University of Birmingham

Interactions with Combinatorics

29 – 30 June 2017

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Thursday 29th June

10:00	*Tea and coffee*	
10:30	Oriol Serra	
11:20	Daniel Horsley	
12:10	*Lunch*	
13:45	Zdeněk Dvořák	
14:35	Mark Jerrum	
	LRA	LRC
15:00	Aleen Sheikh	Anum Khalid
15:25	*Tea and coffee*	
15:45	Amanda Montejano	David Tankus
16:10	Christoph Spiegel	Noam Zeilberger
16:35	Yufei Zhao	

Friday 30th June

10:00	Leslie Goldberg	
10:50	*Tea and coffee*	
11:10	Peter Hegarty	
12:00	*Lunch*	
13:30	Julia Wolf	
14:20	Gi-Sang Cheon	
14:45	*Tea and coffee*	
	LRA	LRC
15:10	Ben Barber	Tássio Naia
15:35	Casey Tompkins	Sandra Kingan
16:00	Oleg Pikhurko	

Invited speakers

Polynomial expansion

Zdeněk Dvořák (Charles University)

A class $\mathcal{C}$ of graphs has polynomial expansion if there exists a polynomial p such that for every graph G from $\mathcal{C}$ and for every integer r, each minor of G obtained by contracting disjoint subgraphs of radius at most r is $p(r)$-degenerate. Classes with polynomial expansion exhibit interesting structural, combinatorial, and algorithmic properties. In the talk, I will survey these properties and propose further research directions.

Amplifiers for the Moran Process

Leslie Ann Goldberg (University of Oxford)

The Moran process, as adapted by Lieberman, Hauert and Nowak, is a model of a population on a graph, evolving in discrete time. Individuals in the population are associated with the vertices of the graph. Certain individuals, called “mutants” have fitness r and other individuals, called “non-mutants” have fitness 1. The state of being a mutant or not can be spread from vertices to neighbours. We focus on the situation where the mutation is advantageous, in the sense that $r > 1$.

If the graph is strongly connected then, with probability 1, the process will either reach the state where there are only mutants (known as fixation) or it will reach the state where there are only non-mutants (known as extinction). A set of (directed or undirected) graphs is said to be strongly amplifying if the extinction probability tends to 0 when the Moran process is run on graphs in this set, starting from the state with a single mutation, at a uniformly-chosen vertex. It turns out that strong amplifiers exist, even in the undirected case.

This talk will tell you what is known about them, including joint work with Galanis, Goebel, Lapinskas and Richerby and also joint work with Lapinskas, Lengler, Meier, Panagiotou and Pfister.

Opinion dynamics

Peter Hegarty (Chalmers University of Technology)

Modern applied mathematics is full of buzzphrases, three which you have probably heard are “big data”, “complex adaptive systems” and “emergence”. The third is a property of the second which have become popular to study because of the first. Emergence basically refers to the idea that reliable patterns in the collective behaviour of large groups of interacting, but autonomous agents can emerge even if each agent obeys only a simple and “local” algorithm, where local means an agent interacts only with its close neighbours (in some metric). Thus, local dynamics should give rise to nice global patterns. Bird-flocking is the usual cocktail party example.

Presumably, most modellers in this field would, if they had to choose, rather have a model which has relevance for the real world than one about which it is actually possible to prove nice theorems. As a mathematician, I am motivated to find those rare gems which have both properties. In this talk, I will describe two classes of models which arose in the study of how individual humans influence one another's opinions, though there are also some other applications. They go under the names of Deffuant-Weisbuch and Hegselmann-Krause models. Both implement the locality feature in there being a so-called confidence bound, meaning that someone will, at any given time, only compromise with those whose opinions are already sufficiently close to their own (the models differ in how this is made mathematically precise). What makes these models interesting to social scientists is that both exhibit phase transition behaviour, depending on the confidence "radius", from a state where all agents eventually reach consensus to one where they cluster into several groups which eventually stop talking to one another. The amazing thing is that it is possible, at least in some cases, to prove rigorously the existence of a phase transition. Indeed, quite a rich body of rigorous results is now in existence, obtained using pretty standard techniques from calculus, linear algebra, combinatorics and probability. There are also many easily stated open problems. I'll present what I consider to be the most important results and remaining challenges.

Locating arrays and disjoint partitions

Daniel Horsley (Monash University)

Covering arrays are combinatorial objects used for designing testing procedures to determine whether faults are present in complex systems. In particular, they are used extensively in software testing. Locating arrays are variants of covering arrays that further allow any faults found to be precisely located without the need for further testing. It turns out that the existence problem for a simple kind of locating array is equivalent to a natural question in extremal set theory concerning families of disjoint partitions. In this talk I will introduce locating arrays, discuss this equivalence, and outline a complete solution to the problem.

Measurable Combinatorics

Oleg Pikhurko (University of Warwick)

We will discuss measurable versions of classical combinatorial problems (vertex/edge colourings, matchings, etc) and their applications. The main object of study will be infinite graphs whose edge set is the union of finitely many measure-preserving matchings on a standard probability space. Such objects appear in various areas such as the limit theory of bounded-degree graphs, measure-preserving group actions, descriptive set theory, etc.

We will mostly concentrate on positive results, where one constructs a measurable function F that satisfies given combinatorial constraints (such as being a proper vertex colouring). Here, a powerful tool for constructing the desired function F is to design a parallel decentralised algorithm that converges to it almost everywhere.

Linear analogues of additive theorems

Oriol Serra (Universitat Politècnica de Catalunya)

Inequalities relating the cardinality of a sumset with the cardinality of summands play a central role in additive combinatorics. Hou, Leng and Xian gave a version of one of the classical results in the area, the theorem of Kneser, in separable extensions of fields, where dimensions of subspaces play the role of cardinalities. One of the nice features of this dimension version of Kneser's theorem is that it gives the classical one as a Corollary.

The talk will discuss several results of the same nature where classical results in additive combinatorics are translated to the linear setting. In particular we will focus on a new combinatorial proof of a strengthenning of the dimensional Kneser theorem and on the simpler example of an inverse theorem, the theorem of Vosper.

Arithmetic regularity lemmas

Julia Wolf (University of Bristol)

The first arithmetic regularity lemma was established by Green in 2005. It stated that, given any subset A of a finite abelian group G, there exists a large (possibly approximate) subgroup of G with respect to which A is Fourier-uniform. This and its subsequent higher-order analogues have led to several important applications to problems related to Szemeredi's theorem. In this talk we shall discuss the arithmetic regularity lemma, its proof and its applications, as well as a new variant established from a model-theoretic perspective in joint work with Caroline Terry (University of Maryland).

Large deviations in discrete random structures

Yufei Zhao (University of Oxford)

What is the probability that the number of triangles in an Erdős–Rényi random graph exceeds its mean by a constant factor?

This problem has been a useful litmus test for concentration bound techniques. Even the order of the log-probability was considered a difficult problem until its resolution a few years ago. We now wish to determine the exponential rate of the tail probability. I will highlight some recent methods and results on this problem and its variants. Thanks to recent developments by Chatterjee, Varadhan, Dembo, and Eldan, this large deviations problem reduces to a natural variational problem, which then can be solved (in some instances) to deduce the asymptotics of the log-probability. I will explain this large deviation principle and its consequences, and discuss recent developments on the related question on the number of arithmetic progressions in a random set of integers.

Contributed talks

Isoperimetry in integer lattices

Ben Barber (University of Bristol)

The curve enclosing the greatest area for its length is a circle; this is the first example of an isoperimetric theorem. We can look for similar results whenever we have a notion of size and boundary: for example, in graphs. The general problem is NP-hard, but we can sometimes obtain good results in graphs with a lot of structure. I'll describe a very general result that solves the isoperimetric problem for all lattice like graphs. This is joint work with Joshua Erde.

Riordan graphs and their properties

Gi-Sang Cheon (Sungkyunkwan University, Republic of Korea)

Let $\kappa[[z]]$ be the ring of formal power series over an integral domain κ. A *Riordan array* denoted (g, f) is an infinite lower triangular matrix constructed out of two functions $g, f \in \kappa[[z]]$ with $f(0) = 0$ in such a way that its kth column generating function is gf^k for $k \geq 0$. In many contexts we see that the Riordan arrays are used as a machine to generate new approaches in combinatorics.

This talk is devoted to introducing new classes of graphs called *Riordan graphs* by Riordan arrays (g, f) over the ring $\mathbb{Z}[[z]]$. There are many reasons to define new classes of graphs. These graphs can be used as computer networks with certain desired properties or to get useful information when designing algorithms to compute values of graph invariants. The Riordan graphs are found to exhibit a number of interesting properties which meet such reasons. We illustrate several examples including Pascal graphs, Catalan graphs, and Toeplitz graphs.

Partial rejection sampling

Mark Jerrum (Queen Mary, University of London)

Suppose $\Phi = C_1 \wedge C_2 \wedge \cdots \wedge C_m$ is a predicate on variables $X_1, \ldots, X_n$. The Lovász Local Lemma gives a lower bound on the probability that Φ is true in the situation where $X_1, \ldots, X_n$ are independent random variables with specified distributions. If this lower bound is greater than zero, then we are assured that there is some assignment to the variables that satisfies Φ. Moser and Tardos described a polynomial-time algorithm for finding a satisfying assignment when the condition of the Lovász Local Lemma holds. It is natural to ask whether it is possible in polynomial time to sample an assignment from the conditional distribution of $(X_1, \ldots, X_n)$ given that Φ is true. We characterise the situations in which the Moser-Tardos approach —which we might describe as "partial rejection sampling"— does indeed output a sample from this conditional distribution. We then explore ways to extend partial rejection sampling to a wider range of examples. Based on joint work with Heng Guo (Queen Mary) and Jingcheng Liu (UC, Berkeley).

Enumerating height-restricted weighted Dyck paths arising in connection with q-tangent and q-secant numbers

Anum Khalid (Queen Mary, University of London)

Consider path diagrams formed by a Dyck path and columns underneath it, weighted according to the length of the path and the sum of the height of the columns. More precisely, we consider two cases, differing by the maximal height of a column below a down step. Their generating functions turn out to be intimately related to q-tangent and q-secant numbers [1]. This work extends the path diagram enumeration to the case of paths with restricted height.

Using the correspondence between path diagrams and continued fractions [2], we derive generating functions for these restricted height path diagrams from the convergents of the continued fractions by solving the recurrence equations for their numerators and denominators. Thereby we determine these generating functions explicitly in terms of basic hypergeometric functions.

When taking the half-plane limit, we arrive at new and very compact expressions for their generating functions. For example, for q-secant numbers we find

$$G(t,q) = (1+\lambda^2)\sum_{k=0}^{\infty} \frac{(-i\lambda\sqrt{q})^k}{(1-i\lambda\sqrt{q}q^k)}$$

where t and q are variables conjugate to length and column height, and λ is a root of $\lambda^2-(1-q)\lambda/t+1=0$. From this, one recovers the q-secant number expression

$$Q_N(q) = \frac{1}{(1-q)^{2N}}\sum_{m=0}^{N} \frac{q^{m^2+m}\left(\sum_{l=0}^{m}(-1)^l q^{-l^2}(2m+1)\binom{2N}{N+m}\right)}{N+m+1}$$

which has been derived previously [3]. We give analogous results for q-tangent numbers.

In the course of this work, we arrive at a hierarchy of seemingly novel q-series identities, the simplest being

$$\frac{1-q}{1-\lambda^2/q} =$$
$$\frac{{}_2\phi_1(\mu,-\mu;-\mu^2;q,q^2)\,{}_2\phi_1\left(\frac{-q}{\mu},\frac{q}{\mu};\frac{-q^2}{\mu^2};q,q^3\right)+\left(\frac{\mu^2}{q}\right){}_2\phi_1\left(\frac{-q}{\mu},\frac{q}{\mu};\frac{-q^2}{\mu^2};q,q^2\right){}_2\phi_1(\mu,-\mu;-\mu^2;q,q^3)}{{}_2\phi_1(\mu,-\mu;-\mu^2;q,q)\,{}_2\phi_1\left(\frac{-q}{\mu},\frac{q}{\mu};\frac{-q^2}{\mu^2};q,q^3\right)-\left(\frac{\mu^4}{q^2}\right){}_2\phi_1\left(\frac{-q}{\mu},\frac{q}{\mu};\frac{-q^2}{\mu^2};q,q\right){}_2\phi_1(\mu,-\mu;-\mu^2;q,q^3)}$$

References

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[3] M. Josuat-Vergs, "A q-enumeration of alternating permutations," *European Journal of Combinatorics*, vol. 31, no. 7, pp. 1892 – 1906, 2010.

Quasiregular Matroids

Sandra Kingan (Brooklyn College, CUNY)

Regular matroids are binary matroids with no minors isomorphic to the Fano matroid F_7 or its dual F_7^*. Seymour proved that the 3-connected regular matroids are either graphs, cographs, or R_{10}, or else can be decomposed along a non-minimal exact 3-separation induced by R_{12}. Quasiregular matroids are binary matroids with no minor isomorphic to the self-dual binary matroid E_4. The class of quasiregular matroids properly contains the class of regular matroids. In this talk I will present a decomposition result for quasiregular matroids that is similar to the decomposition of regular matroids.

Zero-sum over $\mathbb{Z}$ and the story of K_4

Amanda Montejano (Universidad Nacional Autónoma de México)

Zero-sum Ramsey theory is a newly established area in combinatorics. It brings to Ramsey theory algebraic flavor. Zero-sum problems can be formulated as follows: Suppose the elements of a combinatorial structure are mapped into a finite group Γ. Does there exists a prescribed substructure, such that the sum of the weights of its elements is the neutral element of Γ? Extending classical zero-sum Ramsey theory, Caro and Yuster study functions $f : E(K_n) \to \{-r, \dots, 0, \dots, r\}$ seeking zero-sum copies of a given graph G, subject to the obviously necessary condition that $|\sum_{e \in E(K_n)} f(e)|$ is bounded away from $\binom{n}{2}$, or even in the extreme case where $|\sum_{e \in E(K_n)} f(e)| = 0$. Relying heavily on Pell equations and some classical biquadratic Diophantine equations, we prove the following result: For a positive integer $k \geq 2$, $k \neq 4$, there are infinitely many values of n such that the following holds: There is a weighing function $f : E(K_n) \to \{-1, 1\}$ (and hence a weighing function $f : E(K_n) \to \{-1, 0, 1\}$), such that $\sum_{e \in E(K_n)} f(e) = 0$ but for every copy of K_k in K_n, $\sum_{e \in E(K_k)} f(e) \neq 0$. On the other hand, for every integer $n \geq 5$ and every weighing function $f : E(K_n) \to \{-1, 1\}$ such that $|\sum_{e \in E(K_n)} f(e)| \geq n(n-1)/2 - h(n)$ where $h(n) = 2(n+1)$ if $n \equiv 0 \pmod 4$ and $h(n) = 2n$ if $n \not\equiv 0 \pmod 4$, there is always a copy of K_4 in K_n for which $\sum_{e \in E(K_4)} f(e) = 0$. Our result, not only solve an open problem formulated by Caro and Yuster but also supply a good understanding of the situation concerning K_4, as the value of $h(n)$ is sharp. This is a joint work with Yair Caro and Adriana Hansberg.

Unavoidable trees in tournaments

Tássio Naia (University of Birmingham)

An oriented tree T on n vertices is unavoidable if every tournament on n vertices contains a copy of T. We obtained a sufficient condition for T to be unavoidable, and use this to prove that almost all labelled oriented trees are unavoidable, verifying a conjecture of Bender and Wormald. We additionally proved that every tournament on $n + o(n)$ vertices contains a copy of every oriented tree T on n vertices with polylogarithmic maximum degree, improving a result of Kühn, Mycroft and Osthus. This is joint work with Richard Mycroft.

The Davenport constant of finite abelian groups

Aleen Sheikh (Royal Holloway, University of London)

Let G be a finite abelian group. Define the *sum* of a multiset $\{g_1, \ldots, g_n\}$ of elements $g_i \in G$ to be $g_1 + \cdots + g_n$. A *zero-sum free* multiset over G is a multiset of elements of G with no submultiset whose sum is equal to 0_G. The *Davenport constant* of G indicates the size of the largest zero-sum free multiset over G. The Davenport constant has interesting applications in Number Theory. More precisely, if R is the ring of integers of some algebraic number field with ideal class group isomorphic to G and α is an irreducible element in R, then the Davenport constant of G is the maximal number of prime ideals which occur in the prime ideal decomposition of the ideal aR. It is known that the Davenport constant of G is at least $1 + d^*(G)$ where $d^*(G)$ is a certain constant that is computed using the invariant factor decomposition of G. There was a conjecture that this bound is always tight, but counterexamples are now known for many groups G of rank 4 or more. However, the conjecture has been established for many classes of groups, in particular it is known that $D(G) = 1+d^*(G)$ when G has rank at most 2. Whether the conjecture holds when G has rank 3 is still an open problem. In this talk I will review finite abelian groups for which the precise value of the Davenport constant has been found, present the value of the Davenport constant of the smallest finite abelian group of rank 3 for which the value was previously unknown, and present a new general polynomial upper bound on the Davenport constant of G in terms of $d^*(G)$. The general polynomial upper bound is quadratic in $d^*(G)$ and I will show how it improves to a linear polynomial in $d^*(G)$ if G is isomorphic to a specific infinite class of groups.

Random Strategies are Nearly Optimal for Generalized van der Waerden Games

Christoph Spiegel (Universitat Politècnica de Catalunya)

Beck introduced the *van der Waerden games* as the Maker-Breaker positional games in which Maker attempts to occupy an arithmetic progression of given length in $[n]$. We study the biased version of a strong generalization of this class of games, considering solutions to any abundant, homogeneous system. We determine the threshold biases of them up to constant factors by proving general winning criteria for Maker and Breaker based on the ideas developed by Bednarska and Łuczak. These general criteria also allow us to easily study the hypergraph generalization of the biased H-games. They also show that a random strategy for Maker is the best known strategy, giving further insight into the so-called *probabilistic intuition* in positional games by indicating a strong connection to sparse Turán- and Szemerédi-type statements.

Complexity results for generating subgraphs

David Tankus (Sami Shamoon College of Engineering, Israel)

A graph G is *well-covered* if all its maximal independent sets are of the same cardinality. Assume that a weight function w is defined on its vertices. Then G is *w-well-covered* if all maximal independent sets are of the same weight. For every graph G, the set of weight functions w for which G is w-well-covered is a vector space. That vector space is denoted $WCW(G)$.

Let B be a complete bipartite induced subgraph of G on vertex sets of bipartition B_X and B_Y. Then B is *generating* if there exists an independent set S such that $S \cup B_X$ and $S \cup B_Y$ are both maximal independent sets of G. A *relating edge* is a generating subgraph in the restricted case that $B = K_{1,1}$.

Deciding whether an input graph G is well-covered is **co-NP**-complete. Therefore, finding $WCW(G)$ is **co-NP**-hard. Deciding whether an edge is relating is **NP**-complete. Therefore, deciding whether a subgraph is generating is **NP**-complete as well.

We prove the following:

1. Recognizing relating edges in bipartite graphs is **NP-**complete.
2. Recognizing generating subgraphs in graphs with girth at least 6 is **NP-**complete.
3. Recognizing generating subgraphs in $K_{1,4}$-free graphs is **NP-**complete.
4. Recognizing relating edges and generating subgraphs is a polynomial task when the maximum degree of the graph is bounded.

Ordered graphs avoiding certain even cycles

Casey Tompkins (Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences)

We consider a complete bipartite graph $K_{n,n}$ consisting of two classes A, B of size n drawn in the plane in the following way: $A := \{(1,0),(2,0),...,(n,0)\}$, $B := \{(1,1),(2,1),...,(n,1)\}$, and for every edge ab with $a \in A$ and $b \in B$, we take the line segment joining a and b. We call this graph a geometric $K_{n,n}$ and consider the Turán-type problem of determining how large of a subgraph we can take while avoiding certain geometric subgraphs.

Let H be a subgraph of a geometric $K_{n,n}$. We say that an edge e in H is crossed if it intersects another edge from H and uncrossed otherwise. Observe that there are 6 different cycles of length 6 (up to order-isomorphism) as a subgraph of geometric $K_{n,n}$, and each cycle has at most two uncrossed edges. We partition these six cycles into two sets of three cycles, $\mathcal{C}_6^U$ and $\mathcal{C}_6^C$, where $\mathcal{C}_6^U$ consists of the three cycles with exactly two uncrossed edges. We show that a subgraph of a geometric $K_{n,n}$ with no cycle from C_6^U has at most $O(n^{4/3})$ edges. As a corollary, we get the same bound for C_6^C-free subgraphs. These results extend a classical theorem of Bondy and Simonovits which provided the same upper bound but required forbidding all 6 cycle types.

Finally, we show that for every $k \geq 2$, there exists a graph with $\Omega(n^{1+1/k})$ edges which avoids all cycles of length $2k$ with an uncrossed edge.

Some bridges between lambda calculus and graphs on surfaces

Noam Zeilberger (University of Birmingham)

In recent years, studies of the combinatorics of lambda calculus have revealed some surprising connections to the theory of graphs on surfaces. In the talk, after a quick introduction to lambda calculus, I will give a brief survey of these enumerative connections [1, 2, 3, 4], then focus on the correspondence between rooted trivalent maps and linear lambda terms, explaining how it may be seen as encoding a Tutte decomposition of trivalent maps with boundary.

References

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[2] N. Zeilberger and A. Giorgetti. A correspondence between rooted planar maps and normal planar lambda terms. *Logical Methods in Computer Science*, 11(3:22):1–39, 2015.

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