MSM2P01, Autumn 2013, Exercises 4

Exercise 1. Consider the sequence defined by $a_1 = 1$ and $a_{n+1} = \sqrt{1 + a_n}$.

(a) Assuming for the moment that (a_n) is convergent, by solving a quadratic equation or otherwise make a reasonable 'conjecture' for a value l that might be the limit of the sequence.

(b) Prove that $1 \leqslant a_n < l$ holds for all n, by induction. Your proof must not use the conjecture from (a) that $a_n \to l$, only the value of l itself.

(c) Using part (b), or by induction, prove that the sequence is monotonic nondecreasing.

(d) Which theorem from lectures allows you to deduce from (b) and (c) that a_n has a limit?

(e) Apply the continuity of the function $f(x) = \sqrt{1 + x}$ (considered as a function of positive x into the set of positive reals) to prove that the limit of the sequence (a_n) is l, as conjectured in part (a).

Exercise 2. Find $\sup A$ and $\inf A$ where $A = \{\frac{1}{n} - \frac{1}{k} : n, k \in \mathbb{N}, 1 < n < k\}$.

Exercise 3. (a) Let $p > 0$ in $\mathbb{R}$ and $n \in \mathbb{N}$. Prove that $(1 + p)^n \geqslant 1 + np + n(n - 1)p^2/2$. (Use induction on n.)

(b) If $r \in \mathbb{R}$ with $0 < r < 1$, show that $nr^n \to 0$ as $n \to \infty$. (Hint: write $r = 1/(1 + p)$ and use (a).)

Exercise 4. Prove that if $a_{n+1} = k/(1 + a_n)$ where $k > 0$ and $a_1 > 0$, then the sequence (a_n) converges to the positive root of $x^2 + x = k$.

Definition 5. A *series* is an expression of the form $\sum_{k=1}^{\infty} a_k$ where (a_n) is a sequence. Associated with a series $\sum_{n=1}^{\infty} a_n$ is its *sequence of partial sums* (s_n) given by

$$s_n = \sum_{k=1}^{n} a_k.$$

We say that *the series* $\sum_{k=1}^{\infty} a_k$ *converges* if the sequence s_n converges to some limit $l \in \mathbb{R}$, and if this happens l is the *limit* of the series.

Exercise 6. For the following series $\sum_{k=1}^{\infty} a_k$, use the definition above to determine if the series converges, and if so find the limit. (Hint: use partial fractions to express a_n.)

(a) $a_n = \dfrac{1}{n(n+1)}$

(b) $a_n = \dfrac{1}{n^2 + 2n}$

(c) $a_n = \dfrac{1}{n(n^2 - 1)}$ for $n > 1$ and $a_1 = 0$.

Exercise 7. Repeat 100 times, 'I understand that sequences and series are quite different and I will not muddle them up.'