# MSM203a: Polynomials and rings
# Exercise sheet 1

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Read and attempt all exercises before the exercise class. Not all questions will be assessed. You will be told which exercises should be written up and handed in in the exercise class itself. You will get solutions to all these exercises in due course.

Notation here is standard. $\mathcal{P}(A) = \{U : U \subseteq A\}$ is the power set of $A$; the symbol $\setminus$ denotes set difference; and $\mathbb{R}^+$ is the set $\{x \in \mathbb{R} : x > 0\}$ or positive reals.

**Exercise 1.** Show that $(X, +, \cdot)$ is not a ring, where $X = \mathcal{P}(\mathbb{N})$, $U + V := U \cup V$, and $U \cdot V := U \cap V$.

**Exercise 2.** Show that $(X, +, \cdot)$ is a ring, where $X = \mathcal{P}(\mathbb{N})$, $U + V := (U \setminus V) \cup (V \setminus U)$, and $U \cdot V := U \cap V$.

**Exercise 3.** Show that $(\mathbb{N}, +, \cdot)$ is not a ring, where $+$ and $\cdot$ are the usual operations on $\mathbb{N}$.

**Exercise 4.** Show that $(\mathbb{R}^+, +, \cdot)$ is not a ring where $u + v := uv$ and $u \cdot v := u^v$.

**Exercise 5.** Is $(\mathbb{R}^+, +, \cdot)$ a ring, where $u + v := uv$ and $u \cdot v := u^{\log v}$? (And does it matter to what base logarithms are taken here?)

**Exercise 6.** (Essentially Q2 in the 2005 exam.)
Suppose that $R$ is a ring and 0 is the zero of $R$.

(a) Show that $0y = 0 = y0$ for all $y \in R$.

(b) Say what it means for an element to be a 1 in $R$.

(c) Show that if $R$ has a 1 then the 1 is unique.

(d) Show that if 0 is a 1 in $R$ then $R$ has exactly one element.

**Exercise 7.** (Asked as part of Q2 in the 2007 exam, also Q2 in 2009.)
Give an example of a ring which does not have a 1.

**Exercise 8.** (Asked as part of Q1 in the 2009 exam.)
Give an example of a binary operation defined on $\mathbb{Z}$ which is not associative. Give an example of a finite subset of $\mathbb{C}$ which contains more than two elements and which is closed under complex multiplication.

**Exercise 9.** One of the following equations below is true in all rings (for all $x, y, z, u, v, w$ in the ring) and the other is not. Decide which is true and prove it from the ring axioms. For the other, give an example of a ring and give elements $x, y, z, u, v, w$ in that ring which do not satisfy the equation.

$$(x + y + z)(u + v + w) = (((yv + xu) + zv) + xv) + (xw + ((yu + zu) + yw) + zw)$$

$$(x + y + z)(u + v + w) = (vz + (ux + (vx + (wx + (wy + (uz + (vy + (wz + uy))))))))$$