta$. Then we have that

$$\frac{\pi}{2} \geq \frac{\delta + \gamma}{2} \geq \frac{\delta - \gamma}{2} \geq 0 \tag{2.3.1.14}$$

Since cos is decreasing on $[0, \frac{\pi}{2}]$ (2.3.1.14) implies

$$0 \leq \cos\frac{\delta + \gamma}{2} \leq \cos\frac{\delta - \gamma}{2} \leq 1 \tag{2.3.1.15}$$

Inequality (2.3.1.15) implies

$$\cos\delta + \cos\gamma = 2\cos\frac{\delta + \gamma}{2}\cos\frac{\delta - \gamma}{2} \geq 0 \tag{2.3.1.16}$$

We consider two cases:

(1) $0 < \gamma \leq \delta < \frac{\pi}{2}$.

In this case the inequality of (ii) is trivial.

(2) $\delta \geq \frac{\pi}{2}$ and $\gamma \leq \frac{\pi}{2}$.

Since α, $\delta \geq \pi/2$ by (i) of the lemma we get $0 < da \leq bc$ Using that λ is increasing and $\cos\gamma \geq 0$ we obtain that

$$\lambda(bc)\cos\gamma + \lambda(da)\cos\delta \geq \lambda(da)(\cos\gamma + \cos\delta) \tag{2.3.1.17}$$

By inequalities (2.3.1.16) and (2.3.1.17) we obtain

$$\lambda(bc)\cos\gamma + \lambda(da)\cos\delta \geq 0$$

The following definition is a particular case of definition 2.4 ch.7 [5]:

Definition 2.3.1.6 *Let M be an Hadamard manifold and x, $y \in M$. The distance $d(x, y)$ is defined by $d(x, y)$ = the length of the geodesic joining x to y.*

By the Hadamard theorem there is a unique geodesic joining x to y, hence $d(x, y)$ is well defined.
We recall the following proposition (see Proposition 2.5 ch.7 [5]):

Proposition 2.3.1.7 *With the distance d, M is a metric space, that is:*
1) $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality),
2) $d(x, y) = d(y, x)$,
3) $d(x, y) \geq 0$, and $d(x, y) = 0 \Leftrightarrow x = y$.

We remind the following result (see Proposition 2.6 ch.7 [5]):

Proposition 2.3.1.8 *The topology induced by d on M coincide with the original topology on M.*

Definition 2.3.1.9 *Let M be an Hadamard manifold and $o \in M$. If $\tilde{B}_r(0)$ is the ball of center 0 and radius r in $T_o(M)$, then $\exp_o \tilde{B}_r(0) = B_r(o)$ is called the* geodesic ball *with center o and radius r.*

Since the exponential map is continuous and $\tilde{B}_r(0)$ is compact, we have that $B_r(o)$ is compact.

2.3.2 The complementary vector field of a map

Definition 2.3.2.1 *Let M be an Hadamard manifold and $f : M \to M$. Then the vector field $F \in Sec(TM)$ defined by*

$$F_x = P_x^{f(x)}$$

is called the complementary vector field *of f.*

Definition 2.3.2.1 *Let M be an Hadamard manifold, $f : M \to M$ and $\lambda : [0, \infty[\to [0, \infty[$. Then the vector field $F \in Sec(TM)$ defined by*

$$F_x^{\lambda} = P_x^{\lambda, f(x)}$$

is called the λ-complementary vector field *of f.*

In particular the complementary vector field of f coincides with the id-complementary vector field of f, where "id" is the identity map of M.
By Proposition 2.3.1.1 we have:

Proposition 2.3.2.2 *If $f : M \to M$ and $\lambda : [0, \infty[\to [0, \infty[$ are of class $\mathcal{C}^k$ and $\lambda(0) = 0$ then the λ-complementary vector field F^{λ} of f is of class $\mathcal{C}^k$.*

Remark 2.3.2.3 *Let E be an euclidean space, $f : E \to E$ and $\lambda : [0, \infty[\to [0, \infty[$. Then*

(i) the complementary vector field F of f assigns to each $x \in E$ the tangent vector in 0 of the curve $t \mapsto x + t(x - f(x))$

(ii) the λ-complementary vector field F of f assigns to each $x \in E$ the tangent vector in 0 of the curve $t \mapsto x + t\lambda(\|x - f(x)\|)(x - f(x))$

2.3.3 Projection maps generating monotone vector fields

Definition 2.3.3.1 *Let M be an Hadamard manifold, $x \in M$ and $A \subset M$. Then the nonnegative number*

$$d(x, A) = \inf\{d(x, y)|\ y \in A\}$$

is called the geodesic distance *of x from A.*

Since d is a metric we have the following proposition:

Proposition 2.3.3.2 *Using the notations of the previous proposition, we have that the map d_A; $x \mapsto d(x, A)$ is continuous.*

We remark that for a Hadamard manifold M and a closed convex subset A of M d_A is twice differentiable on $M \setminus A$ (see Corollary 1 [36]).
The following result is a particular case of Theorem 1. [36]. For the simplicity of the ideas we present a new proof of it.

Proposition 2.3.3.3 *Let M be an Hadamard manifold, $x \in M$ and $C \subset M$ a closed convex set. Then the set*

$$\{y \in C|\ d(x,y) = d(x,C)\}$$

contains exactly one element.

Proof. By the definition of $d(x,C)$ there is an $y \in C$ such that

$$d(x,C) \leq d(x,y) < d(x,C) + 1$$

Hence

$$d(x,C) = d(x,B),$$

where $B = C \cap B_x(d(x,C)+1)$ and

$$\{y \in C|\ d(x,y) = d(x,C)\} = \{y \in B|\ d(x,y) = d(x,B)\} \tag{2.3.3.1}$$

Since B is a closed subset of the compact set $B_x(d(x,C)+1)$, B is compact. Hence by (2.3.3.1) and Proposition 2.3.3.2 we have that the set $\{y \in C|\ d(x,y) = d(x,C)\}$ is not empty. Now let us suppose that the set $I = \{y \in C|\ d(x,y) = d(x,C)\}$ contains the elements $p \neq q$. Let u be the midpoint of the geodesic arc joining p to q (see fig. 2.5).

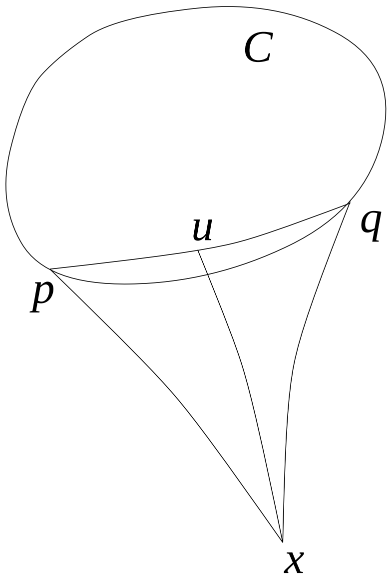

Figure 2.5:

Then by Corollary 2.3.1.4 we have that

$$4d(x,u)^2 \leq 2d(x,p)^2 + 2d(x,q)^2 - d(p,q)^2 < 2d(x,p)^2 + 2d(x,q)^2 \tag{2.3.3.2}$$

But since p, $q \in M$ we have that

$$d(x,p) = d(x,q) = d(x,C) \tag{2.3.3.3}$$

Inserting (2.3.3.3) in (2.3.3.2) we obtain

$$d(x,u)^2 < d(x,C)^2. \tag{2.3.3.4}$$

But since C is convex $u \in C$. Hence (2.3.3.4) is in contradiction with the definition of $d(x,C)$. Thus I contains exactly one element.

The following definition is a particular case of the definition given in Theorem 1. [36].

Definition 2.3.3.4 *Let M be an Hadamard manifold, $x \in M$ and $C \subset M$ a closed convex set. By proposition 2.3.3.3 there is exactly one point $y \in M$ such that $d(x,y) = d(x,C)$. Then y is denoted by $\pi_C(x)$ and is called the* projection *of x to C. The map π_C: $x \mapsto \pi_C(x)$ is called* the projection map *to C.*

It is easy to see that $\pi_C^2 = \pi_C$
The following result generalize inequality (1.2) of [38] and can be found in [36]. For the completeness of the ideas we give a proof of it.

Lemma 2.3.3.5 *Let $(M,\, g)$ be an Hadamard manifold, $C \subset M$ a closed convex set, $x \in M \setminus C$ and $y \in C$. Then the angle $\psi(x,y)$ between the geodesic arc joining $\pi_C(x)$ to x and the geodesic arc joining $\pi_C(x)$ to y is nonacute.*

Proof. For p and q in M denote by $\gamma_{pq} : [0, d(p,q)] \to M$ the unit speed geodesic arc joining p to q ($\gamma_{pq}(0) = p$, $\gamma_{pq}(d(p,q)) = q$) and by e_{pq} the unit tangent vector $\dot{\gamma}_{pq}(0)$ of γ_{pq} in 0. Let $y \in C$ and $x \in M \setminus C$. Suppose that $\psi(x,y) < \pi/2$. This is equivalent to

$$g(P^x_{\pi_C(x)},\, e_{\pi_C(x)y}) < 0. \tag{2.3.3.5}$$

Using (2.3.3.5) and the smoothness of P^x we obtain that

$$g(P^x_{\gamma_{\pi_C(x)y}(t_0)},\, e_{\gamma_{\pi_C(x)y}(t_0)y}) < 0, \tag{2.3.3.6}$$

if $t_0 \neq 0$ is sufficiently small.
Applying Lemma 2.1.4.3 (i) to the geodesic triangle of vertices x, $\pi_C(x)$ and $\gamma_{\pi_C(x)y}(t_0)$ (see fig. 2.6) we get

$$\begin{aligned} d(x,\pi_C(x))^2 &\geq d(x,\gamma_{\pi_C(x)y}(t_0))^2 + d(\pi_C(x),\gamma_{\pi_C(x)y}(t_0))^2 \\ &-2d(\pi_C(x),\gamma_{\pi_C(x)y}(t_0))g(P^x_{\gamma_{\pi_C(x)y}(t_0)},\, e_{\gamma_{\pi_C(x)y}(t_0)y}) \end{aligned} \tag{2.3.3.7}$$

By (2.3.3.6) and (2.3.3.7) we obtain that

$$d(x,C) = d(x,\pi_C(x)) > d(x,\gamma_{\pi_C(x)y}(t_0)). \tag{2.3.3.8}$$

But inequality (2.3.3.8) is in contradiction with the definition of $d(x,y)$. Hence $\psi(x,y) \geq 0$.

The following theorem is an easy consequence of Lemma 2.3.3.5:

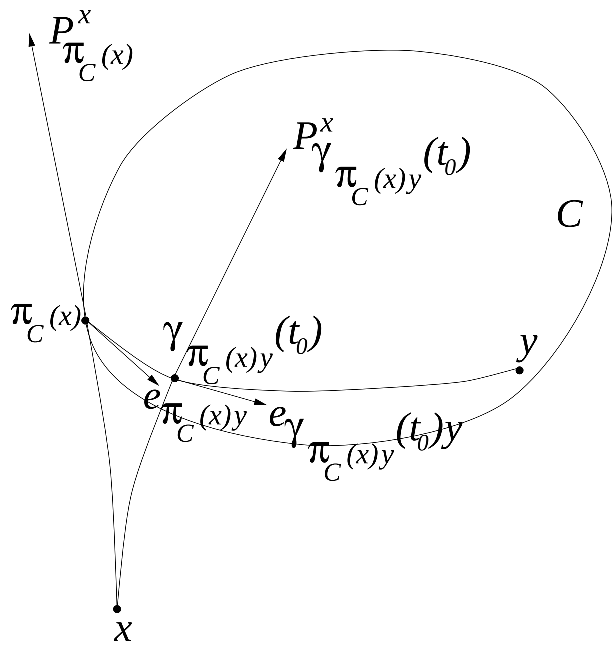

Figure 2.6:

Theorem 2.3.3.6 *Let $(M,\, g)$ be an Hadamard manifold, $C \subset M$ a closed convex set, $x \in M \setminus C$ and $y \in M$, such that $x \neq y$. Then the angle $\theta(x,y)$ between the geodesic arc joining $\pi_C(x)$ to x and the geodesic arc joining $\pi_C(x)$ to $\pi_C(y)$ is nonacute.*

Theorem 2.3.3.6 is a generalization of inequality (1.4) of [38].

We remark that for an Hadamard manifold M the nonexpansivity of a map $\phi :\, M \to M$ (definition 2.2.1.1) is equivalent to the inequality

$$d(\phi(x), \phi(y)) \leq d(x,y), \quad \forall x \in M.$$

Inequality (2.2.1.1) of definition 2.2.1.1 is the local form of the above inequality. The following theorem is a generalization of the nonexpansivity of the projection map to closed convex sets in Hilbert spaces (see Lemma 1.3 [38]) and is a consequence of Lemma 1 [36]. We give here an original proof.

Theorem 2.3.3.7 *Let $(M,\, g)$ be an Hadamard manifold, $C \subset M$ a closed convex set, $x,\, y \in M$. Then we have that*

$$d(\pi_C(x), \pi_C(y)) \leq d(x,y),$$

i.e π_C is nonexpansive.

Proof.
1. If $x,\, y \in C$ then the statement of the theorem is obvious, since $\pi_C(x) = x$ and $\pi_C(y) = y$.

2. If $x \in C$ and $y \in M \setminus C$ (or conversely) then $\pi_C(x) = x$ ($\neq$) and $\pi_C(y) \neq y$ ($=$) and the statement of the theorem is a consequence of Lemma 2.1.4.3 (i) applied to the triangle of vertices x, y, $\pi_C(x)$ ($\pi_C(y)$).

3. If x, $y \in M \setminus C$, then the statement of the theorem is a consequence of Lemma 2.3.1.5 (i) and Theorem 2.3.3.6.
By Proposition 2.3.1.8 Theorem 2.3.3.7 has the following corollary (this result can be also found in [36]):

Corollary 2.3.3.8 *If M is an Hadamard manifold and C a closed convex subset of M, then the projection map π_C is continuous on M and differentiable on $M \setminus C$.*

By Proposition 2.3.2.2 and Corollary 2.3.3.8 we have that:

Proposition 2.3.3.9 *Let M be an Hadamard manifold, C a closed convex subset of M and $\lambda : [0, \infty[\to [0, \infty[$ a continuous function with $\lambda(0) = 0$. Then the complementary vector field Π_C of the projection map π_C is continuous on M and differentiable on $M \setminus C$.*

Proposition 2.3.3.10 *Let (M, g) be an Hadamard manifold, C a closed convex subset of M and $\lambda : [0, \infty[\to [0, \infty[$ an increasing function. Then the λ-complementary vector field Π_C^λ of the projection map π_C is monotone.*

Proof. Let $x \neq y \in M$.

1. If at least one of these points is in C, then by lemma 2.1.4.3 (i) and Theorem 2.3.3.6 we trivially have that:

$$g(\Pi_{C_x}^\lambda, \dot\gamma(0)) \leq g(\Pi_{C_y}^\lambda, \dot\gamma(l)), \tag{2.3.3.9}$$

where $\gamma : [0, l] \to M$ is the unit speed geodesic joining x to y, such that $\gamma(0) = x$ and $\gamma(l) = y$ and $l = d(x, y)$.

2. If x, $y \in M \setminus C$, then by Theorem 2.3.3.6 and corollary 2.3.1.3 applied to the guadrilateral Q of consecutive vertices $\pi_C(x)$, $\pi_C(y)$, y, x we have either $\alpha \leq \pi/2$ or $\beta \leq \pi/2$, where α, β are the angles of Q corresponding to the vertices x, y respectively. Then by Lemma 2.3.1.5 (ii) we have that

$$g(\Pi_{C_x}^\lambda, \dot\gamma(0)) \leq g(\Pi_{C_y}^\lambda, \dot\gamma(l)). \tag{2.3.3.10}$$

Inequalities (2.3.3.9) and (2.3.3.10) proves the theorem.

2.4 Nonexpansive maps

In this section we suppose that M is an Hadamard manifolds. If f is (geodesically) nonexpansive, we prove that the complementary vector field of f is monotone. Since a projection map onto a closed convex set of an Hadamard manifold is nonexpansive, and compositions of nonexpansive maps are nonexpansive, we can take $f = p_1 \circ \ldots \circ p_n$, where $p_1, \ldots, p_n$ are projection maps onto closed convex sets of M. For $n = 1$ this is proved in the previous section. Since in general the composition of projection maps is not a projection map [39], the monotonicity of the complementary vector field of $f = p_1 \circ \ldots \circ p_n$ cannot be reduced to the mentioned result of section 2.3.

2.4.1 Some other consequences of the Comparison theorems

First we prove the following lemma:

Lemma 2.4.1.1 *Consider $\mathbf{R}^2$ endowed with the canonical scalar product $\langle\,.\,,\,.\,\rangle$. Let $abcd$ be a quadrilateral in $\mathbf{R}^2$ such that $dc \geq ab$. Denote by α, β, γ, δ the angles of the vertices a, b, c, d, respectively. Then*

$$ad \cos\delta + bc \cos\gamma \geq 0 \tag{2.4.1.1}$$

(This holds even if $abcd$ degenerates to a triangle.)

Proof. From $dc \geq ab$ and the Schwartz inequalty we have that

$$\langle d - c,\, a - b\rangle \leq dc^2,$$

which is equivalent to

$$\langle c - d,\, a - d\rangle + \langle d - c,\, b - c\rangle \geq 0. \tag{2.4.1.2}$$

It is easy to see that (2.4.1.2) implies (2.4.1.1).

The following lemma is a generalization of Lemma 2.4.1.1:

Lemma 2.4.1.2 *Let M be an Hadamard manifold and $abcd$ be a quadrilateral in M such that $dc \geq ab$. Denote by α, β, γ, δ the angles of the vertices a, b, c, d, respectively. Then*

$$ad \cos\delta + bc \cos\gamma \geq 0$$

(This holds even if $abcd$ degenerates to a triangle.)

Proof. We identify T_aM with $\mathbf{R}^n$, where $n = \mathrm{dim}M$.
If $\delta, \gamma > \pi/2$ then Lemma 2.3.1.5 (i) implies $ab > cd$ which contradicts $cd \geq ab$. Hence we have either $\delta \leq \pi/2$ or $\gamma \leq \pi/2$. We can suppose without loss of generality that

$$\gamma \leq \pi/2. \tag{2.4.1.3}$$

The lengths of the sides of a geodesic triangle satisfy the triangle inequalities. Hence there exist the points b', c', d' of $T_a(M)$ such that $ad' = ad$, $ac' = ac$, $d'c' = dc$, $ab' = ab$, $b'c' = bc$ and b' is contained in the plane of $ad'c'_\triangle$, such that

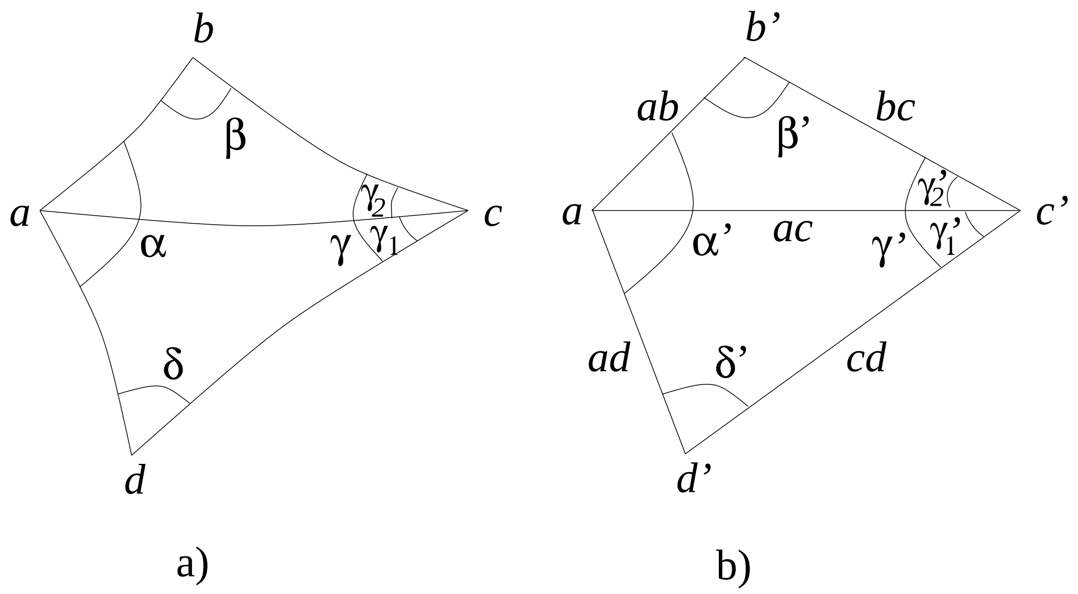

Figure 2.7:

b' and d' are contained in different half planes defined by the straight line in $T_a(M)$ joining a and c'. Let $\alpha' = \angle d'ab'$, $\beta' = \angle ab'c'$, $\gamma' = \angle b'c'd'$, $\delta' = \angle c'd'a$, $\gamma_1' = \angle ac'd'$ and $\gamma_2' = \angle ac'b'$ (see fig. 2.7 b)).
Using Lemma 2.4.1.1 to the quadrilateral $ab'c'd'$ we obtain

$$ad' \cos \delta' + b'c' \cos \gamma' \geq 0. \tag{2.4.1.4}$$

Denote by γ_1, γ_2 the angles of the vertex c in the triangles $adc_\triangle$, $abc_\triangle$, respectively (see fig. 2.7 a)). Then we have, by (i) of Lemma 3.1, p.259 [5] that

$$\delta' \geq \delta. \tag{2.4.1.5}$$

and

$$\gamma_1' + \gamma_2' \geq \gamma_1 + \gamma_2 \geq \gamma. \tag{2.4.1.6}$$

We consider two cases:

1) $\gamma_1' + \gamma_2' \leq \pi$.

We have

$$\gamma' = \gamma_1' + \gamma_2'. \tag{2.4.1.7}$$

Relations (2.4.1.6) and (2.1.4.7) imply

$$\gamma' \geq \gamma. \tag{2.4.1.8}$$

Since $ad' = ad$, $b'c' = bc$ and the cos function is strictly decreasing on $]0, \pi]$ (2.4.1.4), (2.4.1.5) and (2.4.1.8) imply

$$ad \cos \delta + bc \cos \gamma \geq 0. \ \clubsuit$$

2) $\gamma_1' + \gamma_2' > \pi$.

If $\delta \leq \pi/2$ then $ad\cos\delta + bc\cos\gamma \geq 0$ holds trivially, since $\gamma \leq \pi/2$. We suppose that $\delta > \pi/2$. By (2.4.1.5) we have that $\delta' > \pi/2$. Hence

$$\gamma_1' \leq \pi/2. \tag{2.4.1.9}$$

We also have

$$\gamma_2' \leq \pi. \tag{2.4.1.10}$$

Relations (2.4.1.9) and (2.4.1.10) imply

$$2\pi - \gamma' = \gamma_1' + \gamma_2' \leq \frac{3\pi}{2}. \tag{2.4.1.11}$$

By (2.4.1.3) and (2.4.1.11) we have $0 < \gamma \leq \gamma' \leq \pi$. Since the cos function is strictly decreasing on $]0,\pi]$ we have

$$\cos\gamma \geq \cos\gamma'. \tag{2.4.1.12}$$

Similarly (2.4.1.5) implies

$$\cos\delta \geq \cos\delta'. \tag{2.4.1.13}$$

By $ad' = ad$, $b'c' = bc$, (2.4.1.4) (2.4.1.12) and (2.4.1.13) we have

$$ad\cos\delta + bc\cos\gamma \geq 0. \ \clubsuit$$

2.4.2 Nonexpansive maps generating monotone vector fields

The following theorem is an immediate consequence of Lemma 2.4.1.2 and Proposition 2.3.2.2.

Theorem 2.4.2.1 *Let M be an Hadamard manifold and $f : M \to M$ a nonexpansive map. Then the complementary vector field F of f is a continuous monotone vector field.*

By Theorem 2.3.3.7 we obtain the following corollary of Theorem 2.4.4.1:

Corollary 2.4.2.2 *Let M be an Hadamard manifold and $f = p_1 \circ \ldots \circ p_n$, where $p_1, \ldots, p_n$ are projection maps of M onto closed convex sets. Then the complementary vector field of f is a continuous monotone vector field.*

2.5 Generalized monotonicities

In this section we introduce the pseudomonotone, strictly pseudomonotone and quasimonotone vector fields. For each type of (generalized) monotonicity there is a corresponding type of (generalized) convexity. In this way we can speak about pseudoconvex, strictly pseudoconvex and quasiconvex functions on a Riemannian manifold [31]. More precisely a gradient vector field is pseudomonotone, strictly pseudomonotone, quasimonotone if and only if, the underlying function is pseudoconvex, strictly pseudoconvex, quasiconvex, respectively. These results extebd some results of [11].

2.5.1 Pseudomonotone vector fields

Definition 2.5.1.1 *A function f is quasiconvex* [31] *on a convex subset K of a Riemannian manifold M if, for all $x, y \in K$, $x \neq y$, $t \in [0,1]$ and every geodesic arc $\gamma : [0,1] \to M$ joining x to y ($\gamma(0) = x$, $\gamma(1) = y$)*

$$f(y) \leq f(x) \text{ implies } f(\gamma(t)) \leq f(x).$$

Proposition 2.5.1.2 *If a differentiable function f is quasiconvex on an open convex set K of a Riemannian manifold (M, g) then, for every pair of distinct points $x, y \in K$ and every geodesic arc $\gamma : [0,1] \to M$ joining x to y ($\gamma(0) = x$, $\gamma(1) = y$) and contained in K we have*

$$f(y) \leq f(x) \text{ implies } g(\nabla f_x, \dot\gamma(0)) \leq 0. \tag{2.5.1.1}$$

Proof. Suppose that f is quasiconvex and $x, y \in K$, $x \neq y$ such that $f(y) \leq f(x)$. Then, using the quasiconvexity of f we have

$$\frac{f(\gamma(t)) - f(x)}{t} \leq 0,$$

for all $t \in [0,1]$. If we tend with t to zero we obtain

$$g(\nabla f_x, \dot\gamma(0)) \leq 0.$$

Definition 2.5.1.3 *A vector field X is pseudomonotone on a convex subset K of a Riemannian manifold (M, g) if, for every pair of distinct points $x \in K$, $y \in K$ and every geodesic arc $\gamma : [0,1] \to M$ joining x to y ($\gamma(0) = x$, $\gamma(1) = y$) contained in K, we have*

$$g(X_x, \dot\gamma(0)) \geq 0 \text{ implies } g(X_y, \dot\gamma(1)) \geq 0. \tag{2.5.1.2}$$

Definition 2.5.1.4 *A differentiable function f on an open convex subset K of a Riemannian manifold (M, g) is pseudoconvex on K* [31] *if, for every pair of distinct points $x \in K$, $y \in K$ and every geodesic arc joining x to y ($\gamma(0) = x$, $\gamma(1) = y$) contained in K, we have*

$$g(\nabla f_x, \dot\gamma(0)) \geq 0 \text{ implies } f(y) \geq f(x).$$

Lemma 2.5.1.5 *Let K be convex subset of a Riemannian manifold (M, g) and $f : G \to \mathbf{R}$ differentiable at $\overline{x}$, where G is an open set containing K. Then*

$$f(\overline{x}) = \max_{x \in K} f(x) \text{ implies } g(\nabla f_{\overline{x}}, \dot{\gamma}(0)) \leq 0,$$

for all geodesic arcs $\gamma : [0,1] \to M$ contained in K and satisfying $\gamma(0) = \overline{x}$.

Proof. Let $\gamma : [0,1] \to M$ be an arbitrary geodesic arc contained in K and satisfying $\gamma(0) = \overline{x}$. Since f is differentiable at $\overline{x}$ and $f(\overline{x}) = \max_{x \in K} f(x)$, we have that

$$0 \geq f(\gamma(t)) - f(\overline{x}) = tg(\nabla f_{\overline{x}}, \dot{\gamma}(0)) + t\alpha(t),$$

where $\lim_{t \to 0} \alpha(t) = 0$. Hence

$$g(\nabla f_{\overline{x}}, \dot{\gamma}(0)) + \alpha(t) \leq 0,$$

for $0 < t \leq 1$. By taking the limit as t approaches zero, we get

$$g(\nabla f_{\overline{x}}, \dot{\gamma}(0)) \leq 0.$$

Proposition 2.5.1.6 *Let K be an open convex set in a Riemannian manifold (M, g), and let $f : K \to \mathbf{R}$. If f is pseudoconvex on K, then f is quasiconvex on K.*

Proof. Let f be pseudoconvex on K. We shall assume that f is not quasiconvex on K and exhibit a contradiction. If f is not quasiconvex on K, then there exist $x, y \in K$ and a geodesic $\gamma : [0,1] \to M$ contained in K satisfying $\gamma(0) = x$, $\gamma(1) = y$, such that

$$f(y) \leq f(x),$$

and

$$f(\hat{x}) > f(x),$$

where $\hat{x} = \gamma(\hat{t})$, for some $\hat{t} \in]0,1[$. Hence there exists an $\overline{t} \in]0,1[$ such that

$$f(\overline{x}) = \max_{z \in \gamma([0,1])} f(z),$$

where $\overline{x} = \gamma(\overline{t})$. By Lemma 2.5.1.5 we have then that

$$g(\nabla f_{\overline{x}}, \dot{\gamma}_1(0)) \leq 0,$$

where $\gamma_1 : [0,1] \to M$ is the geodesic arc defined by $\gamma_1(t) = \gamma(-\overline{t}t + \overline{t})$. Similarly

$$g(\nabla f_{\overline{x}}, \dot{\gamma}_2(0)) \leq 0,$$

where $\gamma_2(t) = \gamma((1-\overline{t})t + \overline{t})$. Since $\dot{\gamma_1}(0) = -\overline{t}\dot{\gamma}(\overline{t})$ and $\dot{\gamma_2}(0) = (1-\overline{t})\dot{\gamma}(\overline{t})$ we get

$$g(\nabla f_{\overline{x}},\, \dot{\gamma_1}(0)) = -\overline{t}g(\nabla f_{\overline{x}},\, \dot{\gamma}(\overline{t})) \leq 0,$$

and

$$g(\nabla f_{\overline{x}},\, \dot{\gamma_2}(0)) = (1-\overline{t})g(\nabla f_{\overline{x}},\, \dot{\gamma}(\overline{t})) \leq 0.$$

Hence

$$g(\nabla f_{\overline{x}},\, \dot{\gamma}(\overline{t})) = 0$$

and

$$g(\nabla f_{\overline{x}}, \dot{\gamma_2}(0)) = (1-\overline{t})g(\nabla f_{\overline{x}},\, \dot{\gamma}(\overline{t})) = 0.$$

But by the pseudoconvexity of f on K, it follows that

$$f(y) \geq f(\overline{x}),$$

since $g(\nabla f_{\overline{x}},\, \dot{\gamma_2}(0)) = 0$, $\gamma_2(0) = \overline{x}$ and $\gamma_2(1) = y$. This yields

$$f(y) = f(\overline{x}),$$

since

$$f(\overline{x}) = \max_{z\in\gamma([0,1])} f(z)$$

Now from

$$f(\hat{x}) > f(x) \geq f(y) = f(\overline{x})$$

we obtain

$$f(\hat{x}) > f(\overline{x}),$$

contradicting $f(\overline{x}) = \max_{z\in\gamma([0,1])} f(z)$. Hence f is quasiconvex on K.

Lemma 2.5.1.7 *Let K be an open convex subset of a Riemannian manifold $(M,\, g)$. A vector field X is pseudomonotone on K if and only if, for every pair of distinct points $x \in K$, $y \in K$ and every geodesic arc joining x to y ($\gamma(0) = x$, $\gamma(1) = y$) and contained in K, we have*

$$g(X_x,\, \dot{\gamma}(0)) > 0 \textit{ implies } g(X_y,\, \dot{\gamma}(1)) > 0.$$

Proof. In view of (2.5.1.2) pseudomonotonicity is equivalent to

$$g(X_y,\, \dot{\gamma}_-(1)) < 0 \textit{ implies } g(X_x,\, \dot{\gamma}_-(0)) < 0, \tag{2.5.1.3}$$

where $\gamma_- : [0,1] \to M$ is a geodesic arc contained in K, with $\gamma_-(0) = x$, $\gamma_-(1) = y$. Let $\gamma(t) = \gamma_-(1-t)$. Interchanging x and y in (2.5.1.3) we obtain that the pseudomonotonicity is equivalent to

$$g(X_x,\, \dot{\gamma}(0)) > 0, \textit{ implies } g(X_y,\, \dot{\gamma}(1)) > 0,$$

where $\gamma : [0,1] \to M$ is an arbitrary geodesic arc contained in K, joining x to y ($\gamma(0) = x$, $\gamma(1) = y$).

Proposition 2.5.1.8 *Let f be a differentiable function on an open convex subset K of a Riemannian manifold (M, g). Then f is pseudoconvex on K if and only if ∇f is pseudomonotone.*

Proof. Suppose that f is pseudoconvex on K. Let x and y be two arbitrary distinct points in K with

$$g(\nabla f_x, \dot{\gamma}(0)) \geq 0,$$

where $\gamma : [0,1] \to M$ is an arbitrary geodesic contained in K with $\gamma(0) = x$, $\gamma(1) = y$. We want to show that

$$g(\nabla f_y, \dot{\gamma}(1)) \geq 0.$$

From the pseudoconvexity of f we have

$$f(x) \leq f(y). \tag{2.5.1.4}$$

But by Proposition 2.5.1.6, every pseudoconvex function on a convex set is also quasiconvex. Hence, using Proposition 2.5.1.2, (2.5.1.4) implies

$$g(\nabla f_y, \dot{\gamma}_-(0)) \leq 0, \tag{2.5.1.5}$$

where $\gamma_-(t) = \gamma(1-t)$. Since $\dot{\gamma}_-(0) = -\dot{\gamma}(1)$, (2.5.1.5) implies

$$g(\nabla f_y, \dot{\gamma}(1)) \geq 0.$$

Hence ∇f is pseudomonotone.

Now, suppose that ∇f is pseudomonotone on K. Let x and y be two arbitrary points in K and $\gamma : [0,1] \to K$ an arbitrary geodesic arc contained in K joining them ($\gamma(0) = x$, $\gamma(1) = y$), such that

$$g(\nabla f_x, \dot{\gamma}(0)) \geq 0. \tag{2.5.1.6}$$

We want to show that

$$f(y) \geq f(x)$$

Assume to the contrary, that is,

$$f(y) < f(x) \tag{2.5.1.7}$$

From the mean value theorem, applied to $\phi(t) = f(\gamma(t))$, we have

$$f(y) - f(x) = g(\nabla f_{\overline{x}}, \dot{\gamma}(\overline{t})),$$

where $\overline{x} = \gamma(\overline{t})$, for some $0 < \overline{t} < 1$. Since $\dot{\gamma}(\overline{t}) = -\frac{1}{\overline{t}}\dot{\tilde{\gamma}}(0)$, where $\tilde{\gamma}(t) = \gamma(-\overline{t}t + \overline{t})$, we have that

$$f(y) - f(x) = -\frac{1}{\overline{t}} g(\nabla f_{\overline{x}}, \dot{\tilde{\gamma}}(0)) \tag{2.5.1.8}$$

From (2.5.1.7) and (2.5.1.8), we have

$$g(\nabla f_{\overline{x}},\, \dot{\tilde{\gamma}}(0)) > 0. \tag{2.5.1.9}$$

Since ∇f is pseudomonotone and $\tilde{\gamma}(0) = \overline{x}$, $\tilde{\gamma}(1) = x$, it follows from (2.5.1.9) and Lemma 2.5.1.7 that

$$g(\nabla f_x,\, \dot{\tilde{\gamma}}(1)) > 0$$

But $\dot{\tilde{\gamma}}(1) = -\overline{t}\dot{\gamma}(0)$. Hence

$$g(\nabla f_x,\, \dot{\gamma}(0)) < 0,$$

which contradicts (2.5.1.6)

2.5.2 Strictly pseudomonotone vector fields

Definition 2.5.2.1 *A vector field X is strictly pseudomonotone on a convex subset K of a Riemannian manifold $(M,\, g)$ if, for every pair of distinct points $x \in K$, $y \in K$ and every geodesic arc $\gamma : [0,1] \to M$ joining x to y ($\gamma(0) = x$, $\gamma(1) = y$) contained in K, we have*

$$g(X_x,\, \dot{\gamma}(0)) \geq 0 \textit{ implies } g(X_y,\, \dot{\gamma}(1)) > 0.$$

Definition 2.5.2.2 *A differentiable function f on an open convex subset K of a Riemannian manifold $(M,\, g)$ is strictly pseudoconvex on K [31] if, for every pair of distinct points $x \in K$, $y \in K$ and every geodesic arc $\gamma : [0,1] \to M$ joining x to y ($\gamma(0) = x$, $\gamma(1) = y$) contained in K, we have*

$$g(\nabla f_x, \dot{\gamma}(0)) \geq 0 \textit{ implies } f(y) > f(x).$$

The equivalent of Proposition 2.5.1.8 for strictly pseudoconvex functions, is the following proposition:

Proposition 2.5.2.3 *Let f be differentiable on a open convex subset K of a Riemannian manifold $(M,\, g)$. Then, f is strictly pseudoconvex on K if and only if, ∇f is strictly pseudomonotone on K.*

Proof. Suppose that f is strictly pseudoconvex on K. Let $x, y \in K$, $x \neq y$, and $\gamma : [0,1] \to M$ a geodesic arc contained in K with $\gamma(0) = x$, $\gamma(1) = y$ be such that

$$g(\nabla f_x,\, \dot{\gamma}(0)) \geq 0. \tag{2.5.2.1}$$

We want to show that

$$g(\nabla f_y,\, \dot{\gamma}(1)) > 0.$$

Assume the contrary that

$$g(\nabla f_y,\, \dot{\gamma}(1)) \leq 0. \tag{2.5.2.2}$$

Given (2.5.2.1), strict pseudoconvexity of f implies that

$$f(y) > f(x). \tag{2.5.2.3}$$

On the other hand, if $\gamma_-(t) = \gamma(1-t)$, (2.5.2.2) can be written as

$$g(\nabla f_y,\, \dot{\gamma}_-(0)) \geq 0.$$

From strict pseudoconvexity of f, it follows that

$$f(x) > f(y),$$

which contradicts (2.5.2.3).

Conversely suppose that ∇f is strictly pseudomonotone on K. Let $x, y \in K$, $x \neq y$ and $\gamma : [0,1] \to M$ a geodesic arc with $\gamma(0) = x$, $\gamma(1) = y$, be such that

$$g(\nabla f_x,\, \dot{\gamma}(0)) \geq 0. \tag{2.5.2.4}$$

We want to show that

$$f(y) > f(x).$$

Assume to the contrary that

$$f(y) \leq f(x). \tag{2.5.2.5}$$

From the mean value theorem, applied to $\phi = f \circ \gamma : [0,1] \to \mathbf{R}$, we have

$$f(y) - f(x) = g(\nabla f_{\overline{x}},\, \dot{\gamma}(\overline{t})), \tag{2.5.2.6}$$

where

$$\overline{x} = \gamma(\overline{t}),$$

for some $0 < \overline{t} < 1$. Now from (2.5.2.5) and (2.5.2.6), we have

$$g(\nabla f_{\overline{x}},\, \dot{\gamma}(\overline{t})) \leq 0. \tag{2.5.2.7}$$

Let $\tilde{\gamma}(t) = \gamma(-\overline{t}t + \overline{t})$. Then

$$\dot{\tilde{\gamma}}(0) = -\overline{t}\dot{\gamma}(\overline{t}), \tag{2.5.2.8}$$

and

$$\dot{\tilde{\gamma}}(1) = -\overline{t}\dot{\gamma}(0). \tag{2.5.2.9}$$

Inserting (2.5.2.8) in (2.5.2.7) we obtain

$$g(\nabla f_{\overline{x}},\, \dot{\tilde{\gamma}}(0)) \geq 0. \tag{2.5.2.10}$$

Since $\tilde{\gamma}(0) = \overline{x}$ and $\tilde{\gamma}(1) = x$ equation (2.5.2.10) implies, by the strict pseudomonotonicity of ∇f, that

$$g(\nabla f_x, \dot{\tilde{\gamma}}(1)) > 0. \tag{2.5.2.11}$$

Equations (2.5.2.9) and (2.5.2.11) imply

$$g(\nabla f_x,\, \dot{\gamma}(0)) < 0,$$

which contradicts (2.5.2.4).

2.5.3 Quasimonotone vector fields

In view of definition 2.5.1.3, Lemma 2.5.1.7 and definition 2.5.2.1, there is still the case to be considered, where the first inequality in (2.5.1.2) is a strict inequality.

Definition 2.5.3.1 *A vector field X is quasimonotone on a convex subset K of a Riemannian manifold (M, g) if, for every pair of distinct points $x \in K$, $y \in K$ and every geodesic arc $\gamma : [0,1] \to M$ joining x to y ($\gamma(0) = x$, $\gamma(1) = y$) contained in K, we have*

$$g(X_x, \dot\gamma(0)) > 0 \textit{ implies } g(X_y, \dot\gamma(1)) \geq 0. \tag{2.5.3.1}$$

The following proposition relates the quasimonotone vector fields to the quasiconvex functions:

Proposition 2.5.3.2 *Let f be a differentiable function on an open convex set K of a Riemannian manifold (M, g). Then f is quasiconvex on K if and only if ∇f is quasimonotone on K.*

Proof. Suppose that f is quasiconvex. Let $x, y \in K$, $x \neq y$ and let $\gamma : [0,1] \to M$ be a geodesic contained in K with $\gamma(0) = x$, $\gamma(1) = y$ be such that

$$g(\nabla f_x, \dot\gamma(0)) > 0. \tag{2.5.3.2}$$

The inequality $f(y) \leq f(x)$ is not possible, since then

$$g(\nabla f_x, \dot\gamma(0)) \leq 0,$$

according to (2.5.1.1) which contradicts (2.5.3.2). Hence, we have

$$f(y) > f(x).$$

According to (2.5.1.1), $f(x) < f(y)$ implies that

$$g(\nabla f_y, \dot\gamma_-(0)) \leq 0,$$

where $\gamma_-(t) = \gamma(1-t)$. Since $\dot\gamma_-(0) = -\dot\gamma(1)$, we obtain

$$g(\nabla f_y, \dot\gamma(1)) \geq 0. \tag{2.5.3.3}$$

Since we have shown that (2.5.3.2) implies (2.5.3.3), ∇f is quasimonotone.

Conversely, suppose that ∇f is quasimonotone. Assume that f is not quasiconvex. Then, there exists $x, y \in K$, $x \neq y$, $\overline{t} \in]0,1[$ and a geodesic arc $\gamma : [0,1] \to M$ contained in K with $\gamma(0) = x$, $\gamma(1) = y$, such that

$$f(y) \leq f(x),$$

and

$$f(\overline{x}) > f(x) \geq f(y), \tag{2.5.3.4}$$

where

$$\overline{x} = \gamma(\overline{t}).$$

The mean-value theorem applied to $\phi = f \circ \gamma$ implies the existence of $\hat{t}$, and t^* such that

$$f(\overline{x}) - f(y) = (\overline{t} - 1)g(\nabla f_{\hat{x}}, \dot{\gamma}(\hat{t})), \tag{2.5.3.5}$$

and

$$f(\overline{x}) - f(x) = \overline{t}g(\nabla f_{x^*}, \dot{\gamma}(t^*)), \tag{2.5.3.6}$$

where $0 < t^* < \overline{t} < \hat{t} < 1$ and $\hat{x} = \gamma(\hat{t})$, $x^* = \gamma(t^*)$. Then (2.5.3.4) and (2.5.3.5) imply

$$g(\nabla f_{\hat{x}}, \dot{\gamma}(\hat{t})) < 0. \tag{2.5.3.7}$$

Similarly (2.5.3.4) and (2.5.3.6) imply

$$g(\nabla f_{x^*}, \dot{\gamma}(t^*)) > 0, \tag{2.5.3.8}$$

Let $\tilde{\gamma} : [0,1] \to M$, be the geodesic arc defined by $\tilde{\gamma}(t) = \gamma(-(\hat{t} - t^*)t + \hat{t})$, Then we have

$$\dot{\tilde{\gamma}}(0) = -(\hat{t} - t^*)\dot{\gamma}(\hat{t})), \tag{2.5.3.9}$$

and

$$\dot{\tilde{\gamma}}(1) = -(\hat{t} - t^*)\dot{\gamma}(t^*). \tag{2.5.3.10}$$

Since $\hat{t} > t^*$ equations (2.5.3.7) and (2.5.3.9) imply

$$g(\nabla f_{\hat{x}}, \dot{\tilde{\gamma}}(0)) > 0. \tag{2.5.3.11}$$

Similarly equations (2.5.3.8) and (2.5.3.10) imply

$$g(\nabla f_{x^*}, \dot{\tilde{\gamma}}(1)) < 0. \tag{2.5.3.12}$$

Since $\tilde{\gamma}(0) = \hat{x}$ and $\tilde{\gamma}(1) = x^*$ equations (2.5.3.11) and (2.5.3.12) contradicts the quasimonotonicity of ∇f. Hence f is quasiconvex.

In view of the proof of Proposition 2.5.3.2 we can show that the converse of Proposition 2.5.1.2 is also true.

Proposition 2.5.3.3 *Suppose that the differentiable function f, defined on an open convex set K of a Riemannian manifold (M, g), satisfies the relation*

$$f(y) \leq f(x) \text{ implies } g(\nabla f_x, \dot{\gamma}(0)) \leq 0.$$

for every pair of distinct points $x, y \in K$ and every geodesic arc $\gamma : [0,1] \to M$ joining x to y ($\gamma(0) = x$, $\gamma(1) = y$) and contained in K. Then f is quasiconvex.

Proof. The relation of Proposition 2.5.3.3 coincides with (2.5.1.1). On the other hand, following with attention the first part of the proof of Proposition 2.5.3.2, relation (2.5.1.1) implies that ∇f is quasimonotone. Now from the second part of the mentioned proof, we obtain that f is quasiconvex.

Remark 2.5.3.4 *Let $F : \mathbf{R}^n \to \mathbf{R}^n$ be a map and X the vector field on $\mathbf{R}^n$ which assigns to each x the tangent vector in 0 of the curve $x + tF(x)$. Then X is monotone, strictly monotone, pseudomonotone, strictly pseudomonotone, quasimonotone if and only if F is monotone, strictly monotone, pseudomonotone, strictly pseudomonotone, quasimonotone, respectively.*

Hence the relations between the different types of monotone vector fields are the same as the relations between the corresponding types of monotone maps in [11].
If we consider just Hadamard manifolds,then the relations between the corresponding vector fields are the same as in the general case. This assertion is the consequence of the following theorem:

Proposition 2.5.3.5 *Let M be an Hadamard manifold, $o \in M$ and $\lambda : [0, \infty[\to [0, \infty[$ a function. If λ is monotone, strictly monotone, pseudomonotone, strictly pseudomonotone, quasimonotone, then the λ-position vector field of M at o monotone, strictly monotone, pseudomonotone, strictly pseudomonotone, quasimonotone, respectively.*

The proof is similar to the proof of Theorem 2.1.4.5.

Proposition 2.5.3.6 *Let M be an Hadamard manifold, C a convex closed subset of M and $\lambda : [0, \infty[\to [0, \infty[$. If λ is monotone, pseudomonotone, then the λ-complementary vector field of the projection map onto C is a monotone, pseudomonotone vector field, respectively.*

The proof is similar to the proof of Theorem 2.3.3.10.

2.6 Zeros of monotone vector fields

Let M be a manifold and X a vector field on M. We recall that $a \in M$ is called a *zero* of X if X vanishes at a i.e. $X_a = 0$.
It is known that the set of zeros of a monotone map $A : G \to \mathbf{R}^n$, where G is an open convex subset of $\mathbf{R}^n$, is convex (see for example [40] Theorem 32.C.(b)). We generalize this result for Hadamard manifolds.

Theorem 2.6.0.1 *Let (M, g) be an Hadamard manifold, G an open convex set of M and X a smooth monotone vector field on G. Then the set of zeros of X is convex.*

Proof. If X has at most one fix point we have nothing to prove. Therefore suppose that X has at least two distinct zeros. Let a, $b \in M$; $a \neq b$ be two arbitrary zeros of X and $\gamma : [0, l] \to M$ be the unit speed geodesic arc begining at a ($\gamma(0) = a$) and ending at b ($\gamma(1) = b$). We must prove that for all $s \in [0, l]$ $\gamma(s)$ is a zero of X. Suppose that $X_{\gamma(s)} \neq 0$. The monotonicity of X implies that

$$g(X_{\gamma(s)}, \dot{\gamma}(s)) = 0. \tag{2.6.0.1}$$

Let $\phi_t : M \to M$ be the one parameter transformation group generated by X. Then by Theorem 2.2.1.8 (i) ϕ_t is nonexpansive for $t < 0$. Since G is open $\gamma(s) \neq \phi_t(\gamma(s))$ and $\phi_t(\gamma(s)) \in G$ for $t < 0$ sufficiently small. Fix such a t_0. By (2.6.0.1) the trajectory $\phi_t(\gamma(s))$ is perpendicular to γ. Hence the points a, b, $c_s = \phi_{t_0}(\gamma(s))$ determine a unique nondegenerate geodesic triangle abc_s. By the triangle inequality we have

$$ac_s + bc_s > ab. \tag{2.6.0.2}$$

On the other hand the nonexpansivity of ϕ_{t_0} implies

$$ac_s + bc_s \leq a\gamma(s) + b\gamma(s) = ab. \tag{2.6.0.3}$$

But (2.6.0.3) is in contradiction with (2.6.0.2). Hence we must have $X_{\gamma(s)} = 0$. Thus for all $s \in [0, 1]$ $\gamma(s)$ is a zero of X.

Summary.

Our dissertation presents some geometrical results concerning the Minty-Browder monotonicity. In the first chapter different computational methods are presented for determining the domains of monotonicity of an operator $f : \mathbf{R}^n \to \mathbf{R}^n$. Most of them are connected to a new notion introduced by us the scalar derivative. In the 2-dimensional case the connection of the scalar derivative with the complex derivative and Kasner circle is outlined.

In the second chapter the notion of Minty-Browder monotonicity is extended to Riemannian manifolds, introducing the monotone vector fields. The monotone vector fields of gradient type were analyzed by C. Udrişte and T. Rapcsák in [34] [35] and [31]. However the nongradient case was not considered yet. We deal mostly with this general case.

Connections with first variations of the length of a geodesic, closed geodesics, one parameter expansive transformation groups, integral submanifolds, auto-parallel submanifolds, the Lie derivative, conformal vector fields, projection maps, nonexpansive maps, singularities of vector fields are exhibited. Throughout the dissertation many examples are given, and different computation methods are presented for determinining the domains of monotonicity of a vector field. Most of them are connected with a new notion introduced by us the geodesic scalar derivative. We generalized the notion of monotonicity, introducing the pseudomonotone, strictly pseudomonotone and quasimonotone vector fields. Connections with the corresponding types of convexities are presented.

Összefoglaló

A disszertációnk bemutat néhány geometriai eredményt a Minty-Browder monotonitásra vonatkozóan. Az első fejezetben különböző számolási módszereket adtunk egy $f : \mathbf{R}^n \to \mathbf{R}^n$ operátor monotonitási tartományainak a meghatározására. A legtöbb közülük egy általunk bevezetett új fogalomhoz, a skaláris deriválthoz kapcsolódik. A két dimenziós esetben bemutatjuk a skaláris derivált kapcsolatát a komplex deriválttal és a Kasner körrel.

A második fejezetben kiterjesztettük a Minty-Browder monotonitás fogalmát Riemann sokaságokra. A gradiens tipusú monoton vektor mezőket Udrişte Constantin és Rapcsák Tamás vizsgálta a [34], [35] és [31]-ben. Azonban a nem gradiens esetet még nem vizsgálták. Mi főleg ezzel az esettel foglalkoztunk. Bemutattuk a geodetikusok ívhosszának első variációjával, a zárt geodetikusokkal, az integrál részsokaságokkal, az auto-párhuzamos részsokaságokkal, a Lie deriválttal, a konformális vektormezőkkel, a projekciós leképezésekkel, a nemexpanzív leképezésekkel és a vektormezők szingularitásaival való kapcsolatot. A disszertációban számos példát adtunk és különböző számolási módszereket mutattunk be egy vektormező monotonitási tartományainak meghatározására. A legtöbb közülük egy általunk bevezetett új fogalomhoz, a geodetikus skaláris deriválthoz kapcsolódik. Általánosítottuk a monotonitás fogalmát, bevezetve a pszeudomonoton, szigorúan pszeudomonoton és kvázimonoton vektormező fogalmát, megadva a megfelelő konvexitási fogalmakkal való kapcsolatukat.

Publications on the topic of the dissertation

1. S.Z. Németh. Geodesic monotone vector fields. *Submitted to Acta Scientarium, Szeged.*

2. S.Z. Németh. Monotone vector fields. *Accepted for publication at Publicationes Mathematicae, Debrecen.*

3. S.Z. Németh. Monotonicity of λ-complementarity vector fields of a projection map. *Submitted to Annales Univ. Sci. Budapest.*

4. S.Z. Németh. Monotonicity of the complementarity vector field of a nonexpansive map. *Accepted for publication at Acta Mathematica Hungarica, it will appear in* vol. 85(3), 1999.

5. S.Z. Németh. Five kinds of monotone vector fields. *Accepted for publication at PUMA, University of Economics, Budapest. It will appear in the* november issue of 1999.

6. S.Z. Németh. Scalar derivatives and conformity. *Annales Univ. Sci. Budapest* 40 (1997), p. 105-111.

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Other publications

1. S.Z. Németh, D.F. Parker. Some unsteady pseudo-plane motions for the Navier-Stokes equations. *ARI (The Bulletin of the Technical University of Istanbul)* 50/3 (1998) 161-168.

2. S.Z. Németh. Bäcklund transformations of constant torsion curves in 3-dimensional constant curvature spaces. *Accepted for publication at the Italian Journal of Pure and Applied Mathematics.*

3. S.Z. Németh. Bäcklund transformations of n-dimensional constant torsion curves. *Accepted for publication at the Publicationes Mathematicae Debrecen, it will appear in* vol. 53, 1998.

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1. S.Z. Németh. Monotonicity properties of projection maps on Hadamard manifolds.

2. S.Z. Németh. Zeros of monotone vector fields.

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