MSM203a: Polynomials and rings
Exercise sheet 4

Richard Kaye

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Exercise 1. Read your notes carefully. Then put them away and only peek at them if you are stuck.

(a) State what it means for $f\colon R \to S$ to be a ring homomorphism. Define $\ker f$ and show it is an ideal of R.

(b) Let I be an ideal of R. Say what the elements of R/I are. Explain why R/I is a partition of R.

(c) Define addition and multiplication on R/I. Prove your definitions are well-defined.

(d) Show that R/I is a ring.

(e) Define a ring homomorphism $g\colon R \to R/I$ with $\ker g = I$. Prove g has the properties you claim.

Exercise 2. $\mathbb{F}_5 = \{0, 1, 2, 3, 4\}$ is the field of five elements, with addition and multiplication modulo 5, isomorphic to $\mathbb{Z}/5\mathbb{Z}$. Find polynomials $q(X), r(X)$ in $\mathbb{F}_5[X]$ such that

$$X^7 + 2X^6 + 3X^5 + 4X^4 + X^2 + 2X + 3 = q(X) \cdot (X^2 + 4) + r(X)$$

where $r(X)$ has degree at most 1.

Exercise 3. For the following pairs of polynomials $f(X)$ and $g(X)$ find the quotient and remainder on dividing $g(X)$ by $f(X)$.

(a) $g(X) = X^7 - X^3 + 5$, $f(X) = X^3 + 7$ over $\mathbb{Q}$.

(b) $g(X) = X^2 + 1$, $f(X) = X^2$ over $\mathbb{Q}$.

(c) $g(X) = 4X^3 - 17X^2 + X - 3$, $f(X) = 2X + 5$ over $\mathbb{R}$.

(d) $g(X) = X^3 + 2X^2 + 2X + 1$, $f(X) = X + 2$ over $\mathbb{F}_3$.

(e) $g(X) = X^7 + 3X^6 + X^3 + 4X + 5$, $f(X) = 2X^3 + 5$ over $\mathbb{F}_7$.

Exercise 4. (2007, Q4) Suppose that R is a ring and I, J are ideals in R. Define the subset $I + J$ of R and prove that it is an ideal. Let $R = \mathbb{Z}$, $I = 12\mathbb{Z}$, $J = 18\mathbb{Z}$. Find $k \in \mathbb{Z}$ such that $I + J = k\mathbb{Z}$.

Exercise 5. Show that $X^4 + 7$ is irreducible in $\mathbb{F}_{17}[X]$. How many elements are there in $\mathbb{F}_{17}[X]/(X^4 + 7)$? Suggestion:

(a) Start to draw up a table of values x, x^2, x^4 for $x \in \mathbb{F}_{17}$. Fill in the values of x^2 for $x = 1, 2, \ldots, 8$.

(b) Using $x^2 \equiv (17-x)^2 \bmod 17$, find the remaining values for x^2. Use your table of values of x^2 to find the values for x^4.

(c) Deduce that $X^4 + 7$ has no root in $\mathbb{F}_{17}$.

(d) Show that if X^4+7 is reducible in $\mathbb{F}_{17}[X]$ there are $a, b \in \mathbb{F}_{17}$ with $b \neq 0$ and $X^4+7 = (X^2+aX+b)(X^2-aX+7b^{-1})$. Write down two equations for a, b and partially solve them to get $a^3 = 2ab$. Deduce either $a = 0, b^2 = 10$ or $a \neq 0, b^2 = 7$ and conclude from your table there are no such values.

Exercise 6. For each of the following polynomials (over the given field) say, with justification, whether it is reducible or irreducible.

(a) $X^2 + X + 1$ in $\mathbb{F}_5[X]$.

(b) $X^2 + X + 1$ in $\mathbb{F}_7[X]$.

Exercise 7. (a) Let

$$J = \{p(X) : p(X) \in \mathbb{Z}[X],\ p_0 \in 2\mathbb{Z}\}$$

be the set of integral polynomials whose constant term is even. Show that J is an ideal of $\mathbb{Z}[X]$. What can you say about the factor ring $\mathbb{Z}[X]/J$?

(b) Let

$$K = \{p(X) : p(X) \in \mathbb{Z}[X],\ p_n \in 2\mathbb{Z} \text{ for all } n \geqslant 1\}$$

be the set of integral polynomials whose coefficients are all even except possibly the constant term. Is K a subring of $\mathbb{Z}[X]$? an ideal of $\mathbb{Z}[X]$?

Exercise 8. (2005, Q11) Suppose R is a ring and J is an ideal of R with $R \neq J$. Which of the following statements are true and which false? Give a proof or counterexample in each case.

(a) If R has a 1, then J must contain it.

(b) Every subring of a ring is an ideal.

(c) If R is commutative, then R/J is commutative.

(d) If R is non-commutative, then R/J is non-commutative.

(e) If R is a field then R/J is a field.

Exercise 9. Let $R = \mathbb{Z}$ and $I = 5\mathbb{Z}$ and $J = 3\mathbb{Z}$. Show that

(a) $I + J = R$

(b) $I \cap J = 15\mathbb{Z}$

Deduce from the second isomorphism theorem that $5\mathbb{Z}/15\mathbb{Z}$ is isomorphic to $\mathbb{F}_3$. (For (a) you may find is helpful to quote a theorem from MSM1B.)