MSM2P01, Autumn 2013, Exercises 2

For this sheet you will need the definition of absolute value, $|x|$, and the definition of convergence, $a_n \to l$ as $n \to \infty$. These were given earlier. If you don't use them your answers will be wrong. You do not need a calculator.

Exercise 1. Prove that

(a) $1 - \dfrac{1}{\sqrt{n}} \to 1$ as $n \to \infty$

(b) $\dfrac{1}{n(n+1)} \to 0$ as $n \to \infty$

(c) $\dfrac{\sin n}{n(n-3/2)} \to 0$ as $n \to \infty$

(In (c), say precisely what properties of the sin function you are using.)

Exercise 2. Let $x < y < z$ and l be real numbers. Show that $|y - l| < \max(|x - l|, |z - l|)$. (Hint: consider the cases when $y \geqslant l$ and when $y < l$ separately.)

Exercise 3. Three sequences (a_n), (b_n) and (c_n) are given. You are told that $a_n \to l$ and $c_n \to l$ as $n \to \infty$, and that $a_n > b_n > c_n$ for all $n \in \mathbb{N}$.

(a) Is this possible? Show it is possible by giving an example of sequences (a_n), (b_n) and (c_n) such that all these statements are statisfied. (Define your sequences properly and prove the properties.)

(b) Write down the $\varepsilon - N$ definitions of what it means that $a_n \to l$ and $c_n \to l$ as $n \to \infty$.

(c) Write down the $\varepsilon - N$ definition of what it means that $b_n \to l$ as $n \to \infty$, and using Exercise 2 and Exercise 3(b) together with '$a_n > b_n > c_n$ for all $n \in \mathbb{N}$' prove your statement.

The result proved in Exercise 3 can be stated as follows.

Theorem 4 (Squeeze Rule)**.** If (a_n), (b_n) and (c_n) are sequences and $a_n \to l$ and $c_n \to l$ as $n \to \infty$, and also that $a_n > b_n > c_n$ for all $n \in \mathbb{N}$, then $b_n \to l$ as $n \to \infty$.

(I.e. a sequence squeezed between two sequences that both converge to the same limit also converges to the same limit.) You may find it useful to learn this theorem and quote it in future.

Ex1: Answers, commentary and feedback