Decisions, decisions ...

Decision are everywhere to make...

...and some of them are pretty hard!

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1 Some organisational stuff

Assessment:

- 2 class tests of 25% each for 3GTMDM and 22.5% each for 4GTMDM (i.e., fourth year version of the module).
- 80% examination.
- 20% continuous assessment
  - four problem sheets of 5% each for 3GTMDM
  - four problem sheets of 2.5% each + an extra task of 10% for 4GTMDM

Course material:

- The course will be presented using
  - virtual and physical white board,
  - computer presentation/visualiser.
- Lecture notes will be provided in advance on Canvas.
- Problem sheets: available on Canvas on the week before each Guided Study.
- Solution hints to all problems will be available on Canvas at the end of each Guided Study.
Recommended Literature


http://www.convexoptimization.com/wikimization/index.php/Farkas'_lemma#Extended_Farkas.27_lemma

2 What is Multicriteria Decision Making?

Multicriteria Decision Making is the mathematical theory of decision making with conflicting objectives and/or decision makers which conflict each other.

Typical examples include

- portfolio optimization (return ↔ risk)
- power plant control (cost ↔ wear & tear)
- hazardous material deposing/routing (cost ↔ risk)
- health care management (efficiency ↔ cost)
- material optimization (stability ↔ cost)
- parameter estimation (each equation one objective function)
- any model including at least two of the following: cost, risk, stability, etc. ...
What is Multicriteria Decision Making for?

- Understand why the "decision makers" behave as they do. (Thereby understanding, e. g., economic decisions better.)
- Help the decision maker (or even replace him with an automated procedure!) find the "best" decision (if there is a "best" way).

(The second reason is the more practical one...)

Some other names for the topic:

Multicriteria Decision Making, Multiple Criteria Decision Making, Multicriteria Optimisation, Multiobjective Decision Making, Multiobjective Optimisation, Vector Optimisation, Decision Theory, ...

2.1 Decision Making in the Real World

When the Naskapi Indians (Canada) had to send a hunting party out, they had to decide upon a search direction for the party.

For this, the shoulder bones of a caribou was held over hot coals causing cracks in the bone which are then used to direct a hunting party.

This procedure worked!

"Worked" means "a decision has been made", not "the best decision has been made".

The fact that a decision is made is a relief, a comfort; it is satisfying to remove uncertainty and indecision. Any outcome can be satisfying.

Can we do better??

A (very first) example: buying a car

Please help me in buying a car.

There are (only) the criteria prize and style.

The market has been boiled down to the following models with the following prices:

\[ A = \text{Mini: } £12K \]
\[ B = \text{Golf: } £14K \]
\[ C = \text{Alfa Romeo: } £21K \]
\[ D = \text{Mitsubishi: } £9K \]
\[ E = \text{Fiat: } £8K \]
\[ F = \text{Jaguar: } £25K \]

**Style**: let’s do a poll →

Question 1: which car would you NOT choose?
Question 2: which car would you choose?

Let us now use some more criteria: price, size, reliability, style (aesthetics).
But let us simplify the market to four models: $\alpha, \beta, \gamma, \delta$.

For each criterium, one can rank each car ($1 = \text{best}$, ..., $4 = \text{worst}$).

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<td>style</td>
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Which car would you choose??

This is just a simple example: add criteria comfort, risk of theft, weekly costs, safety, depreciation, ...

...and replace style by status and colour...

**Another motivating example**

Suppose a decision maker can invest a certain amount of money on the stock market. Let $x$ denote the decision taken.

It will usually be the case that the decision maker tries to optimise several objectives at once, i. e.

- reward (expected return): $\varphi(x) = f_1(x) \rightarrow \max$
- risk (variance of return): $R(x) = f_2(x) \rightarrow \min$
- dividends: $f_3(x) \rightarrow \max$
- social responsibility: $f_4(x) \rightarrow \max$
- number of securities: $f_5(x) \rightarrow \max$
- short selling: $f_6(x) \rightarrow \min$
- securities sold short: $f_7(x) \rightarrow \min$
- liquidity: $f_8(x) \rightarrow \max$
- skewness: $f_9(x) \rightarrow \min$
- turnover: $f_{10}(x) \rightarrow \min$
- amount invested in R&D: $f_{11}(x) \rightarrow \max$
- growth in sales: $f_{12}(x) \rightarrow \max$

**And another motivating example**

(Based on a true story, Illinois, USA, 1977)

A region of undeveloped land (close to a suburban area of Chicago) is under consideration. It is divided into several different planning regions. Goal: assign a certain land use (small residential, large residential, commercial, offices, manufacturing, schools, open space) to each region.

Objectives:

Maximise compatibility of assignment. (Not every region is as usable as every other.)

Minimise expected total transportation.
Minimise tax load (operating cost).
Minimise environmental impact.
Minimise facility cost.

So far we have seen:

Multicriteria Decision Making looks more difficult than ”standard” decision making.

We will see:

Multicriteria Decision Making IS more difficult than ”standard” decision making ...

...but it is still within our reach!

3 How to model a Multicriteria Decision Making problem?

3.1 Some general ideas

**General scenario:** we have $n > 1$ functions $f_1, \ldots, f_n : X \rightarrow \mathbb{R}$ (”criteria”) to be minimised. (Ex.: $X$: set of feasible portfolios).

(We deal only with minimisation, just for simplicity. For maximisation, multiply criteria with $-1$.)

We also write $f : X \rightarrow \mathbb{R}^n$, with $f(x) = [f_1(x), \ldots, f_n(x)]^\top$.

**Example:** above, $n = 2$, $f_1 = -\varrho$, $f_2 = R$.

**Observation:** only in very special cases does there exist an $x \in X$ which minimises all criteria $f_1, \ldots, f_n$ simultaneously.

$\implies$

We need a new concept of solution.

What is a solution to a multicriteria problem??

**Example:** $n = 2$, $X = [-1, 1]$, $f_1(x) = -x/2$, $f_2(x) = -(5 - x^2)^{1/2}$.

$\implies$

**Some solution ideas:**

1. Build tradeoff model: choose tradeoff parameter $\mu$ and solve $\min_{x \in X} \mu f_1(x) + f_2(x)$.
2. Bound one function value: choose parameter $\varrho$ and minimise $f_2(x)$ subject to $x \in X$, $f_1(x) \leq \varrho$.
3. Bound the other function value: choose parameter $\varrho$ and minimise $f_1(x)$ subject to $x \in X$, $f_2(x) \leq \varrho$.
4. Every $x \in [0, 1]$ is a solution: changing $x$ makes at least one $f_i$ larger.
5. Minimise $f_1$ first, then $f_2$.
6. Minimise $f_2$ first, then $f_1$.
7. Worst-case approach: solve $\min_{x \in X} \max_i f_i(x)$. 
8. Weight importance in the worst case approach: choose weights \( \mu_i \) and solve
\[
\min_{x \in X} \max_i \mu_i f_i(x).
\]
9. Solve \( \min_{x \in X} \left( (f_1(x))^2 + (f_2(x))^2 \right)^{1/2} \).
10. Solve \( \min_{x \in X} \left( \mu_1 (f_1(x))^2 + \mu_2 (f_2(x))^2 \right)^{1/2} \).

Lots of open questions:
1. Which approach is the right one?
2. Are these different ideas/approaches interconnected with each other?
3. What are the fundamental differences between them?
4. Or are some of them equivalent?
5. What are the advantages and disadvantages of the different approaches?
6. Why are there so many of them??
7. Are there even more???
8. ...

So many different approaches are a clear sign for the need of a good mathematical theory!

- There are (literally!) hundreds of different solution methods for multicriteria decision making.
- We can not discuss all or even many of them.
- Emphasis will be on understanding the main concepts and their theoretical underpinnings.
- If you understand this theory, practical applications and new methods will rarely surprise you.
- Many solution methods need a method from Linear Programming & Combinatorial Optimisation or Nonlinear Programming as a “subroutine”. (Hint: we can’t cover everything in one lecture...)

### 3.2 How to order vectors

Recall the situation in \( \mathbb{R}^1 = \mathbb{R} \) (i.e. \( n = 1 \)): \( f : X \rightarrow \mathbb{R} \) has to be minimised. The image set \( \mathbb{R} \) is equipped with a relation “\( \leq \)” with has a meaning to the decision maker:

Let \( f(x) = v \) and \( f(y) = w \) and \( v \leq w \). Then \( v \) (resp. \( x \)) is at least as good as \( w \) (resp. \( y \)).

(Or, if \( v \neq w \), the value \( v \) (resp. \( x \)) is preferred to \( w \) (resp. \( y \)): \( \leq \) is a preference relation. Or, if \( v \neq w \), the value \( v \) (resp. \( x \)) dominates \( w \) (resp. \( y \)).)

\( x, y \): decisions

\( v = f(x), w = f(y) \): values of the decisions

\[ \Rightarrow \]

We are in search for a relation (order) \( \preceq \) on \( \mathbb{R}^n \) (with \( n > 1 \)) that has some meaning for the decision-maker.

(And that also has nice mathematical and computational properties, see below!)

Decision makers want, first and all, two properties:
1. $\preceq$ should be translation invariant, i.e. for all $v, w \in \mathbb{R}^n$:

\[ v \preceq w \implies v + u \preceq w + u \text{ for all } u \in \mathbb{R}^n \]  

("compatible with addition")

2. $\preceq$ should be scale invariant, i.e. for all $v, w \in \mathbb{R}^n$:

\[ v \preceq w & \lambda > 0 \implies \lambda v \preceq \lambda w. \]  

("compatible with positive scalar multiplication")

Let $K \subseteq \mathbb{R}^n$ be given. Define the relation $\leq_K$ by

\[ v \leq_K w :\iff w - v \in K \]  

(The order $\leq_K$ is induced by $K$.)

**Example:** $n = 1$, $K = \mathbb{R}_+$, standard order $\leq$.

(Idea: $K$ simple $\implies x \leq_K y$ easy to check.)

**Example:** $\rightarrow$

**Theorem 1.** A relation $\preceq$ is translation invariant if and only if it is induced by a set $K$, i.e. if $\preceq = \leq_K$ holds.

**Proof.** Suppose that $\preceq$ is translation invariant. Then, $x \preceq y$ implies $0 = x + (-x) \preceq y + (-x) = y - x$. Hence, $y - x \in K := \{z \in \mathbb{R}^n \mid 0 \preceq z\}$ and thus $x \leq_K y$. On the other hand $x \leq_K y$ implies $y - x \in K$ and thus $0 \preceq y - x$. Hence, $x = 0 + x \preceq (y - x) + x = y$. In conclusion $x \preceq y$ is equivalent to $x \leq_K y$ and therefore $\preceq = \leq_K$.

Conversely, suppose that $\preceq = \leq_K$, for some set $K \subseteq \mathbb{R}^n$. Hence, $x \preceq y$ implies $x \leq_K y$ and thus $(y + z) - (x + z) = y - x \in K$. It follows that $x + z \leq_K y + z$ which implies $x + z \preceq y + z$, for any $z \in \mathbb{R}^n$. Therefore, $\preceq$ is translation invariant. \qed

**Definition.** A set $K \subseteq \mathbb{R}^n$ is called a cone if for all $v \in K$ and all $\lambda \in \mathbb{R}$, $\lambda > 0$, we have $\lambda v \in K$. A cone $K$ is called pointed if $K \cap -K \subseteq \{0\}$.

We remark that a cone $K$ is pointed if and only if there is no $a \neq 0$ such that $a, -a \in K$.

**Corollary 1.** The relation $\leq_K$ is scale invariant if and only if $K$ is a cone.

**Proof.** Suppose that $\leq_K$ is scale invariant. Let $x \in K$ and $\lambda \in \mathbb{R}$ with $\lambda > 0$. Then, $0 \leq_K x$. Hence, $0 = \lambda 0 \leq_K \lambda x$, which implies $\lambda x \in K$. Hence, $K$ is a cone.

Conversely, suppose that $K$ is a cone. Let $x, y \in \mathbb{R}^n$ with $x \leq_K y$ and $\lambda \in \mathbb{R}$ with $\lambda > 0$. Then, $y - x \in K$ and since $K$ is a cone we have $\lambda y - \lambda x = \lambda(y - x) \in K$. Hence, $\lambda x \leq_K \lambda y$. Therefore, $\leq_K$ is scale invariant. \qed

**Example:** The nonnegative quadrant

\[ K = \mathbb{R}^2_+ := \{x = (x_1, x_2)^\top \in \mathbb{R}^2 \mid x_1 \geq 0 \text{ and } x_2 \geq 0\} \]
is a cone which is a convex set. Such cones are simply called \textit{convex cones}. The relation induced by \( K \) is defined by \( v \leq_K w \iff w-v \in K \iff w_1 \geq v_1 \) and \( w_2 \geq v_2 \). It is easy to check that \( \leq_K \) is a scale invariant relation.

\textbf{Example:} The set
\[
K = \{ x = (x_1, x_2)^\top \in \mathbb{R}^2 \mid x_1 \geq 0 \text{ and } -x_2 \leq x_1 \}
\]
is a cone which is a convex set, simply called \textit{convex cone}. The relation induced by \( K \) is defined by \( v \leq_K w \iff w-v \in K \iff w_1 - v_1 \geq 0 \) and \( w_2 - v_2 \leq w_1 - v_1 \). It is easy to check that this is a scale invariant relation. Indeed, \( w_1 - v_1 \geq 0 \) and \( v_2 - w_2 \leq w_1 - v_1 \) implies
\[
\lambda w_1 - \lambda v_1 = \lambda(w_1 - v_1) \geq 0 \quad \text{and} \quad \lambda v_2 - \lambda w_2 = \lambda(v_2 - w_2) \leq \lambda(w_1 - v_1) = \lambda w_1 - \lambda v_1.
\]
Hence, \( \lambda v \leq_K \lambda w \).

\textbf{Example:} The set
\[
K = \{ (t,0)^\top \in \mathbb{R}^2 \mid t \geq 0 \} \cup \{ (0,t)^\top \in \mathbb{R}^2 \mid t \geq 0 \}
\]
is a cone, but not a convex cone. It is easy to check that \( K \) is a cone. \( K \) is not convex because \( a := (2,0)^\top \in K, b := (0,2)^\top \in K, \) but \( (1/2)(a + b) = (1,1)^\top \notin K \).

\textbf{Example:} The set
\[
K = \{ x = (x_1, x_2)^\top \in \mathbb{R}^2 \mid x_1 \geq 0 \text{ or } x_2 \geq 0 \}
\]
is a cone, but not a convex cone. It is easy to check that \( K \) is a cone. \( K \) is not convex because \( a := (-2,0)^\top \in K, b := (0,-2)^\top \in K, \) but \( (1/2)(a + b) = (-1,-1)^\top \notin K \).

\textbf{Example:} The set
\[
K = \{ x = (x_1, x_2)^\top \in \mathbb{R}^2 \mid 0 \leq x_1 \leq 1 \text{ and } 0 \leq x_2 \leq 1 \}
\]
is not a cone. Indeed, \( (1,1)^\top \in K, \) but \( 2(1,1)^\top = (2,2)^\top \notin K \). Observe that \( K \) is a bounded set. The relation induced by \( K \) is defined by \( v \leq_K w \iff w-v \in K \iff 0 \leq w_1 - v_1 \leq 1 \) and \( 0 \leq w_2 - v_2 \leq 1 \). As expected this relation is not scale invariant because \( (0,0)^\top \leq_K (1,1)^\top \) but \( (0,0)^\top = 2(0,0)^\top \not\leq_K 2(1,1)^\top = (2,2)^\top \).

\textbf{Example:} The set
\[
K = \{ x = (x_1, x_2)^\top \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \leq 1 \}
\]
is not a cone. Indeed \( (1/2,1/2)^\top \in K, \) but \( 2(1/2,1/2)^\top = (1,1)^\top \notin K \). Observe that \( K \) is a bounded set.

\textbf{Example:} The only nonempty bounded cone in \( \mathbb{R}^n \) is the trivial cone \( K = \{0\} \). Indeed, \( \{0\} \) is a bounded cone. If \( K \neq \{0\} \) is a nonempty cone than \( \exists v \in K \setminus \{0\} \). Hence, \( v^k := kv \in K \), for any positive integer \( k \). But \( \|v^k\| = k\|v\| \to \infty \), when \( k \to \infty \). Thus, \( K \) is not bounded.

\textbf{Example:} The nonnegative orthant
\[
K = \mathbb{R}^n_+ := \{ x = (x_1, \ldots, x_n)^\top \in \mathbb{R}^n \mid x_1 \geq 0, \ldots, x_n \geq 0 \}
\]
is a \textit{convex cone}.

The relation induced by \( K \) is defined by \( v \leq_K w \iff w-v \in K \iff w_1 \geq v_1, \ldots, w_n \geq v_n \). It is easy to check that \( \leq_K \) is a scale invariant relation. This relation is called \textit{standard} or \textit{componentwise} order.

For \( A, B \subset \mathbb{R}^n, v \in \mathbb{R}^n \) and \( \lambda \in \mathbb{R} \) denote
\[
\lambda A := \{ \lambda a \mid a \in A \},
\]
\[
-A = (-1)A = \{ -a \mid a \in A \},
\]
\begin{align*}
A + B & := \{a + b \mid a \in A, \ b \in B\}, \\
A - B & := A + (-B) = \{a - b \mid a \in A, \ b \in B\}, \\
v + B & := \{v\} + B = \{v + b \mid b \in B\}, \\
v - B & := v + (-B) = \{v - b \mid b \in B\}.
\end{align*}

**Remark 1.** Let \( K \subseteq \mathbb{R}^n \). Note that
\[
K = \{x \in \mathbb{R}^n \mid 0 \leq_K x\} = \{x \in \mathbb{R}^n \mid 0 \text{ is at least as good as } x\},
\]
\[
-K = \{x \in \mathbb{R}^n \mid x \leq_K 0\} = \{x \in \mathbb{R}^n \mid x \text{ is at least as good as } 0\},
\]
\[
v + K = \{w \in \mathbb{R}^n \mid v \leq_K w\} = \{w \in \mathbb{R}^n \mid v \text{ is at least as good as } w\},
\]
\[
v - K = \{w \in \mathbb{R}^n \mid w \leq_K v\} = \{w \in \mathbb{R}^n \mid w \text{ is at least as good as } v\},
\]
\[
v + K \setminus \{0\} = \{w \in \mathbb{R}^n \mid v \leq_K w \& \ v \neq w\} = \{w \in \mathbb{R}^n \mid w \text{ is dominated by } v\},
\]
\[
v - K \setminus \{0\} = \{w \in \mathbb{R}^n \mid w \leq_K v \& \ v \neq w\} = \{w \in \mathbb{R}^n \mid w \text{ is dominating } v\}.
\]

**Reminder.** Recall the following properties of the relation \( \leq \) on \( \mathbb{R}^n \):

1. The relation \( \leq \) is called **reflexive** if \( v \leq v, \forall v \in \mathbb{R}^n \).
2. The relation \( \leq \) is called **transitive** if \( u \leq v \text{ and } v \leq w \implies u \leq w \).
3. The relation \( \leq \) is called **antisymmetric** if \( v \leq w \text{ and } w \leq v \implies v = w \).
4. The relation \( \leq \) is called **total** if \( v \leq w \text{ or } w \leq v, \forall v, w \in \mathbb{R}^n \) (in other words any two elements in \( \mathbb{R}^n \) are comparable).

The properties of reflexivity, transitivity, antisymmetry and totality are also “desirable” properties for \( \leq_K \). When do these properties hold? The answer is given by the next theorem.

**Theorem 2.** Let \( K \subseteq \mathbb{R}^n \) be a nonempty set (but not necessarily a cone) and \( \leq_K \) be the binary relation defined by (1). Then, the following holds:

1. The order \( \leq_K \) is **reflexive** if and only if \( 0 \in K \).
2. The relation \( \leq_K \) is **transitive** if and only if \( \{v + w \mid v, w \in K\} =: K + K \subseteq K \).
3. The relation \( \leq_K \) is **antisymmetric** if and only if \( K \cap -K \subseteq \{0\} \). In particular if \( K \) is a cone, then \( \leq_K \) is **antisymmetric** if and only if \( K \) is pointed.
4. The order \( \leq_K \) is **total** if and only if the equality \( K \cup -K = \mathbb{R}^n \) holds.
5. The set \( K \) is closed if and only if the relation \( \leq_K \) is "continuous at 0" in the following sense: For all \( b \in \mathbb{R}^n \) and all sequences \( (w^i)_{i \in \mathbb{N}} \) in \( \mathbb{R}^n \) with \( \lim_{i \to \infty} w^i = b \) and \( 0 \leq_K w^i \) for all \( i \in \mathbb{N} \) it follows that \( 0 \leq_K b \) holds.

**Proof.**

1. Suppose that \( \leq_K \) is reflexive. Then, \( 0 \leq_K 0, \) or equivalently \( 0 - 0 \in K \). Hence, \( 0 \in K \).

Conversely, suppose that \( 0 \in K \). It follows that \( v - v \in K, \forall v \in \mathbb{R}^n \). Hence, \( v \leq_K v, \forall v \in \mathbb{R}^n \), or equivalently \( \leq_K \) is reflexive.
2. Suppose that $\leq_K$ is transitive. Let $v + w \in K + K$ be arbitrary. This means that $v, w \in K$ are arbitrary. We have $-w \leq_K 0$ and $0 \leq_K v$, which by the transitivity of $\leq_K$ implies $-w \leq_K v$. Hence, $v + w \in K$. Therefore, $K + K \subseteq K$.

Conversely, suppose that $K + K \subseteq K$. Then, $u \leq_K v$ and $v \leq_K w$ imply $v - u \in K$ and $w - v \in K$, respectively. Hence, $w - u = (w - v) + (v - u) \in K + K \subseteq K$. Thus, $u \leq_K w$. Therefore, $\leq_K$ is transitive.

3. Suppose that $K \cap -K \subseteq \{0\}$. Then, $v \leq_K w$ and $w \leq_K v$ imply $w - v \in K$ and $v - w \in K$, respectively. Thus, $w - v \in K$ and $w - v \in -K$. Hence, $w - v \in K \cap -K \subseteq \{0\}$. Therefore, $w - v = 0$, or equivalently $v = w$.

Conversely, suppose that $\leq_K$ is antisymmetric. If $K \cap -K = \emptyset$, then obviously $K \cap -K \subseteq \{0\}$. If $K \cap -K \neq \emptyset$, then let $v \in K \cap -K$ be arbitrary. Hence, $v \in K$ and $v \in -K$. Thus, $v \in K$ and $-v \in K$, which imply $0 \leq_K v$ and $v \leq_K 0$. Since $\leq_K$ is antisymmetric, we obtain $v = 0 \in \{0\}$. Therefore, $K \cap -K \subseteq \{0\}$.

4. Suppose that $\leq_K$ is total. We only need to show that $\mathbb{R}^n \subseteq K \cup -K$, because $K \cup -K \subseteq \mathbb{R}^n$ trivially holds. Let $v \in \mathbb{R}^n$ be arbitrary. Since $\leq_K$ is total, we have $0 \leq_K v$ or $v \leq_K 0$. Hence, $v \in K$ or $-v \in K$. Thus, $v \in K$ or $v \in -K$, which implies $v \in K \cup -K$. Therefore, $\mathbb{R}^n \subseteq K \cup -K$.

Conversely, suppose that $K \cup -K = \mathbb{R}^n$. Let $v, w \in \mathbb{R}^n$ be arbitrary. Then, $w - v \in \mathbb{R}^n = K \cup -K$.

Hence, $w - v \in K$ or $w - v \in -K$. Thus, $w - v \in K$ or $w - v \in K$, which implies $v \leq_K w$ or $w \leq_K v$, respectively. Therefore, $\leq_K$ is total.

5. The set $K$ is closed if and only if for all $b \in \mathbb{R}^n$ and all sequences $(w^i)_{i \in \mathbb{N}}$ in $\mathbb{R}^n$ with $\lim_{i \to \infty} w^i = b$ and $w^i \in K$ for all $i \in \mathbb{N}$ it follows that $b \in K$ holds. Hence, the equivalence follows by observing that $w^i \in K$ and $b \in K$ are equivalent to $0 \leq_K w^i$ and $0 \leq_K b$, respectively.

\[ \square \]

**Definition.** A set $S \subseteq \mathbb{R}^n$ is called connected if $\emptyset \neq A, B \subseteq S$, $S = A \cup B$ and $A \cap B = \emptyset$ imply that at least one of $A$ and $B$ is not closed.

Recall that if $A \cap B = \emptyset$ and $S = A \cup B$, then we say that $S$ is the disjoint union of $A$ and $B$. Hence, if $S$ can be written as a disjoint union of two nonempty closed sets, then it is not connected.

**Example:** If $n \geq 2$, then it is known that the unit sphere $S := \{v \in \mathbb{R}^n \mid \|v\|_2 = 1\}$ is connected.

**One can’t have everything:**

Suppose that $n \geq 2$ and $K \subseteq \mathbb{R}^n$ is a pointed cone such that $\leq_K$ is total.

Then: $v \in K, v \neq 0 \implies -v \notin K$.

Consider $S := \{v \in \mathbb{R}^n \mid \|v\|_2 = 1\}$.

$\leq_K$ total $\implies S = S \cap \mathbb{R}^n = S \cap (K \cup -K) = (S \cap K) \cup (S \cap -K)$.

But $(S \cap K) \cap (S \cap -K) = S \cap (K \cap -K) \subseteq S \cap \{0\} = \emptyset$ (because $0 \notin S$).

Therefore $S = S \cap (K \cup -K) = (S \cap K) \cup (S \cap -K)$ is a disjoint union.
Consequence: $K$ is not closed (otherwise $S$ would be a disjoint union of two closed sets, which is impossible because $S$ is connected).

$\implies$

$\leq_K$ is not continuous at 0.

One can not have translation invariance, scale invariance, antisymmetry, totality as well as continuity!

**Example:** The componentwise order in $\mathbb{R}^n$ seems to satisfy all properties listed before. Hang on a minute, maybe not...? If not, then which property is not satisfied?

**Example:** The lexicographic cone in $\mathbb{R}^n$ is defined by

$$L := \{(x_1, \ldots, x_n)^\top \in \mathbb{R}^n \mid \begin{align*}
x_1 &> 0 \\
(x_1 = 0 & x_2 > 0) &\text{ or } \\
(x_1 = x_2 = 0 & x_3 > 0) &\text{ or } \\
&\vdots \\
(x_1 = x_2 = \ldots x_{n-1} = 0 & x_n > 0) &\text{ or } \\
x & = 0 \end{align*}\}$$

The relation $\leq_L$ is called lexicographic order. The relation $\leq_L$ seems to satisfy all properties listed before. Hang on a minute, maybe not...? If not, then which property is not satisfied?

**Example:** Let $K = \{x \in \mathbb{R}^n \mid x_n \geq 0\}$. The relation $\leq_L$ seems to satisfy all properties listed before. Hang on a minute, maybe not...? If not, then which property is not satisfied?

**Discussion**

- Translation invariance & scale invariance too important: *use a cone $K$.*
- Continuity more important than totality: *use a closed set $K$.*
- Transitivity is natural: *use a set $K$ with $K + K \subseteq K$.*

**Theorem 3.** A cone $K$ is convex if and only if $K + K \subseteq K$ holds.

**Proof.** Let $K$ be a cone.

Suppose that $K$ is convex. Take an arbitrary $x + y \in K + K$, which means that $x, y \in K$ are arbitrary. Since $K$ is a cone $2x, 2y \in K$ and since $K$ is convex $x + y = (1/2)(2x + 2y) \in K$. Therefore, $K + K \subseteq K$.

Conversely, suppose that $K + K \subseteq K$ holds. Let $0 < \lambda < 1$ and $x, y \in K$ be arbitrary. Then, $\lambda > 0$ and $1 - \lambda > 0$. Hence, $\lambda x \in K$ and $(1 - \lambda)y \in K$. Thus, $\lambda x + (1 - \lambda)y \in K + K \subseteq K$. Therefore, $K$ is convex. \[\square\]
Conclusion
From now on, the decision maker should choose (somehow) a closed convex cone $K$ to define his preference relation $\leq_K$.

Assumption: Let $K \neq \emptyset$ be a closed convex cone with $K \neq \{0\}$ and $K \neq \mathbb{R}^n$.

3.3 Classical examples of convex cones

3.3.1 Polyhedral cones
Let $u^1, \ldots, u^m \in \mathbb{R}^n$. Then, the set

$$K = \{ \lambda_1 u^1 + \ldots + \lambda_m u^m \mid \lambda_1 \geq 0, \ldots, \lambda_m \geq 0 \}$$

is denoted by $\text{cone}\{u^1, \ldots, u^m\}$ and it is a closed convex cone, called the \textit{polyhedral cone} generated by the vectors $u^1, \ldots, u^m$. The vectors $u^1, \ldots, u^m$ are called the \textit{generators} of the polyhedral cone. It can be proved that each polyhedral cone is the intersection of a finite number of closed half spaces (highly nontrivial to show). This is from where it follows that the $\text{cone}\{u^1, \ldots, u^m\}$ is closed. However, the convexity of $\text{cone}\{u^1, \ldots, u^m\}$ can be shown easily from its definition as follows. If

$$x = \lambda_1 u^1 + \ldots + \lambda_m u^m \in K,$$

$$y = \mu_1 u^1 + \ldots + \mu_m u^m \in K$$

and $\lambda > 0$, then

$$\lambda x = (\lambda \lambda_1) u^1 + \cdots + (\lambda \lambda_m) u^m \in K,$$

(because $\lambda \lambda_1 \geq 0 \ldots \lambda \lambda_m \geq 0$) and

$$x + y = (\lambda_1 + \mu_1) u^1 + \cdots + (\lambda_m + \mu_m) u^m \in K$$

(because $\lambda_1 + \mu_1 \geq 0, \ldots, \lambda_m + \mu_m \geq 0$). Let $u^1, \ldots, u^m \in \mathbb{R}^n$. Then,

$$U = \{ x \in \mathbb{R}^n \mid (u^1)\top x \geq 0, \ldots, (u^m)\top x \geq 0 \}$$

is a closed convex cone. This follows easily because $U$ is an intersection of closed half spaces which are also closed convex cones. Indeed, it is easy to see from the corresponding definitions that the properties “closed”, “convex” and “cone” of a set are preserved by intersection. In fact $U$ is a polyhedral cone (highly nontrivial to show). If you change some of the inequalities in the definition of $U$ into strict ones you will also get a convex cone.

3.3.2 Simplicial cones
Let $u^1, \ldots, u^n \in \mathbb{R}^n$ be linearly independent. Then,

$$K = \text{cone}\{u^1, \ldots, u^n\}$$

is called a \textit{simplicial cone} which is a particular polyhedral cone. The name follows from the observation that cutting $K$ with a hyperplane which does not contain the origin one obtains a simplex: triangle in 3 dimensions, tetrahedron in 4 dimensions, .... The convex hull of $w^1, w^2, \ldots, w^n, w^{n+1} \in \mathbb{R}^n$ such that $w^2 - w^1, \ldots, w^n - w^1, w^{n+1} - w^1$ are linearly independent is called a simplex. It can be shown that simplicial cones are of the form

$$K = \{ x \in \mathbb{R}^n \mid (v^1)\top x \geq 0, \ldots, (v^n)\top x \geq 0 \},$$

where $v^1, \ldots, v^n \in \mathbb{R}^n$ are linearly independent.
Classical simplicial cones

- **Nonnegative orthant:**
  \[ \mathbb{R}_n^+ = \{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 \geq 0, \ldots, x_n \geq 0 \} \]

- **Monotone nonnegative cone:**
  \[ \mathbb{R}_n^{\geq+} = \{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 \geq x_2 \geq \cdots \geq x_n \geq 0 \} \]

### 3.3.3 The Lorentz cone

The Lorentz cone is also called second order cone or icecream cone. It is defined by

\[ L = \{ (x_{m+1}) \in \mathbb{R}^{m+1} \mid x_{m+1} \geq \|x\| \} \]

where \( \| \cdot \| \) is the Euclidean norm in \( \mathbb{R}^m \). From the definition it can be easily seen that \( L \) is closed. The set \( L \) is a convex cone. Indeed, if \( (x_{m+1}) \in L \) and \( \lambda > 0 \), then \( \lambda (x_{m+1}) = (\lambda x_{m+1}) \in L \) because \( x_{m+1} \geq \|x\| \) implies \( \lambda x_{m+1} \geq \lambda \|x\| = \|\lambda x\| \). Hence \( L \) is a cone. Moreover, if \( (x_{m+1}) \in L \) and \( (y_{m+1}) \in L \), then \( (x_{m+1} + y_{m+1}) = (x_{m+1}) + (y_{m+1}) \in L \) because \( x_{m+1} + y_{m+1} \geq \|x\| + \|y\| \geq \|x+y\| \), where we used the triangle inequality. Hence, \( L \) is a convex cone. The Lorentz cone is an example of a convex cone which is not polyhedral.

![Figure: Lorentz cone in 3 dimensions. The angle between \( Oa \) and \( Ox_3 \) is 45 degrees.](image)

### 3.3.4 Cone generated by a set

If \( A \subset \mathbb{R}^n \) is a nonempty set, then denote by \( \text{closure}(A) \) the smallest closed set (with respect to the inclusion of sets) containing \( A \). Then,

\[ \text{cone}(A) := \text{closure}\{ \lambda_1 u^1 + \cdots + \lambda_m u^m \mid m \in \mathbb{N} \text{ and } u^1, \ldots, u^m \in A \text{ and } \lambda_1, \ldots, \lambda_m > 0 \} \]

is a convex cone, called the **cone generated by** \( A \). It can be shown that it is the smallest closed convex cone (with respect to the inclusion of sets) containing \( A \).
3.4 The set of efficient points

Remark For $v, w \in \mathbb{R}^n$ we might have $v \not\leq_K w$ and $w \not\leq_K v$: no totality!

If $n = 1$ and $K = \mathbb{R}_+$, we know what ”minimal” means.

What about the more general case here??

Definition. Let $M \subseteq \mathbb{R}^n$ be a set. The set of minimal elements of $M$ with respect to $\leq_K$ (resp. $K$) is

$$E(M, K) := \{v \in M \mid \exists w \in M : w \leq_K v \& w \neq v\}.$$

Remark Minimal elements are also called ”efficient”, ”preferred”, ”dominating”, ”Pareto-optimal”, ”Pareto-minimal”, ”Edgeworth-Pareto-optimal”, etc.

We have

$$E(M, K) = \{v \in M \mid \exists w \in M : w \leq_K v \& w \neq v\} = \{v \in M \mid \forall w \in M, w \neq v : w \not\leq_K v\} = \{v \in M \mid \forall w \in M, w \neq v : w - v \not\in K\} = \{v \in M \mid (v - M) \cap (K \setminus \{0\}) = \emptyset\} = \{v \in M \mid M \cap (v - K \setminus \{0\}) = \emptyset\}.$$

To show this denote the sets on the right hand sides of the equalities above from top to bottom by $E_1, E_2, E_3, E_4, E_5, E_6$. $E_1$ is the original definition of $E(M, K)$. Hence, we only need to show that $E_1 = E_2 = E_3 = E_4 = E_5 = E_6$.

$E_1$ and $E_2$ logically mean the same thing. Indeed, saying that $\exists w \in M : w \leq_K v \& w \neq v$ (i.e., $v \in E_1$) it is the same as saying that $\forall w \in M : w \not\leq_K v$ or $w = v$, which means that $\forall w \in M, w \neq v : w \not\leq_K v$ (i.e., $v \in E_2$). Hence, $E_1 = E_2$.

The relation $w \not\leq_K v$ is equivalent to $v - w \notin K$, hence $E_2 = E_3$ (because the remaining parts of $E_2$ and $E_3$ are identical).

$E_3$ and $E_4$ logically mean the same thing. Indeed saying that $\forall w \in M, w \neq v : v - w \notin K$ (i.e., $v \in E_3$) it is the same as saying that $\forall w \in M : v - w = 0$ or $v - w \notin K$, which means that $\forall w \in M : v - w \notin K \setminus \{0\}$ (i.e., $v \in E_4$). We used that $v - w \in K \setminus \{0\}$ is equivalent to $v - w \in K$ and $v - w \neq 0$, which negated gives that $v - w \notin K \setminus \{0\}$ is equivalent to $v - w = 0$ or $v - w \notin K$. Hence, $E_3 = E_4$.

The relation $v \notin E_6$ is equivalent to $\exists w \in M \cap (v - K \setminus \{0\})$, which means that $\exists w \in M : w \in v - K \setminus \{0\}$. Hence, $v \notin E_6$ if and only if $\exists w \in M : v - w \in K \setminus \{0\}$, which is equivalent to $\exists w \in M : v - w \in K \& w \neq v$, or to $v \notin E_1$. Thus, $E_1 = E_6$. It can similarly be shown that $E_1 = E_5$.

In conclusion we have $E_1 = E_2 = E_3 = E_4 = E_5 = E_6$.

Example: Let $M = [-1, 1] \times [-1, 1] \subseteq \mathbb{R}^2$ and $K = \mathbb{R}_+^2$. Then, $E(M, K) = \{(-1, -1)^T\}$.

Example: Let $M = \{v \in \mathbb{R}^2 \mid v_1^2 + v_2^2 \leq 1\}$ and $K = \mathbb{R}_+^2$. Then,

$$E(M, K) = \{v \in \mathbb{R}^2 \mid v_1 \leq 0, v_2 \leq 0, v_1^2 + v_2^2 = 1\}$$

Example: Let $M = M_1 \cup M_2 \cup M_3$, where
\(M_1 = \{(x_1, x_2)^\top \in \mathbb{R}^2 \mid x_1 \geq x_2 \text{ and } x_2 \geq 0 \text{ and } x_2 \geq 2x_1 - 2\}\),

\(M_2 = \{(0, \lambda)^\top \in \mathbb{R}^2 \mid -2 \leq \lambda \leq 0\}\),

\(M_3 = \{(\lambda, -2)^\top \in \mathbb{R}^2 \mid 0 \leq \lambda \leq 1\}\)

and

\(K_1 = \{(x_1, x_2)^\top \in \mathbb{R}^2 \mid x_1 > x_2 \text{ and } x_2 \geq 0\}\),

\(K_2 = \{(x_1, x_2)^\top \in \mathbb{R}^2 \mid x_1 \geq x_2 \text{ and } x_2 > 0\}\),

\(K_3 = \{(\lambda, 0)^\top \in \mathbb{R}^2 \mid \lambda \geq 0\} \cup \{(0, \lambda)^\top \in \mathbb{R}^2 \mid \lambda \geq 0\}\),

\(K_4 = \{(x_1, x_2)^\top \in \mathbb{R}^2 \mid x_1 > 0 \text{ or } (x_1 = 0 \& x_2 > 0) \text{ or } x_1 = x_2 = 0\}\)

(Discriminative cone)

Determine the set of efficient points \(E(M, K_i)\) for all \(i \in \{1, 2, 3, 4\}\).

Help me sketch the set set of efficient points on the board!

Note that \(M\) is usually the set of possible values of decisions: \(v, w \in M, v = f(x), w = f(y)\). Hence, solving a decision making problem means that we have a set \(X\) of decisions and we want to choose the optimal decisions. In order to do so, we need to be given (or for a practical problem often identify by ourselves) a set of objective functions \(f_1, \cdots, f_n : X \to \mathbb{R}\). Let \(f := (f_1, \cdots, f_n) : X \to \mathbb{R}^n\). Usually \(X\) is a subset of another finite dimensional space \(\mathbb{R}^m\), or it can be identified with such a subset. Solving our problem means that a cone \(K \subseteq \mathbb{R}^n\) is given (if no cone is given we assume that the cone is \(\mathbb{R}^n_+\); for practical problems we need to decide which cone to use), we need to find the image set

\[M = f(X) := \{f(x) \mid x \in X\},\]

the set of efficient points \(E(M, K)\) of \(M\) with respect to the order \(\leq_K\) induced by \(K\) (or simply with respect to \(K\)) and the decisions

\[f^{-1}(E(M, K)) = \{x \in X \mid f(x) \in E(M, K)\}\]

corresponding to \(E(M, K)\), that is, the preimage set. A part of this process is illustrated in the next figure.
Remark Let $v, w \in E(M, K)$, $v \neq w$. Then $v \not \leq_K w$ and $w \not \leq_K v$: efficient points are not comparable with each other. (This feature is called ”inner stability”.)

The decision maker can not compare efficient elements with each other without resorting to measures outside the mathematical model!

Remark

$$K_1 \subseteq K_2 \implies E(M, K_2) \subseteq E(M, K_1)$$

(”larger cone $\implies$ weaker ordering $\implies$ less minimal elements”)

### 3.5 Topological properties of the set of efficient points

#### Theorem 4.
The set $E(M, K)$ is a subset of the boundary of $M$.

**Proof.** Denote the boundary of $M$ by $\partial M$ and the interior of $M$ by $\text{int}(M)$. We need to show that $E(M, K) \subseteq \partial M$. Suppose to the contrary that there is a $v \in E(M, K)$ such that $v \notin \partial M$. Then, $v \in M \setminus \partial M \subseteq \text{int} M$. Since, $K \neq \emptyset$ and $K \neq \{0\}$ there exists $u \in K$ with $u \neq 0$. Let $w = v - \lambda u$, where $\lambda$ is a positive real number. Since $v \in \text{int}(M)$, $u \neq 0$ and $K$ is a cone, it follows that $v \neq w = v - \lambda u \in M \cap (v - K \setminus \{0\})$ for $\lambda$ sufficiently small. Thus, $w \in M$ and it is dominating $v$, which contradicts $v \in E(M, K)$. Hence, $E(M, K) \subseteq \partial M$. \[\square\]

**Example:**

**Example:**

Reminder: A set $S \subseteq \mathbb{R}^n$ is called _connected_ if there do not exist open sets $U_1, U_2 \subset \mathbb{R}^n$ such that $S \subseteq U_1 \cup U_2$ and $S \cap U_1 \neq \emptyset$, $S \cap U_2 \neq \emptyset$, $S \cap U_1 \cap U_2 = \emptyset$.

**Examples:**

Is $E(M, K)$ connected?

Not always, but:

**Theorem 5.** Let $M \subseteq \mathbb{R}^n$ be convex, and closed. If $K$ is a pointed closed convex cone, then $E(M, K)$ is connected.

(No Proof.)

**Example:**

Reminder (not examinable): A _total order_ on a set $X$ is any binary relation $\leq$ on $X$ that is antisymmetric, transitive, and total, i.e., for all $a, b, c \in X$

- if $a \leq b$ and $b \leq c$ then $a \leq c$ (transitivity);
- if $a \leq b$ and $b \leq a$ then $a = b$ (antisymmetry);
- $a \leq b$ or $b \leq a$ (totality).
A set paired with an associated total order on it is called a \textit{totally ordered set}.

Notice that the totality condition implies \textit{reflexivity}, that is, $a \preceq a$. Thus a total order is also a \textit{partial order}, that is, a binary relation which is reflexive, antisymmetric and transitive.

\textbf{Lemma 1 (Zorn’s lemma).} Every non-empty partially ordered set, in which every totally ordered subset has a lower bound, contains at least one minimal element.

(No Proof.)

Let $X$ be a set with $A = \{A_i\}_{i \in I}$ a family of subsets of $X$. Then the collection $A$ has the \textit{finite intersection property}, if any finite subcollection $J \subset I$ has non-empty intersection $\cap_{i \in J} A_i$. A set $X$ is compact if and only if every collection of closed subsets of $X$ satisfying the finite intersection property has nonempty intersection itself (no proof).

\textbf{End of Reminder}

In which situations do minimal elements exist?

\textbf{Theorem 6.} If $K$ is pointed and there exists a $v \in M$ such that $(v - K) \cap M$ is compact and nonempty, then

$$E(M, K) \neq \emptyset.$$

Proof (not examinable):

Below only a sketch of the proof is presented. You are challenged to fill in the missing details of the proof. This is a real research mini-project for you. Please let me know if you would like to get the missing details from me.

Suppose that $M_v = (v - K) \cap M$ is compact. Let $\{u^i\}_{i \in I}$ be any totally ordered subset of $M_v$. The family of closed subsets $M_{u^i}$ ($i \in I$) has the finite intersection property; that is, every finite subfamily has a nonempty intersection. Since $M_v$ is compact, the family of subsets $M_{u^i}$ ($i \in I$) has a nonempty intersection; that is, there is an element

$$u \in \cap_{i \in I} M_{u^i} = \cap_{i \in I} (u^i - K) \cap M_v.$$

Hence, $u$ is a lower bound of the subset $\{u^i\}_{i \in I}$ and belongs to $M_v$. Then by Zorn’s lemma $M_v$ has at least one minimal element which is also a minimal element of $M$. \hfill \Box

\textbf{Example:} $\rightarrow$

Recall the following very important phrase from before:

“Note again that $M$ is usually the set of possible values of decisions: $v, w \in M$, $v = f(x)$, $w = f(y)$. Hence, solving a decision making problem means that we have a set $X$ of decisions and we want to choose the optimal decisions. In order to do so, we need to be given (or for a practical problem often identify by ourselves) a set of objective functions $f_1, \cdots, f_n : X \to \mathbb{R}$. Let $f := (f_1, \cdots, f_n) : X \to \mathbb{R}^n$. Usually $X$ is a subset of another finite dimensional space $\mathbb{R}^m$, or it can be identified with such a subset. Solving our problem means that a cone $K \subseteq \mathbb{R}^n$ is given (if no cone is given we assume that the cone is $\mathbb{R}^n_+$; for practical problems we need to decide which cone to use), we need to find the \textit{image set}

$$M = f(X) := \{f(x) \mid x \in X\},$$
the set of efficient points $E(M, K)$ of $M$ with respect to the order $\leq_K$ induced by $K$ (or simply with respect to $K$) and the decisions

$$f^{-1}(E(M, K)) = \{ x \in X \mid f(x) \in E(M, K) \}$$

corresponding to $E(M, K)$, that is, the preimage set. A part of this process is illustrated in the next figure.

4 Examples for Multicriteria Decision Making Problems

1. Solve the following decision making problem:

$$\begin{align*}
\text{Minimise} & \quad x_2 \\
\text{Minimise} & \quad x_1 + x_2 \\
\text{Subject to} & \quad x_1 + x_2 \geq 0 \\
& \quad x_2^2 - 2x_1x_2 - x_1^2 \leq 0
\end{align*}$$

with respect to the cone $K = \mathbb{R}^2_+ := \{ (x_1, x_2)^T \in \mathbb{R}^2 : x_1 \geq 0, x_2 \geq 0 \}$.

**Solution:** The decision variables are $x_1, x_2$. The decision set is

$$\{ (x_1, x_2)^T \in \mathbb{R}^2 : x_1 + x_2 \geq 0, x_2^2 - 2x_1x_2 - x_1^2 \leq 0 \}.$$ 

The objective functions are $y_1 = x_2, y_2 = x_1 + x_2$. The image set is

$$M = \{ (y_1, y_2)^T \in \mathbb{R}^2 : y_2 \geq y_1^2 \}.$$ 

The efficient set is

$$E(M, K) = \{ (y_1, y_2)^T \in \mathbb{R}^2 : y_2 = y_1^2, y_1 \leq 0 \}.$$ 

Indeed, $E(M, K) \subseteq \partial M$, where $\partial M$ is the boundary of $M$. Let $(y_1, y_2)^T \in \partial M$. Then, $y_2 = y_1^2$. Suppose $y_1 > 0$. Then, $y_2 > 0$, and hence $(0, y_2)^T \leq_K (y_1, y_2)^T$ (componentwise ordering) and $(y_1, y_2)^T \neq (0, y_2)^T \in M$. Hence $(y_1, y_2)^T \notin E(M, K)$. Suppose $y_1 \leq 0$ and let $(z_1, z_2)^T \in M$ such that $(z_1, z_2)^T \leq_K (y_1, y_2)^T$. Hence, $y_2^2 = y_2 \geq z_2 \geq z_1^2$ and $z_1 \leq y_1 \leq 0$ which implies $-y_1 = |y_1| \geq |z_1| = -z_1$, or equivalently $z_1 \geq y_1$. Thus, $y_1 = z_1$, which by $y_1^2 = y_2 \geq z_2 \geq z_1^2$
implies \( y_2 = z_2 \). Thus, \((z_1, z_2)\top = (y_1, y_2)\top\). So, no point in \( M \) is dominating \((y_1, y_2)\top\). Thus, 
\[(y_1, y_2)\top \in E(M, K)\]. In conclusion,
\[E(M, K) = \{(y_1, y_2)\top \in \mathbb{R}^2 : y_2 = y_1^2, y_1 \leq 0\}\].
The preimage set of \( E(M, K) \) is
\[\{(x_1, x_2)\top \in \mathbb{R}^2 : x_1 + x_2 = x_2^2, x_2 \leq 0\}\].

2. Solve the following decision making problem:
\[
\begin{align*}
\text{Minimise} & \quad x_1 \cos x_2 \\
\text{Minimise} & \quad x_1 \sin x_2 \\
\text{Subject to} & \quad x_2^2 \leq 1
\end{align*}
\]
with respect to the cone \( K = \mathbb{R}_+^2 \).

\textbf{Solution:} Let \( y_1 = x_1 \cos x_2 \) and \( y_2 = x_1 \sin x_2 \). The image set is
\[M = \{(y_1, y_2)\top \in \mathbb{R}^2 : y_1^2 + y_2^2 \leq 1\}\].

For any point \( v \) on the arc \( a, x \) and \( b \) we have \((v - K) \cap M \subseteq \{v\}\) (such points are for example \( a, b \) and \( x \)). For any other point \( u \) on the boundary of \( M \) but not on this arc we have \((u - K) \cap M \nsubseteq \{u\}\). Hence, the efficient set is the arc joining \( a, x \) and \( b \). This can be expressed as:
\[E(M, K) = \{(y_1, y_2)\top \in \mathbb{R}^2 : y_1^2 + y_2^2 = 1, y_1 \leq 0, y_2 \leq 0\}\].
The preimage set of \( E(M, K) \) is
\[\left( \bigcup_{k \in \mathbb{Z}} A_k \right) \cup \left( \bigcup_{k \in \mathbb{Z}} B_k \right),\]
where
\[A_k = \left\{(x_1, x_2)\top \in \mathbb{R}^2 : x_1 = 1, x_2 \in \left[ \pi + 2k\pi, \frac{3\pi}{2} + 2k\pi \right] \right\}\]
and
\[B_k = \left\{(x_1, x_2)\top \in \mathbb{R}^2 : x_1 = -1, x_2 \in \left[ 2k\pi, \frac{\pi}{2} + 2k\pi \right] \right\} .\]
3. Solve the following decision making problem:

\[
\begin{align*}
\text{Minimise} & \quad x_1 + x_2 \\
\text{Minimise} & \quad x_1 - x_2 \\
\text{Subject to} & \quad x_1 - 2 \geq 0 \\
& \quad 3x_1 - 5x_2 + 4 \geq 0 \\
& \quad 2x_2 - 7 \leq 0 \\
& \quad 3x_1 + 5x_2 + 4 \geq 0 \\
& \quad 2x_2 + 7 \geq 0
\end{align*}
\]

with respect to the cone \( K = \mathbb{R}^2_+ \).

**Solution:** Let \( y_1 = x_1 + x_2 \) and \( y_2 = x_1 - x_2 \). Then, \( x_1 = (y_1 + y_2)/2 \) and \( x_2 = (y_1 - y_2)/2 \). Inserting these expressions for \( x_1 \) and \( x_2 \) into the constraints we obtain the image set of the decision set:

\[
M = \{ (y_1, y_2) \in \mathbb{R}^2 : y_1 + y_2 \geq 4, y_1 - 4 \leq 4y_2, y_1 - 7 \leq y_2, 4y_1 + 4 \geq y_2, y_1 + 7 \geq y_2, y_1 \geq 0, y_2 \geq 0 \}.
\]

By drawing the image set and inspecting it we get

\[
E(M, K) = \{ (y_1, y_2) \in \mathbb{R}^2 : y_1 + y_2 = 4, y_1 \geq 0, y_2 \geq 0 \}.
\]

Indeed, the image set is drawn below.
We can observe that for any point $v$ on the straight line segment $[(0, 4)^\top, (4, 0)^\top]$ we have $(v - K) \cap M \subseteq \{v\}$ (for example such points are $a = (0, 4)^\top$, $b = (4, 0)^\top$ and $c$) and for any point $u$ on the boundary of $M$ which is not on this straight line segment we have that $(u - K) \cap M \not\subseteq \{u\}$ (for example such points are $d$, $e$, $f$ and $g$, one for each edge of $M$ different from $[a, b]$). Hence, $E(M, K) = [(0, 4)^\top, (4, 0)^\top]$ which can be written as $E(M, K) = \{(y_1, y_2)^\top \in \mathbb{R}^2 : y_1 + y_2 = 4, y_1 \geq 0, y_2 \geq 0\}$.

The set of the decision variables corresponding to the efficient set (the preimage set of $E(M,K)$) is $\{(x_1, x_2)^\top \in \mathbb{R}^2 : x_1 + x_2 \geq 0, x_1 - x_2 \geq 0\}$, or equivalently $\{(2, x_2)^\top \in \mathbb{R}^2 : -2 \leq x_2 \leq 2\}$.

4. Solve the following decision making problem:

$$
\begin{align*}
\text{Minimise} & \quad 2x_1 + x_2 \\
\text{Minimise} & \quad x_1 - 2x_2 \\
\text{Subject to} & \quad (2a + b)x_1 + (a - 2b)x_2 + c \geq 0 \\
& \quad 2x_1 + x_2 \geq 0 \\
& \quad x_1 - 2x_2 \geq 0
\end{align*}
$$

with respect to the cone $K = \mathbb{R}_2$, where $a, b, c$ are real constants with $a > 0$, $b > 0$ and $c < 0$.

**Solution:** Let $y_1 = 2x_1 + x_2$ and $y_2 = x_1 - 2x_2$. The image set is $M = \{(y_1, y_2)^\top \in \mathbb{R}^2 : ay_1 + by_2 + c \geq 0, y_1 \geq 0, y_2 \geq 0\}$.

The efficient set is $E(M, K) = \{(y_1, y_2)^\top \in \mathbb{R}^2 : ay_1 + by_2 + c = 0, y_1 \geq 0, y_2 \geq 0\}$.

Indeed, the image set $M$ is drawn below
If \( z = (0, -c/b) \) and \( w = (-c/a, 0) \), then \( E(M, K) \) is the straight line segment \([z, w]\). Indeed, for any point \( v \) on this segment (for example such points are \( z \) and \( p \)) we have \((v - K) \cap M \subseteq \{v\}\) and for any other point \( u \) on the boundary (for example such points are \( q \) and \( r \)) we have \((u - K) \cap M \not\subseteq \{u\}\).

The efficient set \( E(M, K) \) can be also written as
\[
E(M, K) = \{(y_1, y_2)^\top \in \mathbb{R}^2 : ay_1 + by_2 + c = 0, y_1 \geq 0, y_2 \geq 0\}.
\]
The preimage set of \( E(M, K) \) is
\[
\{(x_1, x_2)^\top \in \mathbb{R}^2 : (2a + b)x_1 + (a - 2b)x_2 + c = 0, 2x_1 + x_2 \geq 0, x_1 - 2x_2 \geq 0\}.
\]

5. Solve the following decision making problem:

\[
\begin{align*}
\text{Minimise} & \quad x_3^1 \\
\text{Minimise} & \quad x_5^2 \\
\text{Subject to} & \quad x_1 \geq 0 \\
& \quad x_1 \leq 3 \\
& \quad x_2 \geq 0 \\
& \quad x_2 \leq 2
\end{align*}
\]

with respect to the cone \( K = \{(x_1, x_2)^\top \in \mathbb{R}^2 : x_1 \geq x_2, x_2 > 0\} \).

**Solution:** Let \( y_1 = x_3^1 \) and \( y_2 = x_5^2 \). The image set is
\[
M = \{(y_1, y_2)^\top \in \mathbb{R}^2 : y_1 \geq 0, y_1 \leq 27, y_2 \geq 0, y_2 \leq 32\}.
\]
The set \( M \) is a box. For any point \( v \) on the left vertical edge of \( M \) we have \((v - K) \cap M \subseteq \{v\}\) and for any point \( u \) on the bottom horizontal edge of \( M \) we have \((u - K) \cap M \subseteq \{u\}\) (because the horizontal line \\{(x_1, x_2)^\top : x_2 = 0, x_1 > 0\}\) is not part of \( K \). For all other points \( w \) of the boundary of \( M \) we have \((w - K) \cap M \not\subseteq \{w\}\). In fact these ideas are true regardless of the size of the box \( M \), as illustrated below.
Hence, the efficient set is
\[ E(M, K) = \{(y_1, y_2) \in \mathbb{R}^2 : y_1 = 0, y_2 \geq 0, y_2 \leq 32\} \cup \{(y_1, y_2) \in \mathbb{R}^2 : y_2 = 0, y_1 \geq 0, y_1 \leq 27\}. \]

The preimage set of \( E(M, K) \) is
\[ \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = 0, x_2 \geq 0, x_2 \leq 2\} \cup \{(x_1, x_2) \in \mathbb{R}^2 : x_2 = 0, x_1 \geq 0, x_1 \leq 3\}. \]

5 Computing efficient points

5.1 The dual cone

Definition. Define the dual cone \( K^* \) by
\[ K^* := \{z \in \mathbb{R}^n \mid \forall v \in K : z^T v \geq 0\}. \]

We have \( K^{**} = K \) (no proof).

Lemma 2. If \( x \in \text{int}(K) \) and \( y \in K^* \setminus \{0\} \), then \( y^T x > 0 \). If \( x \in K \setminus \{0\} \) and \( y \in \text{int}(K^*) \), then \( y^T x > 0 \).

Proof. Let \( x \in \text{int}(K) \) and \( y \in K^* \setminus \{0\} \). Since \( x \in \text{int}(K) \), \( x - \lambda y \in K \) for \( \lambda > 0 \) sufficiently small. Since \( y \in K^* \) and \( x - \lambda y \in K \), it follows that \( y^T(x - \lambda y) \geq 0 \). Thus, \( y^T x \geq \lambda \|y\|^2 > 0 \), because \( y \neq 0 \). The second statement follows from the first one by using that \( K = (K^*)^* \). \( \square \)

A converse of Lemma 2. If \( \text{int}(K) \neq \emptyset \) and \( y^T x > 0 \) for all \( x \in \text{int}(K) \), then \( y \in K^* \setminus \{0\} \).

Example: \( \rightarrow \)
Example: \( \rightarrow \)
5.2 Examples of classical dual cones

5.2.1 Geometrical interpretation of dual cones

Note that the scalar product of $z$ and $v$ is $z^Tv = z_1v_1 + \cdots + z_nv_n$, where $z = (z_1, \ldots, z_n)^T \in \mathbb{R}^n$ and $v = (v_1, \ldots, v_n) \in \mathbb{R}^n$.

Define the dual of the set $C \subseteq \mathbb{R}^n$ by

$$
C^* = \{ z \in \mathbb{R}^n | \forall v \in C : z^Tv \geq 0 \}.
$$

The dual of any set $C$ is a closed convex cone. Indeed the figure below shows that $C^* = \bigcap_{v \in C} \{ z \in \mathbb{R}^n | z^Tv \geq 0 \}$ which is an intersection of closed half spaces which are closed convex cones.

Convexity, closedness and “being a cone” are all properties preserved by intersection of any number of sets. Hence $C^*$ is a closed convex cone. It is easy to see from definition as well that $C^*$ is a convex cone. Indeed, $z \in C^*$ and $\lambda > 0$ implies $(\lambda z)^Tv = \lambda(z^Tv) \geq 0$, for all $v \in C$. Hence, $\lambda z \in C$, which shows that $C^*$ is a cone. If $x, y \in C^*$, then $(x+y)^Tv = x^Tv + y^Tv \geq 0$, for all $v \in C$. Hence, $x + y \in C^*$.

Note that if $K$ is a closed convex cone and $L = K^*$, then $L^* = K^{**} = K$. Therefore, $K$ and $L$ are called mutually dual cones. The cones $K \subseteq \mathbb{R}^n$ with the property $K \subseteq K^*$ are called subdual and the cones with the property $K^* \subseteq K$ are called superdual. Cones $K \subseteq \mathbb{R}^n$ which are both supdual and superdual, that is $K = K^*$, are called self-dual. Note that if $n > 2$, then there are many cones $K \subseteq \mathbb{R}^n$ which are neither subdual nor superdual. However, if $K$ is a subdual closed convex cone, then $L := K^*$ is superdual. Indeed, $L^* = K$ and thus $L^* = K \subseteq K^* = L$.

5.2.2 Dual of closed convex cones in $\mathbb{R}^2$

If $K \subseteq \mathbb{R}^2$ is a closed convex cone, then either $K \subseteq K^*$ or $K^* \subseteq K$. So any closed convex cones in $\mathbb{R}^2$ are either subdual or superdual (see the figure below).
In this figure $K = \text{cone}\{a, b\}$ and $L = \text{cone}\{c, d\}$; where $a \perp c$ and $b \perp d$. It can be seen that $K^* = L$, $L = K^*$, $K$ is subdual and $L$ is superdual.

5.2.3 Dual of the nonnegative orthant

If $K = \mathbb{R}_+^n$, then $K^* = K$, that is, $K$ is self-dual. Indeed, if $z \in K = \mathbb{R}_+^n$, then $z^\top v \geq 0$ for any $v \in K = \mathbb{R}_+^n$ because $z^\top v = z_1v_1 + \cdots + z_nv_n \geq 0$, where $z = (z_1, \ldots, z_n)^\top$ and $v = (v_1, \ldots, v_n)^\top$. Hence, $z \in K^*$ which implies $K \subseteq K^*$, that is, $K$ is subdual. Conversely, let $z = (z_1, \ldots, z_n)^\top \in K^*$. Denote $e^i$ the vector which has all components 0 except the $i$-th component which is 1. Then, $e^i \in K = \mathbb{R}_+^n$ for any $i \in \{1, \ldots, n\}$. Hence, $z^\top e^i = z_i \geq 0$ for any $i \in \{1, \ldots, n\}$. Thus, $z \in \mathbb{R}_+^n = K$. Therefore, $K^* \subseteq K$, that is, $K$ is superdual. It follows that $K$ is self-dual, that is, $K^* = K$.

5.2.4 Dual of the monotone nonnegative cone

It can be seen that if $C$ and $D$ are two cones, such that $C \subseteq D$, then $D^* \subseteq C^*$. Indeed, let $x \in D^*$. Then, for any $p \in D$ we have $x^\top p \geq 0$. Thus, for any $q \in C \subseteq D$ we have $x^\top q \geq 0$. Hence, $x \in C^*$ which implies $D^* \subseteq C^*$. In particular if $D$ is self-dual, then $C$ is subdual, because $C \subseteq D = D^* \subseteq C^*$. Let

$$K = \mathbb{R}_+^n = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 \geq x_2 \geq \cdots \geq x_n \geq 0\}$$

be the monotone nonnegative cone. Hence, since $K = \mathbb{R}_+^n \subseteq \mathbb{R}_+^n$ and $\mathbb{R}_+^n$ is self-dual, it follows that $K = \mathbb{R}_+^n$ is subdual. So, $K^*$ is superdual. Let $y = (y_1, \ldots, y_n) \in K^*$. Note that $(1, 0, \ldots, 0)^\top \in K$, $(1, 1, 0, \ldots, 0)^\top \in K$, $(1, 1, 1, \ldots, 1, 1)^\top \in K$ and $(1, 1, 1, 1, 1)^\top \in K$. Hence, $y^\top (1, 0, \ldots, 0) = y_1 \geq 0$, $y^\top (1, 1, 0, \ldots, 0)^\top = y_1 + y_2 \geq 0$, $\ldots$, $y^\top (1, 1, \ldots, 1, 1)^\top = y_1 + y_2 + \cdots + y_{n-1} + y_n \geq 0$. Thus $K^* \subseteq L$, where

$$L = \{y = (y_1, y_2, \ldots, y_{n-1}, y_n)^\top \in \mathbb{R}^n \mid y_1 \geq 0, y_1 + y_2 \geq 0, \ldots, y_1 + y_2 + \cdots + y_{n-1} + y_n \geq 0\}.$$ 

Now let $y = (y_1, y_2, \ldots, y_{n-1}, y_n)^\top \in L$. Then, the identity

$$y_1(x_1 - x_2) + (y_1 + y_2)(x_2 - x_3) + \cdots + (y_1 + y_2 + \cdots + y_{n-1})(x_{n-1} - x_n) + (y_1 + y_2 + \cdots + y_{n-1} + y_n)x_n = x_1y_1 + x_2y_2 + \cdots + x_{n-1}y_{n-1} + x_ny_n = y^\top x$$

shows that for any $x \in K$ and any $y \in L$ we have $y^\top x \geq 0$. Hence, $L \subseteq K^*$. It follows that $K^* = L$. 26
5.2.5 Dual of the Lorentz cone

Let

$$L = \{ (x_{m+1}) \in \mathbb{R}^{m+1} \mid x_{m+1} \geq \|x\| \},$$

be the Lorentz cone. Then, $L^* = L$. Indeed, let $(x_{m+1}) \in L^*$. Suppose $x = 0$. We have $(0) \in L$. Hence, $0 \leq (0^\top, x_{m+1}^\top)(0) = x_{m+1}$. Thus, $x_{m+1} \geq 0 = \|x\|$ and therefore $(x_{m+1}) \in L$. Suppose $x \neq 0$. Then, $(x_{m+1}^\top) \in L$. Hence, $0 \leq (x^\top, x_{m+1}) (x_{m+1}) = -\|x\|^2 + x_{m+1}\|x\|$. It follows that $x_{m+1}\|x\| \geq \|x\|^2$, or equivalently, $x_{m+1} \geq \|x\|$. Thus, $(x_{m+1}) \in L$ and therefore $L^* \subseteq L$.

Conversely, let $(x_{m+1}) \in L$. Then, for any $(y_{m+1}) \in L$ we have $(x^\top, x_{m+1})(y_{m+1}) = x^\top y + x_{m+1}y_{m+1} \geq 0$ (by using the Cauchy inequality $\|x\|\|y\| \geq \|x^\top y\|$). Hence, $(x_{m+1}) \in L^*$ and therefore $L \subseteq L^*$. In conclusion, $L^* = L$, that is, the Lorentz cone is self-dual.

5.2.6 Dual of polyhedral cones

**Theorem A.** Let $u^1, \cdots, u^m \in \mathbb{R}^n$, $K = \text{cone}\{u^1, \ldots, u^m\}$ and

$$L = \{x \in \mathbb{R}^n \mid (u^1)^\top x \geq 0, \ldots, (u^m)^\top x \geq 0\}.$$  

Then, $K$ and $L$ are mutually dual cones, that is, $K^* = L$ and $L^* = K$.

**Proof.** Since, $K$ is a closed convex cone, it is enough to show that $K^* = L$. Let $x \in K^*$. Then, since $u^1, \ldots, u^m \in K$, we obtain $(u^1)^\top x \geq 0, \ldots, (u^m)^\top x \geq 0$. Hence, $K^* \subseteq L$. Suppose that $x \in L$. Then, for all $y = \lambda_1 u^1 + \cdots + \lambda_m u^m \in K$ we have

$$y^\top x = (\lambda_1 u^1 + \cdots + \lambda_m u^m)^\top x = \lambda_1 (u^1)^\top x + \cdots + \lambda_m (u^m)^\top x \geq 0.$$  

It follows that $x \in K^*$. Hence, $L \subseteq K^*$. In conclusion, $K^* = L$. □

**Numerical examples**

1. Determine the dual cone of the following cones

   (a) $K_1 = \{(x_1, x_2)^\top \in \mathbb{R}^2 \mid 2x_1 + 5x_2 \geq 0, 3x_1 - 4x_2 \leq 0\}$,

   (b) $K_2 = \text{cone}\{(1, 4)^\top, (3, 2)^\top\} \subseteq \mathbb{R}^2$,

   (c) $K_3 = \{(x_1, \cdots, x_7)^\top \in \mathbb{R}^7 \mid x_1 + x_7 \leq 0, x_2 + x_5 \leq 0\}$,

   (d) $K_4 = \text{cone}\{(1, 0, 0, -1)^\top, (0, -1, 1, 0)^\top\} \subseteq \mathbb{R}^4$.

**Solution**

   (a) $K_1^* = \text{cone}\{(2, 5)^\top, (-3, 4)^\top\}$

   (b) $K_2^* = \{(x_1, x_2)^\top \in \mathbb{R}^2 \mid x_1 + 4x_2 \geq 0, 3x_1 + 2x_2 \geq 0\}$

   (c) $K_3^* = \text{cone}\{(-1, 0, 0, 0, 0, -1)^\top, (0, -1, 0, 0, -1, 0)^\top\}$

   (d) $K_4^* = \{(x_1, x_2, x_3, x_4)^\top \in \mathbb{R}^2 \mid x_1 - x_4 \geq 0, -x_2 + x_3 \geq 0\}$

2. Let $K_5 = \text{cone}\{(2, 7)^\top, (7, 2)^\top\} \subseteq \mathbb{R}^2$. Determine the vectors $u, v \in \mathbb{R}^2$ such that $u^\top u = 53$, $v^\top v = 53$, and $K_5^* = \text{cone}\{u, v\}$. 

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3. Write the dual cone of $K_6 = \{(x_1, x_2)\top \in \mathbb{R}^2 \mid x_1 + 5x_2 \geq 0, 2x_1 - x_2 \leq 0\}$ as an intersection of half spaces.

**Solution**

2. We have

$$K_5^* = \{(x_1, x_2)\top \in \mathbb{R}^2 \mid 2x_1 + 7x_2 \geq 0, 7x_1 + 2x_2 \geq 0\}.$$ 

The generators of $K_5^*$ can be obtained by solving the equations $2x_1 + 7x_2 = 0, 7x_1 + 2x_2 = 0$. Hence, the generators of $K_5^*$ are of the form $u = (7\lambda, -2\lambda)\top$ and $v = (2\mu, -7\mu)\top$ with $\lambda, \mu \in \mathbb{R} \setminus \{0\}$. Since $u \in K_5^*$ and $(7, 2)\top \in K_5$, we have $0 \leq 7(7\lambda) + 2(-2\lambda) = 45\lambda$. Hence, $\lambda > 0$. Since $v \in K_5^*$ and $(2, 7)\top \in K_5$, we have $0 \leq 2(2\mu) + 7(-7\lambda) = -45\mu$. Hence, $\mu < 0$. Since $53 = u\top u = 53\lambda^2$ and $\lambda > 0$ it follows that $\lambda = 1$. Therefore, $u = (7, -2)\top$. Since $53 = v\top v = 53\mu^2$ and $\mu < 0$, it follows that $\mu = -1$. Hence, $v = (2, 7)\top$.

3. A set of generators of $K_6$ can be obtained by solving the equations $x_1 + 5x_2 = 0, 2x_1 - x_2 = 0$. Hence a set of generators of $K_6$ can be searched in the form $u = (5\lambda, -\lambda)\top$ and $v = (\mu, 2\mu)\top$ with $\lambda, \mu \in \{-1, 1\}$. Since $u \in K_6$ and $(-2, 1)\top \in K_6^*$, we have $0 \leq (-2)5\lambda - \lambda = -11\lambda$. Hence $\lambda = -1$ and $u = (-5, 1)\top$. Since $v \in K_6$ and $(1, 5)\top \in K_6^*$, we have $0 \leq \mu + 5(2\mu) = 11\mu$. Hence, $\mu = 1$ and $v = (1, 2)\top$. Therefore,

$$K_6 = \text{cone}\{(-5, 1)\top, (1, 2)\top\}$$

and

$$K_6^* = \{(x_1, x_2)\top \in \mathbb{R}^2 \mid -5x_1 + x_2 \geq 0, x_1 + 2x_2 \geq 0\}.$$ 

**Another way to find this** is as follows: We have

$$K_6^* = \text{cone}\{(1, 5)\top, (2, 1)\top\}.$$ 

Hence,

$$K_6^* = \{(x_1, x_2)\top \in \mathbb{R}^2 \mid x_1 = \lambda - 2\mu, x_2 = 5\lambda + \mu \text{ where } \lambda, \mu \geq 0\}.$$ 

So, we get $-5x_1 + x_2 = 11\mu \geq 0$ and $x_1 + 2x_2 = 11\lambda \geq 0$. Therefore,

$$K_6^* = \{(x_1, x_2)\top \in \mathbb{R}^2 \mid -5x_1 + x_2 \geq 0, x_1 + 2x_2 \geq 0\}.$$ 

5.2.7 Any dual of a set is a dual of a closed convex cone

If $C$ is not a cone, then

$$C^* = (\text{cone}(C))^*.$$ 

For example see the figure below:
In this figure we have 3 circular disks. A is tangent to the horizontal axis in the origin. B is tangent to both the horizontal and vertical axis and C is in the interior of the nonnegative quadrant. We have $\text{cone}(A) = \{(x_1, x_2)^\top \in \mathbb{R}^2 \mid x_2 \geq 0\}$ and $\text{cone}(B) = \mathbb{R}^2_+$ and in general $\text{cone}(C)$ is the conic hull of the tangents from the origin to C, that is, $\text{cone}(C) = \text{cone}\{a, b\}$. If the origin $O$ is in the interior of a circular disk $D$, then $\text{cone}(D) = \mathbb{R}^2$.

**Dual of the lexicographic cone**

Let $L \subseteq \mathbb{R}^n$ be the lexicographic cone.

$$\text{cone}\{L\} = \text{closure}\{L\} = \{x \in \mathbb{R}^n \mid x_1 \geq 0\} \implies L^* = (\text{cone}(L))^* = \text{cone}\{(1, 0, \ldots, 0)^\top\}.$$

### 5.3 Conditions for the efficiency of a point

**Separation Theorem.** Let $K \subseteq \mathbb{R}^n$ be a closed convex cone and $S \subseteq \mathbb{R}^n$ a convex set with $\text{int}(K) \cap S = \emptyset$. Then, there exists a $z \in \mathbb{R}^n$ with $z^Tv > 0$, $\forall v \in \text{int}(K)$ and $z^Tw \leq 0$, $\forall w \in S$.

**Theorem 7.** Let $\text{int}(K) \neq \emptyset$, $M$ be convex and $v \in M$. Then $v \in E(M, \text{int}(K) \cup \{0\})$ if and only if there exists an $z \in K^* \setminus \{0\}$ such that for all $w \in M$ we have $z^Tv \leq z^Tw$.

**Illustration of minimisation of linear functionals on a set:**
Example for Theorem 7 and comments

Let $K := \mathbb{R}^2_+$, $C := \text{int } K \cup \{0\}$ and $M$ be a square with sides parallel to the coordinate axes.
E(M, K) = \{v \in M \mid \exists w \in M: w \leq_K v, w \neq v\}

E(M, C) = E(M, \text{int}(K) \cup \{0\}) = \{v \in M \mid \exists w \in \text{int}(K): v, w \neq v\}

Therefore, instead of handling E(M, K) (respectively E(M, \text{int}(K) \cup \{0\})) directly, do as follows:

1. Choose a \(z \in K^* \setminus \{0\}\),
2. Solve \(\min_{w \in M} z^Tw\) (a standard optimisation problem considerably simpler to solve),
3. result: \(v \in M\),
4. Use Theorem 7 to conclude that \(v\) is efficient,
5. GOTO 1.

Recall the following very important phrase from before:

"Note again that \(M\) is usually the set of possible values of decisions: \(v, w \in M, v = f(x), w = f(y)\). Hence, solving a decision making problem means that we have a set \(X\) of decisions and we want to choose the optimal decisions. In order to do so, we need to be given (or for a practical problem often identify by ourselves) a set of objective functions \(f_1, \cdots, f_n: X \to \mathbb{R}\). Let \(f := (f_1, \cdots, f_n): X \to \mathbb{R}^n\). Usually \(X\) is a subset of another finite dimensional space \(\mathbb{R}^m\), or it can be identified with such a subset. Solving our problem means that a cone \(K \subseteq \mathbb{R}^n\) is given (if no cone is given we assume that the cone is \(\mathbb{R}^n_+\); for practical problems we need to decide which cone to use), we need to find the image set

\[ M = f(X) := \{f(x) \mid x \in X\}, \]

the set of efficient points \(E(M, K)\) of \(M\) with respect to the order \(\leq_K\) induced by \(K\) (or simply with respect to \(K\)) and the decisions

\[ f^{-1}(E(M, K)) = \{x \in X \mid f(x) \in E(M, K)\} \]

corresponding to \(E(M, K)\), that is, the preimage set. A part of this process is illustrated in the next figure.
Hence, the standard procedure described before for finding an efficient point by minimising a linear functional has to be rephrased as follows:

\[\begin{align*}
\text{2'. Solve } & \min_{y \in X} z^T f(y), \\
\text{3'. result: } & x \in X, f(x) = v. \\
\end{align*}\]

**Remark:**

Be careful $z$ might not be unique. See the next figure which should be self-explanatory.

**Proof of Theorem 7.**
⇒” Let \( v \in E(M, \text{int}(K) \cup \{0\}) \), i.e., \((v-M) \cap \text{int}(K) = \emptyset\). Use the Separation Theorem with \( S = v-M \). Then, \( \exists z \in \mathbb{R}^n : z^\top u > 0, \forall u \in \text{int}(K) \), i.e., \( z \in K^* \setminus \{0\} \) (by Lemma 2 and its converse); and \( z^\top (v-w) \leq 0, \forall w \in M \), i.e., \( z^\top v \leq z^\top w, \forall w \in M \).

“⇐” Conversely, suppose that

\[ \exists z \in K^* \setminus \{0\} : z^\top v \leq z^\top w, \forall w \in M. \]

We want to prove that \( v \in E(M, \text{int}(K) \cup \{0\}) \). Suppose to the contrary, that

\[ v \notin E(M, \text{int}(K) \cup \{0\}). \]

Then, \( \exists w \in (v - \text{int}(K)) \cap M \). It follows that \( v - w \in \text{int}(K) \). Hence, Lemma 2 implies that \( z^\top (v-w) > 0 \). It follows that

\[ z^\top v > z^\top w, \]

which is a contradiction. Therefore, \( v \in E(M, \text{int}(K) \cup \{0\}) \).

\[ \square \]

Other examples for Theorem 7

Example:

Let \( M \subseteq \mathbb{R}^2 \) be a circular disk around 0 with radius 1 and \( K = \mathbb{R}^2_+ \). Then,

\[ E(M, \text{int}(K) \cup \{0\}) = \{v \in \mathbb{R}^2 \mid v_1 \leq 0, v_2 \leq 0, v_1^2 + v_2^2 = 1\}. \]

Let \( v \in E(M, \text{int}(K) \cup \{0\}) \). In order to use Theorem 7 it is enough to choose \( z = -v \in K^* \), as illustrated in the next figure.

![Diagram](image)

Example:

Let \( M = \{x \in \mathbb{R}^2 \mid -1 \leq x_1 \leq 1, -1 \leq x_2 \leq 1\} \) and \( K = \mathbb{R}^2_+ \). Then, it is easy to see that

\[ E(M, \text{int}(K) \cup \{0\}) = \{v \in M \mid v_1 = -1 \text{ or } v_2 = -1\}. \]
(a) To find the solutions with \( v_1 = -1 \) by using Theorem 7, choose \( z = (1, 0) \top \in K^* \).

(b) To find the solutions with \( v_2 = -1 \) by using Theorem 7, choose \( z = (0, 1) \top \in K^* \).

(c) To find the solution with \( v = (-1, -1) \top \) by using Theorem 7, choose any \( z \in K^* \setminus \{0\} \).

In (a) one solves \( \min_{w \in M} z \top w \), i.e.,

\[
\begin{align*}
\text{Minimise} & \quad w_1 \\
\text{Subject to} & \quad -1 \leq w_1 \leq 1, \\
& \quad -1 \leq w_2 \leq 1.
\end{align*}
\]

In (b) one solves \( \min_{w \in M} z \top w \), i.e.,

\[
\begin{align*}
\text{Minimise} & \quad w_2 \\
\text{Subject to} & \quad -1 \leq w_1 \leq 1, \\
& \quad -1 \leq w_2 \leq 1.
\end{align*}
\]

### 5.3.1 Finding efficient points by using \( K \)-monotone functions

**Definition.** A function \( s : \mathbb{R}^n \rightarrow \mathbb{R} \) is called \( K \)-monotone, if

\[ v \leq_K w \implies s(v) \leq s(w). \]

The function \( s \) is called strictly \( K \)-monotone, if

\[ v \leq_K w \text{ and } v \neq w \implies s(v) < s(w). \]

**Example:** Let \( z \in K^* \setminus \{0\} \) and define \( s(v) := z ^\top v \). Then \( s \) is \( K \)-monotone.

**Proof.** Indeed, \( v \leq_K w \) implies \( w - v \in K \). Hence, \( z ^\top (w - v) \geq 0 \) (because \( z \in K^* \) and \( w - v \in K \)), or equivalently

\[ s(v) = z ^\top v \leq z ^\top w = s(w). \]

□

**Example:** Let \( z \in \text{int}(K^*) \) and define \( s(v) := z ^\top v \). Then \( s \) is strictly \( K \)-monotone.

**Proof.** Indeed, \( v \leq_K w \) \& \( v \neq w \) implies \( w - v \in K \setminus \{0\} \). Since \( z \in \text{int}(K^*) \) and \( w - v \in K \setminus \{0\} \), Lemma 2 implies \( z ^\top (w - v) > 0 \), or equivalently

\[ s(v) = z ^\top v < z ^\top w = s(w). \]

□

**Example:** Let \( K = \mathbb{R}^2_+ \), \( z = (1, 1) \top \) and \( s(u) = z ^\top u \). Then, \( z \in \text{int}(K) = \text{int}(K^*) \). One has \( s(u) = z ^\top u = u_1 + u_2 \). Let \( v \leq_K w \), i.e., \( w - v \in K \), i.e., \( v_1 \leq w_1 \) \& \( v_2 \leq w_2 \). It follows easily that \( s(v) = v_1 + v_2 \leq w_1 + w_2 = s(w) \). Moreover, if \( v \neq w \), then at least one of the inequalities \( v_1 \leq w_1 \) \& \( v_2 \leq w_2 \) is strict, hence \( s(v) = v_1 + v_2 < w_1 + w_2 = s(w) \). Hence, \( s \) is both \( K \)-monotone and strictly \( K \)-monotone.\n
In fact any strictly \( K \)-monotone function is \( K \)-monotone, but the converse is not true, as shown by the next example.

**Example:** Let \( K = \mathbb{R}^2_+ \), \( z = (1, 0) \top \) and \( s(u) = z ^\top u = u_1 \). Then, \( z \in K \setminus \{0\} = K^* \setminus \{0\} \). Let \( v \leq_K w \), i.e., \( w - v \in K \), i.e., \( v_1 \leq w_1 \) \& \( v_2 \leq w_2 \). It follows easily that \( s(v) = v_1 \leq w_1 = s(w) \). Hence, \( s \) is \( K \)-monotone. However, \( s \) is not strictly \( K \)-monotone, because \( a := (0, 0) \top \leq_K (0, 1) =: b ^\top \) and \( a \neq b ^\top \), but \( s(a) = s(b) \).
Theorem 8. Let \( s : \mathbb{R}^n \rightarrow \mathbb{R} \) be a function and let \( v \in M \) be a minimum of \( s \) over \( M \). Then, the following hold.

1. If \( s \) is \( K \)-monotone and \( v \) is the unique minimum of \( s \) over \( M \), then \( v \in E(M,K) \).

2. If \( s \) is strictly \( K \)-monotone, then \( v \in E(M,K) \).

Proof. Let \( s : \mathbb{R}^n \rightarrow \mathbb{R} \) be \( K \)-monotone and \( v \) be a solution of

\[
\begin{align*}
\text{minimise} & \quad s(w) \\
\text{subject to} & \quad w \in M
\end{align*}
\]

Assume \( v \notin E(M,K) \), i.e., \( \exists w \in M, w \neq v: w \leq_K v \). Since \( s \) is \( K \)-monotone, \( s(w) \leq s(v) \).

Case 1 \( v \) is unique: But, \( w \neq v \) is also a minimum; contradiction.

Case 2 \( s \) is strictly \( K \)-monotone: Then, \( w \leq_K v \) and \( w \neq v \) implies \( s(w) < s(v) \). So, \( v \) is not a minimum; contradiction.

\[ \square \]

Corollary 2. Let \( z \in \text{int}(K^*) \). Define \( s(v) = z^\top v \). Solve

\[
\begin{align*}
\text{minimise} & \quad s(w) \\
\text{subject to} & \quad w \in M
\end{align*}
\]

If \( v \) is a solution, then \( v \in E(M,K) \).

Theorem 9. Let \( \text{int}(K^*) \neq \emptyset \) and \( v \in M \). Then \( v \in E(M,K) \) if and only if there exists a \( z \in \text{int}(K^*) \) and a \( \rho \in \mathbb{R}^n \) such that \( v \) solves the following optimisation problem:

\[
\begin{align*}
\text{min} & \quad z^\top w \\
\text{subject to} & \quad w \in M, \\
& \quad w \leq_K \rho.
\end{align*}
\]

Proof.

\( \Leftarrow \) Let \( v \in M \) be a solution of

\[
(P) \quad \begin{align*}
\text{minimise} & \quad z^\top w \\
\text{subject to} & \quad w \in M, \\
& \quad w \leq_K \rho.
\end{align*}
\]

Suppose \( v \notin E(M,K) \), i.e., \( \exists w \in M, w \neq v: w \leq_K v \). By transitivity of \( \leq_K \), \( w \leq_K \rho \), i.e., \( w \) is feasible for \( P \). Since \( z \in \text{int}(K^*) \), \( s(w) = z^\top w \) is strictly \( K \)-monotone. It follows that \( z^\top w < z^\top v \).

Hence \( v \) does not solve \( P \); contradiction.

\( \Rightarrow \) Let \( v \in E(M,K) \). Let \( \rho := v \) and \( z \in \text{int}(K^*) \) arbitrary. Then, \( v \) is the only feasible point for \( P \).

Hence \( v \) solves \( P \).

\[ \square \]

Recall that in general \( M = f(X) \) and a cone \( K \) is given in the image space of \( f \) as illustrated in the next figure. Finding an efficient point \( v \in E(M,K) \) and its corresponding decision \( x \in X \) with \( f(x) = v \) by solving the problem \( \min_{x \in X} s(f(x)) \), where \( s : \mathbb{R}^n \rightarrow R \) is called scalarisation. If \( s(u) = z^\top u \), then the latter problem is called weighted scalarisation and it can be written in the form \( \min_{x \in X} \sum_{i=1}^n z_i f_i(x) \).

Note that if the latter problem provides an efficient point (or efficient points), then we will find the same efficient point(s) if we replace \( z \) with \( \lambda z \), for any \( \lambda > 0 \).
5.4 The set of properly efficient points

**Definition.** The set of properly efficient points $P(M, K)$ is defined by

$P(M, K) := \bigcup_{z \in \text{int}(K^*)} \{v \in M \mid \forall w \in M : z^T v \leq z^T w\}.$

**Example:** Let $M = [-1,1] \times [-1,1] \subseteq \mathbb{R}^2$ and $K = \mathbb{R}_+^2$. Then, $E(M, K) = P(M, K) = \{(−1, −1)^T\}$. To find $(−1, −1)^T \in P(M, K)$ choose any $z \in \text{int}(K^*)$.

**Example:** Let $M = \{v \in \mathbb{R}^2 \mid v_1^2 + v_2^2 \leq 1\}$ and $K = \mathbb{R}_+^2$. Then,

$E(M, K) = \{v \in \mathbb{R}^2 \mid v_1 \leq 0, v_2 \leq 0, v_1^2 + v_2^2 = 1\}$

and $P(M, K) = E(M, K) \setminus \{−(1, 0)^T, (0, −1)^T\}$. To find $v \in P(M, K)$ choose $z = −v$.

**Theorem 10.** The following inclusions hold:

1. $P(M, K) \subseteq E(M, K)$.
2. If $K$ is pointed and $M$ is convex and closed, then $E(M, K) \subseteq \text{closure}(P(M, K))$.

**Proof.** We prove 1. only. It follows from the definition of $P(M, K)$ and Corollary 2. \hfill $\square$

Denote the components of any vector $a \in \mathbb{R}^n$ by $a_1, \ldots, a_n$, that is, $a = (a_1, \ldots, a_n)^T$.

**Remark 2.** If $K$ is pointed and $M$ is convex and closed, then $P(M, K) \subseteq E(M, K) \subseteq \text{closure}(P(M, K))$. This means that a computer cannot distinguish between $E(M, K)$ and $P(M, K)$.

**Theorem 11.** Suppose $K = \mathbb{R}_+^n$ and $M$ is polyhedral, i.e., there exists a matrix $A \in \mathbb{R}^{k \times n}$ and a vector $b \in \mathbb{R}^k$ such that

$M = \{v \in \mathbb{R}^n \mid (Av)_i \leq b_i \ (i = 1, \ldots, k)\}$.

Then $P(M, K) = E(M, K)$.

(No Proof.)
Extension of Theorem 11. Suppose $K$ is a simplicial cone in $\mathbb{R}^n$, that is, $K = \text{cone}\{u^1, \ldots, u^n\}$, where $u^1, \ldots, u^n \in \mathbb{R}^n$ are linearly independent. Also suppose that $M$ is polyhedral, that is, there exists a matrix $A \in \mathbb{R}^{k \times N}$ and a vector $b \in \mathbb{R}^k$ such that

$$M = \{v \in \mathbb{R}^n \mid (Av)_i \leq b_i \ (i = 1, \ldots, k)\}.$$ 

Then $P(M, K) = E(M, K)$.

(No Proof.)

Idea:
- Choose (several) $z \in \text{int}(K^*)$.
- Solve (several) optimization problems of the form $\min_{v \in M} z^\top v$.
- Result: (several??) points in $P(M, K)$.

Caution: $P(M, K) = E(M, K)$ does not mean that this idea works! Consider $K = \mathbb{R}^2_+$,

$$M := \{(v_1, v_2)^\top \mid v_1 + v_2 \geq 4, v_1 \geq 1, v_2 \geq 1\}.$$

Indeed, $P(M, K) = E(M, K) = \{(v_1, v_2)^\top \mid v_1 + v_2 = 4, v_1 \geq 1, v_2 \geq 1\}$ and a point $v \in P(M, K)$ with $v_1 > 1, v_2 > 1$ can be obtained by choosing a positive scalar multiple of $z = (1, 1)^\top$ only. It is unlikely that a computer will find this unique direction by randomly generating several $z \in \text{int}(K^*)$.

5.4.1 Geometrical interpretation of the set of properly efficient points

Recall that the scalar product of $z$ and $v$ is $z^\top v = z_1v_1 + \cdots + z_nv_n$, where $z = (z_1, \ldots, z_n)^\top \in \mathbb{R}^n$ and $v = (v_1, \ldots, v_n) \in \mathbb{R}^n$.

Definition. A hyperplane (with normal vector $z \neq 0$ and through $v \in \mathbb{R}^n$) is a set of form

$$H(z, v) = \{w \in \mathbb{R}^n \mid z^\top w = z^\top v\}.$$

A hyperplane $H(z, v)$ determines two closed halfspaces $H_-(z, v)$ and $H_+(z, v)$ of $\mathbb{R}^n$, defined by

$$H_-(z, v) = \{w \in \mathbb{R}^n \mid z^\top w \leq z^\top v\},$$

and

$$H_+(z, v) = \{w \in \mathbb{R}^n \mid z^\top w \geq z^\top v\},$$

A hyperplane $H(z, v)$ is a supporting hyperplane of $M$ at $v \in M$ if $M \subseteq H_+(z, v)$.

See the following figure for the geometric meaning of the supporting hyperplane.
From Definition 3 it follows that \( v \in P(M, K) \) if and only if there exists \( z \in \text{int } K^* \) such that \( H(z, v) \) is a supporting hyperplane of \( M \) at \( v \).

**Example:** Let \( M = M_1 \cup M_2 \), where

\[
M_1 = \{(x_1, x_2)^\top \in \mathbb{R}^2 \mid 0 \leq x_1 \leq 1 \text{ and } 1 \leq x_2 \leq 2\},
\]

\[
M_2 = \{(x_1, x_2)^\top \in \mathbb{R}^2 \mid 1 \leq x_1 \leq 2 \text{ and } 0 \leq x_2 \leq 2 \text{ and } x_1 + x_2 \geq 2\},
\]

and

\[
K = \{(x_1, x_2)^\top \in \mathbb{R}^2 \mid x_1 \geq x_2 \text{ and } x_2 \geq 0\}.
\]

Determine the set of properly efficient points \( P(M, K) \).

**Solution:** See the next figure.
Let us first determine the set of efficient points $E(M, K)$. It is easy to see that $v \in E(M, K)$ if and only if $(v - K) \cap M \subseteq \{v\}$. The boundary $\partial M$ of $M$ is formed by the straight line segments

$$[(0, 2)^T, (0, 1)^T], [(0, 1)^T, (1, 1)^T], [(1, 1)^T, (2, 0)^T], [(2, 0)^T, (2, 2)^T], [(2, 2)^T, (0, 2)^T].$$

We know that $E(M, K) \subseteq \partial M$. For any point $v$ belonging to one of the closed straight line segments

$$[(2, 0)^T, (2, 2)^T], [(0, 1)^T, (1, 1)^T], [(0, 2)^T, (2, 2)^T]$$

we have $(v - K) \cap M \not\subseteq \{v\}$. However, for any point $v$ belonging to one of the straight line segments

$$[(0, 2)^T, (0, 1)^T], [(1, 1)^T, (2, 0)^T]$$

we have that $(v - K) \cap M = \{v\}$ (note that the second segment does not contain the point $(1, 1)^T$). For this observe that the set $v - K$ is the translation of the set $-K$ into the point $v$. Therefore,

$$E(M, K) = [(0, 2)^T, (0, 1)^T] \cup [(1, 1)^T, (2, 0)^T]$$

$$= \{(0, \lambda)^T \in \mathbb{R}^2 \mid 1 \leq \lambda \leq 2\} \cup \{(1 + \lambda, 1 - \lambda)^T \in \mathbb{R}^2 \mid 0 < \lambda \leq 1\}.$$

We know that $P(M, K) \subseteq E(M, K)$. Observe that $p = (1, 0)^T \in \text{int}(K^*)$. Indeed $(1, 0)(x_1, x_2)^T = x_1 > 0$ for any $(x_1, x_2)^T \in K \setminus \{0\}$, because $x_1 \geq x_2 \geq 0$ and $x_1 = 0$ implies $x_2 = 0$. Minimising $s(v) := p^Tv$ on the set $M$ we obtain the segment $[(0, 2)^T, (0, 1)^T]$. Hence, $[(0, 2)^T, (0, 1)^T] \subseteq P(M, K)$. It is easy to see that except the point $(2, 0)^T$ there is no point on the segment $((1, 1)^T, (2, 0)^T]$ through which there exist a supporting hyperplane for $M$ (that is, there is no line which leaves $M$ on one of its sides or half-planes). The straight line through $(0, 1)^T$ and $(2, 0)^T$ is a supporting hyperplane of $M$. The vector $z = (1, 2)^T$ is a normal vector of the supporting hyperplane which is directed towards its half-plane containing $M$. It can be seen that $z \in \text{int}(K^*)$. Indeed, $K \subseteq \mathbb{R}^2_+$ and hence $(\mathbb{R}^2_+)^* = \mathbb{R}^2_+ \subseteq K^*$. Thus, $z \in \text{int}(\mathbb{R}^2_+)^* \subseteq \text{int}(K^*)$. Therefore $(2, 0)^T \in P(M, K)$. In conclusion

$$P(M, K) = [(0, 2)^T, (0, 1)^T] \cup \{(2, 0)^T\} = \{(0, \lambda)^T \in \mathbb{R}^2 \mid 1 \leq \lambda \leq 2\} \cup \{(2, 0)^T\}.$$
5.5 Finding efficient points by minimising norms

5.5.1 Norms

Reminder: A function \( \eta : \mathbb{R}^n \rightarrow \mathbb{R} \) is called a norm if

1. \( \eta(v) \geq 0 \) for all \( v \in \mathbb{R}^n \).
2. \( \eta(v) = 0 \) implies \( v = 0 \).
3. \( \eta(v + w) \leq \eta(v) + \eta(w) \) for all \( v, w \in \mathbb{R}^n \) (subadditivity of \( \eta \) or triangle inequality)
4. \( \eta(\lambda v) = |\lambda| \eta(v) \) for all \( v \in \mathbb{R}^n, \lambda \in \mathbb{R} \) (homogeneity of \( \eta \)).

5.5.2 Examples of norms

Euclidean norm

- In \( \mathbb{R}^2 \): \( \eta(x) = \|x\|_2 := \sqrt{x_1^2 + x_2^2} \).
- In \( \mathbb{R}^n \): \( \eta(x) = \|x\|_2 := \sqrt{\sum_{i=1}^{n} x_i^2} \).

1-norm

- In \( \mathbb{R}^2 \): \( \eta(x) = \|x\|_1 := |x_1| + |x_2| \).
- In \( \mathbb{R}^n \): \( \eta(x) = \|x\|_1 := \sum_{i=1}^{n} |x_i| \).

\( \infty \)-norm

- In \( \mathbb{R}^2 \): \( \eta(x) = \|x\|_\infty := \max(|x_1|, |x_2|) \).
- In \( \mathbb{R}^n \): \( \eta(x) = \|x\|_\infty := \max_{i=1,\ldots,n} |x_i| \).

\( p \)-norm

Let \( 1 \leq p < \infty \) be a parameter

- In \( \mathbb{R}^2 \): \( \eta(x) = \|x\|_p := (|x_1|^p + |x_2|^p)^{1/p} \).
- In \( \mathbb{R}^n \): \( \eta(x) = \|x\|_p := (\sum_{i=1}^{n} |x_i|^p)^{1/p} \).

Weighted \( p \)-norm

Let \( 1 \leq p \leq \infty \) be a parameter and \( \omega_1, \ldots, \omega_n > 0 \) given weights

- In \( \mathbb{R}^2 \): \( \eta(x) = \|x\|_{p,\omega} := (|\omega_1 x_1|^p + |\omega_2 x_2|^p)^{1/p} \).
- In \( \mathbb{R}^n \): \( \eta(x) = \|x\|_{p,\omega} := (\sum_{i=1}^{n} |\omega_i x_i|^p)^{1/p} \).
- In \( \mathbb{R}^n \): \( \eta(x) = \|x\|_{\infty,\omega} := \max_{i=1}^{n} |\omega_i x_i| \).

Example: If \( \omega_1 = 1/2, \omega_2 = 1 \) and \( p = 2 \), then \( \|(1,0)^T\|_{p,\omega} = ((1/2 \cdot 1)^2 + 0^2)^{1/2} = 1/2 \).
Rotated weighted $p$-norm

In $\mathbb{R}^2$: Let $1 \leq p < \infty$ be a parameter and $\omega_1, \ldots, \omega_n > 0$ given weights and $R = (\cos \alpha \sin \alpha)$. 

$\|x\|_{p, \omega, R} = \|Rx\|_{p, \omega}$.

5.5.3 Any Minkowski functional is a norm

**Definition.** Let $A \subseteq \mathbb{R}^n$. Then, $A$ is called symmetric if $x \in A$ implies $-x \in A$; or equivalently $A = -A$.

**Definition.** Let $A \subseteq \mathbb{R}^n$ be a compact convex symmetric set such that $0 \in \text{int} A$. Then, the function $\rho_A: \mathbb{R}^n \to \mathbb{R}$ defined by 

$$
\rho_A(x) = \begin{cases} 
\min\{r > 0 : x \in rA\} & \text{if } x \neq 0 \\
0 & \text{if } x = 0
\end{cases}
$$

is called the Minkowski functional.

**Lemma 3.** Let $A \subseteq \mathbb{R}^n$ be a convex set and $r, s > 0$. Then $rA + sA \subseteq (r + s)A$.

**Proof.** Since $A$ is convex, $\lambda A + (1 - \lambda)A \subseteq A$, for any $\lambda \in [0, 1]$. Take $\lambda = r/(r + s)$. □ □

**Proposition 1.** Let $A \subseteq \mathbb{R}^n$ be a compact convex symmetric set such that $0 \in \text{int}(A)$. Then, the Minkowski functional is well defined and it is a norm.

**Proof.** Let $x \in \mathbb{R}^n \setminus \{0\}$. Denote 

$$I = \{r > 0 : x \in rA\} \subset \mathbb{R}.$$

The set $I$ is nonempty: Indeed, if $\lambda > 0$ is sufficiently large, then $x/\lambda$ is sufficiently close to the origin, and hence $x/\lambda \in A$; that is $\lambda \in I$.

The set $I$ is closed: Indeed, let $r_n \in I$ with $r_n \to r^*$ as $n \to \infty$. Then, $x/r_n \in A$. Since $A$ is closed, $x/r^* \in A$ too; that is, $r^* \in I$.

The set $I$ is convex: Indeed, if $p, q \in I$ and $0 < t < 1$, then $x \in pA$ and $x \in qA$. Hence, by using the convexity of $A$ and Lemma 3, we get 

$$x = tx + (1-t)x \in tpA + (1-t)qA \subseteq [tp + (1-t)q]A,$$

that is, $tp + (1-t)q \in I$.

Since the set $I$ is nonempty, closed, convex and bounded from below, it is a closed interval, bounded from below. Hence, it has a minimal element. This means that the Minkowski functional is well defined.

The conditions 1. and 2. of a norm are trivially satisfied.

**Subadditivity of $\rho_A$:** Consider $v, w \in \mathbb{R}^n \setminus \{0\}$. Then, $v \in \rho_A(v)A$ and $w \in \rho_A(w)A$. Thus, since $A$ is convex, by Lemma 3 we have 

$$v + w \in \rho_A(v)A + \rho_A(w)A \subseteq [\rho_A(v) + \rho_A(w)]A.$$

Hence, $\rho_A(v + w) \leq \rho_A(v) + \rho_A(w)$.

**Homogeneity of $\rho_A$:** Consider $v \in \mathbb{R}^n \setminus \{0\}$ and $\lambda \in \mathbb{R} \setminus \{0\}$. By the symmetry of $A$ it follows that $\lambda v \in rA$ if and only if $v \in r/|\lambda|A$. Therefore, 

$$\rho_A(\lambda v) = \min\{r > 0 : \lambda v \in rA\} = \min\left\{r > 0 : v \in \frac{r}{|\lambda|}A\right\} = \min\left\{\frac{r}{|\lambda|} : r > 0 : v \in \frac{r}{|\lambda|}A\right\} = |\lambda|\rho_A(v).$$
Miscellaneous remarks related to the Minkowski functional

Let $A \subseteq \mathbb{R}^n$ be a compact convex symmetric set such that $0 \in \text{int}(A)$.

- $rA = \{rx : x \in A\}$.
- Note that for any $x \in \mathbb{R}^n$ we have $x \in \rho_A(x)A$.
- Let $t \in [0,1]$. If $x \in A$, then $tx \in A$. Indeed $x \in A$ implies $tx = tx + (1-t)0 \in A$ because $A$ is convex. This implies $tA \subseteq A$ and hence $A = \{x \in \mathbb{R}^n : \rho_A(x) \leq 1\}$. Indeed, $\rho_A(x) \leq 1$ implies $x \in \rho_A(x)A \subseteq A$ and $x \in A$ implies $\rho_A(x) \leq 1$.
- In the proof of Proposition 1, when showing that $I$ is closed, we can assume $r^* \neq 0$, otherwise $x/r_n \in A$ and $\|x/r_n\| = \|x\|/r_n \to \infty$ which contradict the boundedness of $A$. Here $\| \cdot \|$ is the Euclidean norm.
- In the proof of Proposition 1, when showing that $\rho_A$ is subadditive, we assumed that $v \neq 0$ and $w \neq 0$. If one of them is zero, then the subadditivity is trivial.
- In the proof of Proposition 1, when showing the homogeneity of $\rho_A$, we assumed $v \neq 0$ and $\lambda \neq 0$. Otherwise the homogeneity is trivial.
- It can be shown that any norm in $\mathbb{R}^n$ is a Minkowski functional. More specifically, if $\eta$ is a norm in $\mathbb{R}^n$, then $\eta = \rho_A$, where $A = \{x \in \mathbb{R}^n : \eta(x) \leq 1\}$.

5.5.4 Cone-monotone norms

Let $K$ be a pointed closed convex cone, $\emptyset \neq K \neq \{0\}$, $K \neq \mathbb{R}^n$.

**Definition.** The norm $\eta : \mathbb{R}^n \to \mathbb{R}$ is called $K$-monotone if $0 \leq_K x \leq_K y$ implies $\eta(x) \leq \eta(y)$.

Although this definition seems very similar to the definition of a $K$-monotone function, the two notions are completely different: Please note that there is no $K$-monotone norm which is a $K$-monotone function. Moreover, there is no norm which is a $K$-monotone function. Indeed, since $\emptyset \neq K \neq \{0\}$, there exists an $x \in K \setminus \{0\}$. Then, $-2x \leq_K -x$. If we suppose that the norm $\eta : \mathbb{R}^n \to \mathbb{R}$ is a $K$-monotone function, then we get

$$0 \leq 2\eta(x) = | -2|\eta(x) = \eta(-2x) \leq \eta(-x) = \eta((-1)x) = | -1|\eta(x) = \eta(x),$$

which implies $0 \leq \eta(x) \leq 0$. Thus, $\eta(x) = 0$ and therefore $x = 0$, which is a contradiction. Hence, $\eta$ cannot be a $K$-monotone function.

**Examples of cone-monotone norms**

1. Suppose $K \subseteq \mathbb{R}^n$ is subdual. Then, the Euclidean norm $\| \cdot \| : = \| \cdot \|_2$ is $K$-monotone norm. Indeed, let $0 \leq_K x \leq_K y$. Then, $x \in K \subseteq K^*$ and $y - x \in K$ imply

$$0 \leq x^\top (y - x) = x^\top y - \|x\|^2. \quad (2)$$

Similarly, $y - x \in K \subseteq K^*$ and $y \in K$ implies

$$0 \leq (y - x)^\top y = \|y\|^2 - x^\top y \quad (3)$$

Hence, formulas (2) and (3) imply that $\|x\|^2 \leq x^\top y \leq \|y\|^2$. Therefore, $\|x\| \leq \|y\|$.
2. Let $K = \mathbb{R}^n_+$, $1 \leq p < \infty$ and $\omega = (\omega_1, \ldots, \omega_n)^T$, where $\omega_i > 0$ for any $i \in \{1, \ldots, N\}$. Then, the weighted $p$-norm $\| \cdot \|_{p, \omega}$ is a $K$-monotone norm. Indeed, $0 \leq_K x \leq_K y$ implies $0 \leq x_i \leq y_i$ for any $i \in \{1, \ldots, N\}$, which implies

$$\|x\|_{p, \omega} = \left( \sum_{i=1}^{n} |\omega_i x_i|^p \right)^{\frac{1}{p}} \leq \left( \sum_{i=1}^{n} |\omega_i y_i|^p \right)^{\frac{1}{p}} = \|y\|_{p, \omega}.$$ 

Therefore, $\|x\|_{p, \omega} \leq \|y\|_{p, \omega}$.

3. Let $K = \mathbb{R}^n_+$, and $\omega = (\omega_1, \ldots, \omega_n)^T$, where $\omega_i > 0$ for any $i \in \{1, \ldots, N\}$. Then, the weighted $\infty$-norm $\| \cdot \|_{\infty, \omega}$ is a $K$-monotone norm. Indeed, $0 \leq_K x \leq_K y$ implies $0 \leq x_i \leq y_i$ for any $i \in \{1, \ldots, N\}$, which implies

$$\|x\|_{\infty, \omega} = \max_{i=1, \ldots, N} |\omega_i x_i| = \max_{i=1, \ldots, N} (\omega_i x_i) \leq \max_{i=1, \ldots, N} (\omega_i y_i) = \max_{i=1, \ldots, N} |\omega_i y_i| = \|y\|_{\infty, \omega}.$$ 

Therefore, $\|x\|_{\infty, \omega} \leq \|y\|_{\infty, \omega}$.

**Example of a norm which is not an $\mathbb{R}^2_+$-monotone norm**

Consider the rotated weighted 2-norm $\| \cdot \|_{2, \omega, R}$, where $R = (\begin{smallmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{smallmatrix})$ with $\alpha = \frac{\pi}{4}$, $\omega_1 = \frac{\sqrt{2}}{2}$ and $\omega_2 = 1$, that is,

$$\| \cdot \|_{2, \omega, R}(x) = \|Rx\|_2 = \frac{1}{2} \left( \sqrt{\frac{2}{2}} x_1 + \sqrt{\frac{2}{2}} x_2 \right)^2 + \left( -\sqrt{\frac{2}{2}} x_1 + \sqrt{\frac{2}{2}} x_2 \right)^2$$

$$= \sqrt{\frac{1}{2} \left( \frac{x_1^2}{2} + \frac{x_2^2}{2} + x_1 x_2 \right) + \left( \frac{x_1^2}{2} + \frac{x_2^2}{2} - x_1 x_2 \right)} = \sqrt{\frac{3}{2} (x_1^2 + x_2^2) - 1 \frac{1}{2} x_1 x_2},$$

for any $x = (x_1, x_2)^T \in \mathbb{R}^2$. A simple calculation yields that $\left\| \begin{pmatrix} 0, \sqrt{\frac{2}{3}} \end{pmatrix}^T \right\|_{2, \omega, R} = 1$ and $\left\| \begin{pmatrix} \frac{1}{4}, \sqrt{\frac{2}{3}} \end{pmatrix}^T \right\|_{2, \omega, R} = 0.9916879 < 1$. Hence, $\begin{pmatrix} 0, 0 \end{pmatrix}^T \leq_{\mathbb{R}^2_+} \begin{pmatrix} 0, \sqrt{\frac{2}{3}} \end{pmatrix}^T \leq_{\mathbb{R}^2_+} \begin{pmatrix} \frac{1}{4}, \sqrt{\frac{2}{3}} \end{pmatrix}^T$ and

$$\left\| \begin{pmatrix} 0, \sqrt{\frac{2}{3}} \end{pmatrix}^T \right\|_{2, \omega, R} > \left\| \begin{pmatrix} \frac{1}{4}, \sqrt{\frac{2}{3}} \end{pmatrix}^T \right\|_{2, \omega, R}.$$ 

Therefore, $\| \cdot \|_{2, \omega, R}$ is not an $\mathbb{R}^2_+$-monotone norm. I will not show this example on the whiteboard. Instead I will show you on a whiteboard-figure a more general idea behind this example.

**5.5.5 A necessary and sufficient condition for efficiency**

**Theorem 12.** Let there exist a $u \in \mathbb{R}^n$ with $M \subseteq u + K$. Then, $v \in E(M, K)$ if and only if there exists a $K$-monotone norm $\eta : \mathbb{R}^n \to \mathbb{R}$ such that $v$ is the only solution to the optimisation problem

$$\begin{align*}
\text{minimise} & \quad \eta(w - u) \\
\text{subject to} & \quad w \in M.
\end{align*}$$

(No proof.)

**Example →**

*Problem:* finding an appropriate $\eta$ is, in general, not easy.
5.5.6 How to choose \( u \) when \( K = \mathbb{R}^n_+ \)?

For \( f = (f_1, \ldots, f_n) : X \rightarrow M \) do the following:

Solve \( \min_{x \in X} f_i(x), i = 1, \ldots, n \). The minimisers of these problems are \( x^i, i = 1, \ldots, n \), respectively. The corresponding minimal values are \( f_i(x^i), i = 1, \ldots, n \), respectively.

The point \( u = (f_1(x^1), \ldots, f_n(x^m))^\top \) is a suitable point and it is called *utopia point* (why?).

Theorem 12 is called *goal programming* in the literature.