MSM2P01, Autumn 2013, Exercises 3

Exercise 1. By quoting theorems on boundedness, subsequences, uniqueness of limits, or anything else from the course so far, or by arguing directly from the definition, prove that the following sequences do not converge to any limit.

(a) $a_n = 2^n$

(b) $b_n = \dfrac{(-1)^n}{10000000} + \dfrac{1}{n}$

(c) $c_n = \sin \dfrac{3n\pi}{4}$

(d) $d_n = \sin^2 n$

Exercise 2. A sequence (a_n) is given by $a_1 = 2$ and

$$a_{n+1} = \frac{a_n + \frac{2}{a_n}}{2}.$$

(a) Prove that

$$a_{n+1}^2 - 2 = \frac{1}{4}\frac{(a_n^2 - 2)^2}{a_n^2}$$

for all n.

(b) Deduce that $a_n^2 > 2$ for all $n > 1$.[1]

(c) Using induction, show that

$$a_n^2 - 2 \leqslant \frac{8}{4^n}$$

for each n.

(d) Using $a_n^2 - 2 = (a_n - \sqrt{2})(a_n + \sqrt{2})$ deduce that $a_n \to \sqrt{2}$ as $n \to \infty$.

Exercise 3. Suppose in each case that (a_n) is a sequence whose nth term is given by the expressions below. Prove that (a_n) converges and find the limit. If you use the continuity of any standard functions in your answer, state which function(s) you use that you know to be continuous and at what value $l \in \mathbb{R}$ or $(l, m) \in \mathbb{R}^2$ the continuity of that(those) function(s) is required.

(a) $\dfrac{n^2 + 2n + 1}{2n^2 + 3}$

(b) $\dfrac{1 - 2n - 3n^2}{2 - 3n - 4n^2}$

(c) $\dfrac{(n+1)^2 - (n-1)^2}{n+1}$

(d) $\dfrac{3n}{\sqrt{2n^2 + n + 1} + \sqrt{3n^2 + 1}}$

[1] An earlier version of the sheet has a typo here.

Exercise 4. Prove that the following sequences (a_n) do not converge, using the following method.

Suppose first that the sequence in question converges to some $l \in \mathbb{R}$. Then find convergent sequences (b_n), (c_n),..., and continuous functions f, g,... such that some expression involving f, g,..., (a_n), (b_n), (c_n),... evaluates to a sequence that you in fact know to be non-convergent.

The convergence or nonconvergence of any sequences you use must be justified either by stating that they are 'standard' convergent/nonconvergent sequences discussed in lectures (such as $1/n$ or $(-1)^n$) or by giving a proof.

(a) $(-1)^n(1-2^{-n})+1$.

(b) $(-1)^n(2^{-n})+\frac{n^2}{1+n}$.

(c) $\frac{1}{2}(\cos(n)+(-1)^n\frac{1}{n})$.

(d) $\frac{\sin(n)+n\cos(n)}{1+n}$.

Solutions to selected exercises

The following may be useful as models.

Solution to 3(a). Write

$$\frac{n^2+2n+1}{2n^2+3}=\frac{1+\frac{2}{n}+\frac{1}{n^2}}{2+\frac{3}{n^2}}$$

and observe that $1/n \to 0$ and $1/n^2 \to 0$ as $n \to \infty$, and so by the constant sequences 1,2 and the continuity of $+$ at $(1,0)$ and $\cdot$ at $(2,0)$, we have $1+\frac{2}{n}+\frac{1}{n^2} \to 1$ as $n \to \infty$. Also by the constant sequences 2,3 and the continuity of $+$ at $(2,0)$ and $\cdot$ at $(3,0)$, we have $2+\frac{3}{n^2} \to 2$ as $n \to \infty$. By the continuity of division at $(1,2)$ we have $\frac{n^2+2n+1}{2n^2+3} \to 1/2$.

Solution to 4(a). Suppose $a_n=(-1)^n(1-2^{-n})+1$ defines a sequence converging to $l \in \mathbb{R}$. Consider

$$b_n=\frac{(a_n-1)}{(1-2^{-n})}.$$

Now 1 is a constant sequence, so converges to 1. The sequence 2^{-n} converges to 0 by a standard result from lectures. So by the continuity of $-$ at $(0,1)$, $1-2^{-n} \to 1$ as $n \to \infty$, by the continuity of $-$ at $(l,1)$, $(a_n-1) \to l-1$ as $n \to \infty$ and by the continuity of division at $(l-1,1)$, the sequence b_n converges to $\frac{l-1}{1}$. But this is impossible as $\frac{(a_n-1)}{(1-2^{-n})}=(-1)^n$ which is known from lectures to be nonconvergent.