MSM203a: Polynomials and rings
Additional topic: Noetherian Rings

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Autumn 2013*

This handout is to help you write a short essay on an interesting topic additional to that in the syllabus: Noetherian Rings. You do not have to answer all questions listed here, and you may add other points that you might think are relevant. In fact you do not have to answer any of the questions here directly. Just use them as a start for your own investigation.

I have listed some things for people who might be interested in (a) the history and/or (b) the mathematics, and both are interesting. You have to address the general issues, find out what Noetherian rings are and why they might be interesting. You must go beyond the notes given in lectures and other printed notes, and generally beyond the lecture course in some ways.

Your essay may be written by hand, or written on the computer in any format. If you use the computer you must send source files as well as the final output files, in case these are different (e.g. LaTeX). You must submit work to me directly by the end of the term. (Some time in the last week is fine, either personally or via email at R.W.Kaye@bham.ac.uk. Do not use pigeonholes.)

Your essay should be interesting, and should show that you have investigated the topic(s) independently of lectures, and independently of other students. As usual, some discussion amongst your friends is OK prior to writing-up, but you should not discuss the details of what you write with your friend(s). Assuming you write by hand on widely spaced lines on A4 paper, and that your handwriting is of medium size, I would expect at least 6 pages and probably 10-12 pages.

You can use the internet or books for sources. You must reference any sources you use correctly, saying which ones you used and if appropriate which part of it, and if you quote any part of any source directly, you indicate in your text that it is a quotation and where the quotation comes from. Otherwise, you must write in your own words. If you do repeat whole arguments or proofs of theorems your work should show that you understand them, e.g. by giving examples. Throughout this piece you can restrict your attention throughout to commutative rings with 1, or even to integral domains if you like.

There will be a final mark for this work, and all third years taking polynomials and rings must do it. I intend to mark fairly, in line with the 'research skills' module, but I will bear in mind that you have much less time to address this work. I will also check your marks in the regular fortnightly homeworks and your final mark may reflect these too. So if you have not handed in work to some or all of the homeworks that will reflect badly on your mark, and if you have done very well in the examples classes that may reflect well on you. For most students it is only this exercise that generates the (rather small) continuous assessment element of MSM3P04a.

*Version 1.2 of 2013-10-31

Work will be returned to you via student pigeonholes on the first floor of the Watson building by the first day of the spring term. If you come to see me after Christmas I can give you personal feedback on your work. I will need to keep a copy of your work which may be shown to external examiners etc. to demonstrate that the university regulations of third-year level modules have been met. This will not be used in any other way.

Question 1. Noetherian rings are named for Emmy Noether, perhaps the best known woman mathematician of her period. Investigate and describe her life and work.

Question 2. What is a Noetherian ring? How does the idea of a Noetherian ring that has no nontrivial zero divisors generalise that of a Principal Ideal Domain (PID)? You might want to think about how an ideal can be generated by several elements, just like a vector space is generated by its basis elements. What can you say about ideals in a Noetherian ring and their generators?

Question 3. Show that $\mathbb{Z}[X]$ is not a PID. Show that it is Noetherian.

Question 4. Show that $\mathbb{R}[X, Y]$ is not a PID. Show that it is Noetherian.

Question 5. Define and describe more examples and non-examples.

Question 6. If R is Noetherian and I is an ideal of R show that R/I is Noetherian.

Question 7. The *Hilbert Basis Theorem* says that if R is a Noetherian ring then $R[X]$ is also Noetherian. This generalises the results that $\mathbb{Z}[X]$ and $\mathbb{R}[X, Y]$ are Noetherian. Can you sketch the important parts of the proof, and explain what's going on by way of examples?

Question 8. Who said 'Das ist nicht Mathematik. Das ist Theologie.'? What was he talking about and why was he so upset?