MSM203a: Polynomials and rings
Exercise sheet 2

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Autumn term, 2013

Exercise 1. For each positive $n > 1$ in $\mathbb{Z}$, prove that n is not prime if and only if $\mathbb{Z}/n\mathbb{Z}$ has zero divisors. So $\mathbb{Z}/n\mathbb{Z}$ is an integral domain if and only if n is prime.

Exercise 2 (2008, Q3)**.** If S is a subring of a field F and S contains the 1 of F then S is an integral domain.

Exercise 3 (2005, Q3)**.** Show that if R is a finite integral domain other than $\{0\}$ then it is a field. [Hint: Suppose $|R| = n$ and let $x \neq 0$. You will want to show x^{-1} exists. Suppose it doesn't. Put each element y (there are n of them) into the 'pigeonhole' labelled by $x \cdot y$. Assuming x^{-1} doesn't exist, how many pigeonholes get occupied?]

Exercise 4. A ring-like structure $F = \{0, 1, 2, 3\}$ is given with addition and multiplication defined by

+	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

·	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	3	1
3	0	3	1	2

(a) (Not assessed.) Prove that F is a field.

(b) Evaluate the following in $F[X]$: $(X^2+3X)(2X+1)$; $(X+2)(X+3)$; $(X+1)(X+2)(X+3)$.

Exercise 5. Let S be the ring-like structure $S = P(\{0, 1\})$ with addition and multiplication defined by $X + Y := \{x : x \in X \text{ or } x \in Y \text{ but not both}\}$ and $X \cdot Y := X \cap Y$. Show that S is a commutative ring with 1. Is S an integral domain?

Show that S has three distinct subrings of size 2, each isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

Exercise 6. Given $n, k \in \mathbb{N}$ find $m \in \mathbb{N}$ such that each term in $(X + Y)^m$ contains either X^n as a factor or contains Y^k as a factor.

Let R be a commutative ring and $I \lhd R$. Define $\sqrt{I} = \{x \in R : x^n \in I, \text{ some } n \geqslant 1\}$. Show that $\sqrt{I} \lhd R$.

For $R = \mathbb{Z}$, find $\sqrt{I}$ where I is: (a) $3\mathbb{Z}$; (b) $4\mathbb{Z}$; (c) $12\mathbb{Z}$.