MSM2P01, Autumn 2013, Exercises 5

Exercise 1 (2012, A8). Determine, with justification, which of the following series converge.

(a) $\sum_{n=1}^{\infty} \frac{n + \sin n}{n+1}$

(b) $\sum_{n=1}^{\infty} n^2 \left(\frac{3}{4}\right)^n$

(c) $\sum_{n=1}^{\infty} \frac{n^2+4}{3n^2+4n+6}$

(d) $\sum_{n=1}^{\infty} (-1)^n(\sqrt{n+1} - \sqrt{n})$

In this question you may appeal to general limit theorems and standard convergence tests for series.

Exercise 2 (2011, A7). Determine, with justification, which of the following series converge.

(a) $\sum_{n=1}^{\infty} \frac{n}{3^n}$

(b) $\sum_{n=1}^{\infty} \left(\frac{3n+2}{4n}\right)^{2n}$

(c) $\sum_{n=1}^{\infty} \frac{1}{n \log n}$

In this question you may apeal to general limit theorems and standard convergence tests for series, including the *integral test* that says if $f\colon [1,\infty) \to \mathbb{R}^+$ is decreasing[1] then $\sum_{n=1}^{\infty} f(n)$ converges if and only if the limit $\lim_{n\to\infty} \int_1^n f(x)\,\mathrm{d}x$ exists.

Exercise 3 (Based on 2012, B12(e)). Suppose (a_n) is a sequence of real numbers such that

$$l = \lim_{n\to\infty} n^2|a_n|$$

exists.

(a) Prove that there is $n \in \mathbb{N}$ such that for all $n \geqslant N$, $|n^2 a_n| < 1 + |l|$.

(b) Hence show by comparison with $\sum \frac{1}{n^2}$ that $\sum_{n=1}^{\infty} a_n$ converges absolutely.

[1] This implies the integrals mentioned here all exist, but the theory of integration is beyond the scope of this course.

Exercise 4 (2010, B12). (c) Let (a_n) be a decreasing sequence of positive real numbers and let n_1, n_n be natural numbers satisfying $n_1 < n_2$. Explain why

$$\sum_{n=n_1+1}^{n_2} a_n \geqslant (n_2 - n_1) a_{n_2}$$

(e) Show that

$$\sum_{n=2^{j-1}+1}^{2^j} \frac{1}{n \log n} \geqslant \frac{1}{2 \log 2} \frac{1}{j}$$

for all $j \in \mathbb{N}$. Deduce that the series $\sum_{n=2}^{\infty} \frac{1}{n \log n}$ diverges.

Exercise 5 (2011, B12(d)). Using appropriate series convergence tests, prove that the series

$$\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}}$$

converges if and only if the real number x satisfies $-1 \leqslant x < 1$.

Exercise 6. Find the radius of convergence of the following series.

(a) $\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2} x^n$

(b) $\sum_{n=1}^{\infty} \frac{2^n + 3^n}{4^n} x^n$