Reflective permutation groups

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10th April, 1997

1 Introduction

This work is concerned with a class of closed permutation groups acting on countably infinite sets Ω with particularly rich structure. The motivation and background for this paper is in the relationships between model theory and the theory of infinite permutation groups, in particular between a countably infinite relational stucture M with domain Ω and its automorphism group $G = \operatorname{Aut} M$, considered as a permutation group with its natural action on Ω.

The study of the relationships between countably infinite structures M and their automorphism groups $G \leqslant \operatorname{Sym}\Omega$ has been particularly fruitful in the special case when M is $\aleph_0$-*categorical* (in the standard terminology from model theory). Here, it turns out by a result of Engeler, Ryll-Nardzewski and Svenonius that M is $\aleph_0$-categorical if and only if G is a closed *oligomorphic* subgroup of $\operatorname{Sym}\Omega$, where 'closed' is with respect to a natural topology explained below and 'oligomorphic' means that for all $n \in \mathbb{N}$ there are finitely many G-orbits in the natural action on Ω^n. Macpherson's paper [4] gives much more information on these automorphism groups and in particular shows that such G are 'large' in the sense that G contains the free group of rank $\aleph_0$ as a dense subgroup. See also Cameron [1] for further details and much more information on oligomorphic permutation groups and their relationships with relational structures.

Other interesting results have been discovered for structures which are not $\aleph_0$-categorical, usually in cases when the structure M is sufficiently rich, or 'saturated', and the corresponding automorphism group G is large in some sense. I would like to point out in particular the case when M is *recursively saturated* here. (The definition and some basic properties of recursively saturated structures are given at the end of the next section, full details can be found elsewhere, for example [2].) Recursive saturation is a model theoretic notion that can be regarded as a nonstandard analogue of $\aleph_0$-categoricity [3, pages ??–??] and many of the results originally proved in the $\aleph_0$-categorical case transfer to recursively saturated structures [3]. In particular, Macpherson's result just stated carries over to this situation too, so recursively saturated stuctures have 'large' automorphism groups. Unfortunately, no direct permutation group theoretic characterization of the automorphism groups of countable recursively saturated stuctures is known.

This paper presents a contribution to understanding the connections between the permutation groups $G = \operatorname{Aut} M$ for sufficiently saturated countable structures M and the structural properties of M. In it, I present a weaker notion of saturation (called here *reflectivity*) which includes the recursively saturated and $\aleph_0$-categorical cases but is nevertheless sufficient to show that G is still large in the sense just mentioned. The notion of reflectivity also has the advantage that it has an elegant characterization in terms of the permutation group G, as well as a model theoretic equivalent. We then take up the question of how 'saturated' need reflective stuctures really be.

One possible way of measuring the saturation properties of a structure M (and the one adopted in this paper) is to describe the subgroups H that may be embedded into $G = \operatorname{Aut} M$. Since we are interested here in cases when G is large, we may expect free groups to embed; the question is then what other groups embed—in particular can groups whose presentations require 'many' relations also embed?

In the case of $\aleph_0$-categorical structures M in general, the best possible statement one can make in this direction is the following.

FACT 1.1. Suppose M is countably infinite and $\aleph_0$-categorical and $G = \operatorname{Aut} M$. For every countable right-orderable group H there is an embedding $H \hookrightarrow G$.

(A group H is *right-orderable* if there is a linear order $<$ on H which is respected by the group operation on the right: $g < h$ implies $gk < hk$ for all $g, h, k \in H$.)

The condition that H is right-orderable is in general necessary. For example the structure $(\mathbb{Q}, <)$ of the rationals with its usual order relation is $\aleph_0$-categorical. Also, $G = \operatorname{Aut}(\mathbb{Q}, <)$ is right-orderable (and hence so is every subgroup of G); to see this, take any well-order $\prec$ of $\mathbb{Q}$ and define a linear order $<$ on G by

$$g < h \Leftrightarrow xg < xh \text{ where } x \text{ is the } \prec\text{-least element of } \mathbb{Q} \text{ for which } xg \neq xh.$$

For structures without linear orders one might expect other subgroups to embed. Indeed the structure Ω consisting of a set with no relations is $\aleph_0$-categorical and has as its automorphism group the full symmetric group $\operatorname{Sym} \Omega$.

There are two possible proofs of Fact 1.1, which I shall briefly sketch here for readers with the appropriate model-theoretic background. The first is to use the model theoretic notion of indiscernibles to show that there is an Ehrenfeuct–Mostowski model $M' \supseteq H$ of the theory of M generated by H, where H is our group considered as a linearly ordered set of indiscernibles of M'. Then $M \cong M'$ by $\aleph_0$-categoricity, and the regular action of H on itself lifts to an embedding of H into $\operatorname{Aut} M'$. Alternatively, one can write down a presentation of H, $\mathscr{P}$, and note that M, being $\aleph_0$ saturated, is $\mathscr{P}$-recursively saturated (i.e., recursively saturated, except that the notion of recursive is replaced by the relativized notion 'recursive in $\mathscr{P}$' throughout). Then the corresponding relativized notion of resplendency gives the result. (Note that this second proof immediately shows that if M is recursively saturated, rather than $\aleph_0$-categorical, then every right-orderable H with recursive presentation embeds in $\operatorname{Aut} M$. However, using a deep result on indiscernibles in countable recursively saturated models due to

J. H. Schmerl [5, ?], the first proof sketched here is also valid in this situation, even without any further conditions on H. So Fact 1.1 is true with the weaker hypothesis that M is countably infinite and recursively saturated.)

Note here that, although Fact 1.1 can be stated entirely within the language of permutation group theory (using the notion of a closed oligomorphic permutation group in place of $\aleph_0$-categoricity), every known proof of it uses some amount of model theory—even in the very particular case when H is a free-abelian group of rank $\aleph_0$. (A direct proof using recursive saturation that the free-abelian group of rank $\aleph_0$ embeds in the automorphism group of every countably infinite recursively saturated structure is given in the next section.)

As a test case, the notion of reflectivity is studied in this paper in relation to the question of whether free abelian groups of rank at least two embed in the automorphism group G of the structure. In fact, it turns out that reflectivity is rather weak. A major part of this paper is taken up with providing an example of reflective structures M for which every abelian subgroup H of $G = \mathrm{Aut}\, M$ is cyclic. Thus this example seems to be located at one extreme, being a stucture with a large automorphism group but nevertheless rather weakly saturated—the other extreme being the highly saturated $\aleph_0$-categorical structures and in particular the structure that has $\mathrm{Sym}\,\Omega$ as its automorphism group. Reflectivity does, however, suggest many interesting variations and strenthenings. It is left for future work to develop this notion to give stronger saturation conditions to enable other groups such as free-abelian groups of higher rank to embed, and possibly also to find a permutation group theoretic characterization of recursive saturation. This would in turn have important consequences in terms of recovering structure of a model from its automorphism group, as well as possibly giving new information and insights into the sometimes mysterious workings of recursive saturation.

2 Basic definitions

Throughout this paper, Ω will denote a countably infinite set. This set is both the underlying set (or *domain*) of all models we consider, and the basic set our groups act on. Lower case variables $a, b, c, \ldots, u, v, w, x, y, z, \ldots$ range over elements of Ω, except that if these variables are annotated with a bar, as in $\bar{a}$, they are thought to be elements of Ω^n for some $n \in \mathbb{N}$, where n is either completely general or is determined from the context. Much of sections 5 and 6 is concerned with free groups. In these sections, a *word in* $a, b, c, \ldots$ will mean a finite string of symbols taken from the *letters* $a, a^{-1}, b, b^{-1}, c, c^{-1}, \ldots$. In these contexts, we will usually identify each symbol or word with the permutation it represents.

Let G be a permutation group acting faithfully on Ω, i.e., $G \leqslant \mathrm{Sym}\,\Omega$. Any $G \leqslant \mathrm{Sym}\,\Omega$ acts on Ω^n for $n \in \mathbb{N}$ in the obvious way, and I will often implicitly refer to this action rather than the special case when $n = 1$. Both G and $\mathrm{Sym}\,\Omega$ can be regarded as topological groups in the normal way, by letting the basic open subgroups of $\mathrm{Sym}\,\Omega$ be stabilizers

$$\{g \in \mathrm{Sym}\,\Omega : ag = a \text{ for all } a \in A\}$$

of finite subsets A of Ω, open subsets of $\mathrm{Sym}\,\Omega$ being unions of cosets of these, and giving G the subspace topology. The subspace topology on G is also characterized by saying the basic open subgroups of G are the pointwise stabilizers $G_{(A)} \leqslant G$ of finite sets $A \subseteq \Omega$. (The stabilizer $G_{(A)}$ is also notated $G_{\bar{a}}$, when $A = \{a_1, a_2, \ldots, a_n\}$ and $\bar{a} = (a_1, a_2, \ldots, a_n) \in \Omega^n$, for some $n \in \mathbb{N}$.)

For $A \subseteq \Omega$, the *algebraic closure* of A is defined by

$$\operatorname{acl}(A) = \{b \in \Omega : bG_{\bar{a}} \text{ is finite, for some } \bar{a} \in A\}$$

where $\bar{a}$ ranges over finite tuples of elements from A. So, in the special case when A is finite, $\operatorname{acl} A = \{b : bG_{(A)} \text{ is finite}\}$. A point x is said to be *definable* over A if $xG_{\bar{a}} = \{x\}$ for some $\bar{a} \in A$; the set of points x definable over A is the *definable closure* of A, $\operatorname{dcl}(A)$. (Note that these are *permutation-group* definitions of algebraic and definable closure, and will only agree with the model theoretic definition in rather exceptional circumstances, such as G being oligomorphic.)

Using the identity $\left|bG_{(A)}\right| = [G_{(A)}{:}G_{(Ab)}]$ one can easily show that acl() is a closure operator, in the sense that $\operatorname{acl} X \subseteq \operatorname{acl} Y$ for $X \subseteq Y$ and $\operatorname{acl}\operatorname{acl} A = \operatorname{acl} A$. Note also that $\operatorname{acl} A = \bigcup\{\operatorname{acl} B : B \subseteq A \text{ finite}\}$; however, it is not necessarily true that $\operatorname{acl} \varnothing = \varnothing$.

A *relational structure* (sometimes called *model*) on Ω is a family

$$M = (\Omega, R_i)_{i \in I}$$

where each $R_i \subseteq \Omega^{n_i}$ for some $n_i \in \mathbb{N}$ and I is a (possibly empty) set. The pair (I, f), where $f{:}\, I \to \mathbb{N}$ is given by $f{:}\, i \mapsto n_i$ is called the *similarity type* of M. In this paper, the set I will always be countable.

Much of this paper is concerned with links between relational structures and permutation groups.

Given a relational structure $M = (\Omega, R_i)_{i \in I}$ on Ω, its *automorphism group* $\operatorname{Aut} M$, is defined to be the set of all permutations $g \in \operatorname{Sym}\Omega$ such that

$$\bar{a} \in R_i \Leftrightarrow \bar{a}g \in R_i$$

for all $\bar{a} \in \Omega^{n_i}$ and all $i \in I$.

In the other direction, given $G \leqslant \operatorname{Sym}\Omega$, there is a standard way of obtaining a relational structure M_G on Ω, called the *canonical structure*. Let I_G be the set of G-orbits $\mathscr{O} \subseteq \Omega^n$ for all possible $n \in \mathbb{N}$. Then define $R_{\mathscr{O}}$ to be precisely the orbit $\mathscr{O}$. The structure M_G is the family $(\Omega, R_{\mathscr{O}})_{\mathscr{O} \in I_G}$. According to standard model theoretic terminology, the canonical structure M_G is atomic, i.e., the set of first-order formulas true of any given $\bar{x} \in \Omega^n$ is determined by precisely one such formula, namely $R_{\mathscr{O}}(\bar{x})$ where $\mathscr{O}$ is the G-orbit of $\bar{x}$. It also has elimination of quantifiers.

For any $G \leqslant \operatorname{Sym}\Omega$ we have $G \leqslant \overline{G}$. In fact, $\overline{G}$ is the closure $\overline{G}$ of G in $\operatorname{Sym}\Omega$ (using the topology defined above), so $G = \overline{G}$ if and only if G is closed. In general, we have that

$$\overline{G} = \overline{G} = \{g \in \operatorname{Sym} M : \forall \bar{a} \in \Omega\, \exists h \in G\; \bar{a}g = \bar{a}h\}.$$

We will also be interested in the semigroup of embeddings of a stucture into itself. Given $M = (\Omega, R_i)_{i \in I}$ on Ω, define $\operatorname{Inj} M$, to be the semigroup of all injections $g{:}\, \Omega \to \Omega$ such that

$$\bar{a} \in R_i \Leftrightarrow \bar{a}g \in R_i$$

for all $\bar{a} \in \Omega^{n_i}$ and all $i \in I$. Similarly, we define $\operatorname{Inj}\Omega$ to be the semigroup of all injections $\Omega \to \Omega$. If $G \leqslant \operatorname{Sym}\Omega$ and M_G is the canonical structure, we denote $\overline{\overline{G}}$ by $\overline{\overline{G}}$. It turns out that

$$\overline{\overline{G}} = \{g\colon M \to M : \forall \bar{a} \in \Omega\, \exists h \in G\, \bar{a}g = \bar{a}h\}$$

and $\overline{\overline{G}}$ is the topological closure of G in $\operatorname{Inj}\Omega$ when this is regarded as a topological space in a way completely analogous to the topology on $\operatorname{Sym}\omega$.

In model theoretic terms, $\overline{\overline{G}}$ is the set of *elementary embeddings* of M_G into itself, since M_G has elimination of quantifiers. More generally, if we start with a relational structure M and let $G = \operatorname{Aut} M$, then embeddings in $\operatorname{Inj} M$ need not be elementary; however, $\overline{\overline{G}}$ is the set of elementary embeddings of the original structure M into itself.

The rest of this section concerns some background information on recursive saturation, as referred to in the introduction, and will not be used in the rest of the paper except to motivate definitions and theorems.

Assume that $\mathscr{L}$ is a first-order language with some Gödel-numbering of terms and formulas into $\mathbb{N}$. In the case when $\mathscr{L}$ is finite, we assume the Gödel-numbering is recursive. If there is no such recursive Gödel-numbering, an oracle G is added for this Gödel-numbering (so the word 'recursive' should be interpreted as 'recursive in G' throughout). Where convenient, we identify formulas with their Gödel-numbers, so for example, we may talk about a 'recursive set of formulas'.

A structure M is recursively saturated if for every recursive set $\{\phi_i(\bar{x}, \bar{y}) : i \in \mathbb{N}\}$ of $\mathscr{L}$-formulas in some finite number of free-variables shown, and for every $\bar{a} \in M$, the following infinitary sentence holds in M:

$$\bigwedge_{n\in\mathbb{N}} \exists \bar{x} \bigwedge_{i<n} \phi_i(\bar{x}, \bar{a}) \to \exists \bar{x} \bigwedge_{i\in\mathbb{N}} \phi_i(\bar{x}, \bar{a}).$$

By a result of Ressayre, Barwise and Schlipf [2, Theorem 15.7], a countable model is recursively saturated if and only if whenever $\bar{a} \in M$ and T is a recursively axiomatized theory in some recursive extension $\mathscr{L}'$ of $\mathscr{L} \cup \{\bar{a}\}$ and $T + \operatorname{Th}(M, \bar{a})$ is consistent, then there is an expansion $(M, \bar{a}, \ldots) \vDash T$. (This property is called *resplendency*.) Moreover, the expansion $(M, \bar{a}, \ldots) \vDash T$ here can always be chosen to be recursively saturated, so the process may be iterated. (This strengthening of resplendency is called *chronic resplendency*.)

Chronic resplendency is a very powerful tool for obtaining results about automorphism groups. I would like to illustrate it by sketching a third proof of the result alluded to in the introduction that the free-abelian group of rank ω embeds in $\operatorname{Aut} M$ for all countably infinite recursively saturated M.

First, by chronic resplendency, we may expand M to $(M, <)$ where $<$ is a linear order and $(M, <)$ is recursively saturated. Next, by resplendency again, we may take some nontrivial $g_0 \in \operatorname{Aut}(M, <)$ such that $(M, <, g_0)$ is recursively saturated. Since g_0 preserves a linear order it generates an infinite cyclic group. Suppose $x_0 \in M$ is moved by g_0. By resplendency once again, there is $g_1 \in \operatorname{Aut}(M, <, g_0)$ with $x_0 g_1 > x_0 g_0^n$ for all $n \in \mathbb{Z}$ and, again, we may take $(M, <, g_0, g_1)$ to be recursively saturated. (This is the first place in the argument, perhaps, where consistency isn't immediate, but it is very easy to see: for any finite set of integers S there is some $k \in \mathbb{Z}$ such that $x_0 g_0^k > x_0 g_0^n$ for all $n \in S$.

Thus consistency follows from compactness and taking $g_1 = g_0^k$ for suitable k.) Because g_1 is an automorphism of (M, g_0), g_0 and g_1 commute, and clearly the group generated by g_0, g_1 is not cyclic. Continuing in this way we find generators $g_0, g_1, g_2, \ldots$ for the required abelian subgroup.

3 Proper elementary embeddings

We start our discussion proper with the question: for what permutation groups $G \leqslant \operatorname{Sym}\Omega$ is it the case that there is an embedding in $\overline{\overline{G}}$ which is not a permutation?

The key to answering this question is the notion of algebraic closure. We work with a group $G \leqslant \operatorname{Sym}\Omega$ throughout this section.

Lemma 3.1. Suppose $f \in \overline{\overline{G}}$ and suppose there are $a, \bar{b} \in \Omega$ with $\bar{b} \in \operatorname{im}(f)$ and $a \notin \operatorname{im}(f)$. Then $a \notin \operatorname{acl}(\bar{b})$.

Proof. Suppose otherwise, i.e., suppose $a \in \operatorname{acl}(\bar{b})$, and let $a = a_0, a_1, \ldots, a_{k-1}$ list the $G_{\bar{b}}$-orbit of a. Let $\bar{c}$ be the tuple of elements of Ω such that $\bar{b} = \bar{c}f$ and take $g \in G$ with $\bar{c}g = \bar{b}$.

Now, we claim that

$$\text{for all } h \in G \text{ with } \bar{c}h = \bar{b},\ a \in \{a_0 g^{-1} h, a_1 g^{-1} h, \ldots, a_{k-1} g^{-1} h\}$$

since, given such h, we have $h^{-1}g \in G_{\bar{b}}$ and hence $ah^{-1}g \in \{a_0, a_1, \ldots, a_{k-1}\}$. Applying $g^{-1}h$ gives the claim.

On the other hand, as $a \notin \operatorname{im} f$, $a \notin \{a_0 g^{-1} f, a_1 g^{-1} f, \ldots, a_{k-1} g^{-1} f\}$. It follows that there is no $h \in G$ which agrees with f on the tuple

$$a_0 g^{-1}, a_1 g^{-1}, \ldots, a_{k-1} g^{-1}, \bar{c}$$

contradicting $f \in \overline{\overline{G}}$. □

For the next lemma, choose an enumeration $(x_i : i \in \mathbb{N})$ of Ω.

Lemma 3.2. Suppose $f \in \overline{\overline{G}}$ and $a \notin \operatorname{im} f$. Then there are orbits

$$\mathscr{O}_0 \supseteq \mathscr{O}_1 \supseteq \cdots \supseteq \mathscr{O}_i \supseteq \cdots$$

such that $\mathscr{O}_i$ is an infinite $G_{x_0, x_1, \ldots, x_{i-1}}$-orbit and $\mathscr{O}_0 = aG$, the G-orbit of a.

Proof. Given i, take $g_i \in G$ such that

$$(x_0, x_1, \ldots, x_{i-1}) g_i = (x_0, x_1, \ldots, x_{i-1}) f.$$

In particular, when $i = 0$ the sequence $(x_0, x_1, \ldots, x_{i-1})$ is empty, so g_0 could be any element of G. For this case, we choose $g_0 = 1$.

By the previous lemma,

$$\mathscr{U}_i = aG_{x_0 f, x_1 f, \ldots, x_{i-1} f}$$

is infinite. Let

$$\mathscr{O}_i = a g_i^{-1} G_{x_0, x_1, \ldots, x_{i-1}}$$

and note that

$$\mathscr{O}_i = a g_i^{-1} G_{x_0 f g_i^{-1}, x_1 f g_i^{-1}, \ldots, x_{i-1} f g_i^{-1}} = a g_i^{-1} g_i G_{x_0 f, x_1 f, \ldots, x_{i-1} f} g_i^{-1} = \mathscr{U}_i g_i^{-1},$$

so each $\mathscr{O}_i$ is infinite. It is only required to check that $\mathscr{O}_i \supseteq \mathscr{O}_{i+1}$ for each i. But $g_i g_{i+1}^{-1} \in G_{x_0, x_1, \ldots, x_{i-1}}$, so

$$\mathscr{O}_i = a g_i^{-1} g_i g_{i+1}^{-1} G_{x_0, x_1, \ldots, x_{i-1}} = a g_{i+1}^{-1} G_{x_0, x_1, \ldots, x_{i-1}} \supseteq \mathscr{O}_{i+1}$$

as required. □

Definition 3.3. Given $G \leqslant \operatorname{Sym} \Omega$, G will be said to have the *descending chain condition on suborbits (DCC)* if whenever

$$\mathscr{O}_0 \supseteq \mathscr{O}_1 \supseteq \cdots \supseteq \mathscr{O}_i \supseteq \cdots,$$

each $\mathscr{O}_i$ being $a_i G_{(X_i)}$ for some $a_i \in \Omega$ and some chain $X_0 \subseteq X_1 \subseteq \cdots \subseteq X_i \subseteq \cdots$ of finite sets with $\bigcup_{i \in \mathbb{N}} X_i = \Omega$, then for some $n \in \mathbb{N}$

$$\mathscr{O}_n = \mathscr{O}_{n+1} = \cdots = \mathscr{O}_{n+i} = \cdots.$$

If the chain of suborbits $\mathscr{O}_i$ is eventually constant, then $\mathscr{O}_i$ must be a singleton for all i greater than some n, since $\bigcup_i X_i = \Omega$. Thus, another way of stating the DCC on suborbits is that there is no descending sequence $\mathscr{O}_i$ of $G_{(X_i)}$-orbits with $\bigcap_i \mathscr{O}_i = \varnothing$. (In this alternative formulation, the X_is can be any chain of finite subsets of Ω and not necessarily having union equal to Ω.)

Theorem 3.4. If $G \leqslant \operatorname{Sym} \Omega$ is a permutation group, then $\overline{\overline{G}} = \overline{G}$ if and only if G has the descending chain condition on suborbits.

Proof. First, suppose $\overline{\overline{G}} \neq \overline{G}$ and note that the proof of Lemma 3.2 is independent of the choice of enumeration $(x_i : I \in \mathbb{N})$ of Ω. So if $X_0 \subseteq X_1 \subseteq \cdots \subseteq X_i \subseteq \cdots$ is some chain of finite sets with $\bigcup_{i \in \mathbb{N}} X_i = \Omega$, we may choose an enumeration $(x_i : i \in \mathbb{N})$ of Ω so that the first $|X|_0$ elements of the enumeration are precisely the elements of X_0, the first $|X|_1$ elements of the enumeration are precisely the elements of X_1, and so on. Then the hypothesis that there is some $f \in \overline{\overline{G}} \setminus \overline{G}$ together with Lemma 3.2 implies the existence of $a \in \Omega$ and a descending chain

$$\mathscr{O}_0 \supseteq \mathscr{O}_1 \supseteq \cdots \supseteq \mathscr{O}_i \supseteq \cdots$$

with $\mathscr{O}_0 = aG$ and each $\mathscr{O}_i$ is infinite. Hence no $\mathscr{O}_i$ is a singleton, so the descending chain condition fails.

For the other direction, suppose

$$x_0, x_1, \ldots, x_n, \ldots$$

enumerates Ω and, for each i, $\mathscr{O}_i$ is an infinite $G_{x_0, \ldots, x_{i-1}}$-orbit with

$$\mathscr{O}_0 \supseteq \mathscr{O}_1 \supseteq \cdots \supseteq \mathscr{O}_i \supseteq \cdots.$$

We must use this to obtain a suitable embedding.

In the following inductive construction, at stage $i \in \mathbb{N}$ we will have determined elements $a_0, a_1, \ldots, a_i \in \Omega$ and $g_0, g_1, \ldots, g_{i-1} \in G$ such that $a_j \in \mathscr{O}_j$ for all $j \leqslant i$, and $a_{j+1} = a_j g_j$ and $g_j \in G_{x_0,x_1,\ldots,x_{j-1}}$ for all $j < i$. To start things off, take for a_0 any member of $\mathscr{O}_0$.

The construction is this: given $a_0, a_1, \ldots, a_i$ and $g_0, g_1, \ldots, g_{i-1}$ as above, choose $a_{i+1} \in \mathscr{O}_{i+1}$. Since $\mathscr{O}_{i+1} \subseteq \mathscr{O}_i = a_i G_{x_0,x_1,\ldots,x_{i-1}}$ there is $g_i \in G_{x_0,x_1,\ldots,x_{i-1}}$ such that $a_{i+1} = a_i g_i$.

When the construction is complete for all i, define $f \in \overline{\overline{G}}$ by

$$x_n f = x_n g_n^{-1} g_{n-1}^{-1} \cdots g_0^{-1}.$$

Note that as $x_n g_k = x_n$ for all $k > n$, we have

$$x_n f = x_n g_k^{-1} g_{k-1}^{-1} \cdots g_0^{-1}$$

for all $k \geqslant n$. So, given $\bar{c} \in \Omega^n$, take $n \in \mathbb{N}$ sufficiently large so that $\{\bar{c}\} \subseteq \{x_0, x_1, \ldots, x_n\}$. Then $\bar{c}f = \bar{c}g_n^{-1}g_{n-1}^{-1} \cdots g_0^{-1}$ showing $f \in \overline{\overline{G}}$.

On the other hand, $a_0 \notin \operatorname{im} f$, for if $a_0 = x_n g_n^{-1} g_{n-1}^{-1} \cdots g_0^{-1}$ then $x_n = a_0 g_0 g_1 \cdots g_n = a_{n+1}$ showing that $\mathscr{O}_{n+1} = a_{n+1} G_{x_0,x_1,\ldots,x_n} = \{x_n\}$ is finite, a contradiction. □

Definition 3.5. Let $G \leqslant \operatorname{Sym}\Omega$. A G-orbit aG is said to be *egressive* if there is an enumeration $(x_i : i \in \mathbb{N})$ of Ω and

$$\mathscr{O}_0 \supseteq \mathscr{O}_1 \supseteq \cdots \supseteq \mathscr{O}_i \supseteq \cdots$$

such that $\mathscr{O}_i$ is an infinite $G_{x_0,x_1,\ldots,x_{i-1}}$-orbit and $\mathscr{O}_0 = aG$.

Equivalently, aG is egressive if there is a descending chain

$$\mathscr{O}_0 = aG \supseteq \mathscr{O}_1 \supseteq \cdots \supseteq \mathscr{O}_i \supseteq \cdots$$

with empty intersection, where $\mathscr{O}_i$ is a $G_{(X_i)}$-orbit for each i, and $X_0 \subseteq X_1 \subseteq \cdots \subseteq X_i \subseteq \cdots$ is a chain of finite subsets of Ω.

By preceding results, the orbit aG is egressive if and only if there is $f \in \overline{\overline{G}}$ with $a \notin \operatorname{im} f$. In model theoretic terminology (using the fact that M_G is atomic and has elimination of quantifiers) this becomes

Corollary 3.6. Let $G \leqslant \operatorname{Sym}\Omega$ be closed. Then

(i) If G is the automorphism group of a countable atomic relational structure M, then M is *minimal* (i.e., does not possess a proper elementary substructure) if and only if G has the descending chain condition on suborbits.

(ii) For $a \in \Omega$ and any atomic relational structure M on Ω with $\operatorname{Aut} M = G$, there is $K \prec M$ with $a \notin K$ if and only if aG is egressive.

Proof. For (ii), suppose aG is egressive. Then there is $f \in \overline{\overline{G}}$ with $a \notin \operatorname{im} f$ and, since $G = \operatorname{Aut} M$, the map f is an elementary embedding of M into itself. For the other direction, given an elementary substructure $K \prec M$ with $a \notin K$, use atomicity and the fact that K is elementary in M to obtain an elementary embedding h of M into K, so $a \notin \operatorname{im} h$. Then $h \in \overline{\overline{G}}$ and, by Lemma 3.2, aG is egressive.

Part (i) follows immediately from (ii). □

Consider for example $G = \mathrm{Aut}(\mathbb{Z} \times \mathbb{Z}, <)$, where $\mathbb{Z} \times \mathbb{Z}$ is linearly ordered as $\mathbb{Z}$ blocks of $\mathbb{Z}$. For $a_1, a_2, \ldots, a_n, b \in \mathbb{Z} \times \mathbb{Z}$, the orbit $bG_{a_1,a_2,\ldots,a_n}$ is either a singleton (when b is in the same block as one of the a_i) or equals the whole of the block of b. So, even if $\mathscr{O} = bG_{a_1,a_2,\ldots,a_n}$ is infinite, for all $c \in \mathscr{O}$ we have $cG_{a_1,a_2,\ldots,a_n,b} = \{c\}$. Hence the DCC on suborbits holds.

This example shows that there may be sequences $a_0, a_1, \ldots, a_n$ and $b_0, b_1, \ldots, b_n$ of any finite length such that $b_{i+1} \in b_i G_{a_0,\ldots,a_{i-1}}$ and $b_i G_{a_0,\ldots,a_{i-1}}$ is infinite for all i, even if the orbit in question is not egressive. Note also that the pointwise stabilizers $G_{\bar{a}}$ of finite sets are all nontrivial in this example, so $|G| = 2^{\aleph_0}$.

4 Properties of groups with egressive orbits

Let $G \leqslant \mathrm{Sym}\,\Omega$ and $x, \bar{a} \in \Omega$. Terminology will be abused slightly, saying '$xG_{\bar{a}}$ is egressive' instead of '$xG_{\bar{a}}$ is an egressive $G_{\bar{a}}$-orbit'. The following important property of egressive orbits is clear from the discussion in the previous section.

FACT 4.1. If $xG_{\bar{a}}$ is egressive and $b \in \Omega$ then there is $y \in xG_{\bar{a}}$ such that $yG_{\bar{a},b}$ is egressive. The point y here is the image xg of x under some $g \in G_{\bar{a}}$.

In this section I will use this basic fact to derive some interesting properites of groups $G \leqslant \mathrm{Sym}\,\Omega$ with egressive orbits.

PROPOSITION 4.2. If $G \leqslant \mathrm{Sym}\,\Omega$ has an egressive orbit then for each finite $A \subseteq \Omega$ there is a nontrivial $g \in G_{(A)}$.

Proof. Let xG be egressive. By the fact above there is $y \in xG$ with $yG_{(A)}$ egressive, and $z \in yG_{(A)}$ with $zG_{(A\cup\{y\})}$ egressive. So there is $g \in G_{(A)}$ with $yg = z \neq y$ as required. □

It follows by the usual tree argument that for such G, if G is closed then $|G| = 2^{\aleph_0}$. Theorem 4.4 will improve on this though.

The next proposition shows that infinite cyclic groups embed in the most natural way possible.

PROPOSITION 4.3. Let $G \leqslant \mathrm{Sym}\,\Omega$ be closed and have an egressive orbit aG. Then there is $g \in G$ such that the g-cycle containing a,

$$\{\ldots, ag^{-1}, a, ag, ag^2, \ldots\}$$

is infinite.

Proof. Let $a \in \Omega$ have egressive orbit. The permutation g is constructed by back-and-forth: at any stage of the construction we have $\bar{u}, \bar{v} \in \Omega^{<\omega}$ and $(a_0, a_1, \ldots, a_n) \in \Omega$ with

$$(\bar{u}, a_0, a_1, \ldots, a_{n-1}) \equiv (\bar{v}, a_1, a_2, \ldots, a_n)$$

(where $\bar{r} \equiv \bar{s}$ denotes these tuples are in the same G-orbit on Ω^k for some k) and the $G_{\bar{u},a_0,a_1,\ldots,a_{n-1}}$-orbit of a_n is egressive.

Forth. Let $w \in \Omega$. We require $w' \in \Omega$ and a_{n+1} with

$$(\bar{u}, w, a_0, a_1, \ldots, a_n) \equiv (\bar{v}, w', a_1, a_2, \ldots, a_{n+1})$$

and

$$a_{n+1}G_{\bar{u},w,a_0,a_1,\ldots,a_n}$$

egressive. Let $h \in G$ satisfy $\bar{u}h = \bar{v}$ and $a_ih = a_{i+1}$ all $i < n$. As a_n has an egressive $G_{\bar{u},a_0,\ldots,a_{n-1}}$-orbit, the orbit $a_nhG_{\bar{v},a_1,\ldots,a_n}$ is also egressive, so there is $k \in G_{\bar{v},a_1,\ldots,a_n}$ with

$$a_nhkG_{\bar{u},w,a_0,a_1,\ldots,a_n}$$

egressive. Put $w' = whk$ and $a_{n+1} = a_nhk$.

Back. Suppose again that

$$(\bar{u}, a_0, a_1, \ldots, a_{n-1}) \equiv (\bar{v}, a_1, a_2, \ldots, a_n)$$

and

$$a_nG_{\bar{u},a_0,a_1,\ldots,a_{n-1}}$$

is egressive. Let $w \in \Omega$. We require here $w' \in \Omega$ with

$$(\bar{u}, w', a_0, a_1, \ldots, a_{n-1}) \equiv (\bar{v}, w, a_1, a_2, \ldots, a_n)$$

and

$$a_nG_{\bar{u},w',a_0,a_1,\ldots,a_{n-1}}$$

egressive. Let $h \in G$ with $\bar{u}h = \bar{v}$ and $a_ih = a_{i+1}$ all $i < n$. As $a_nG_{\bar{u},a_0,a_1,\ldots,a_{n-1}}$ is egressive there is $k \in G_{\bar{u},a_0,a_1,\ldots,a_{n-1}}$ with $a_nkG_{\bar{u},wh^{-1},a_0,a_1,\ldots,a_{n-1}}$ egressive, so $a_nG_{\bar{u},wh^{-1}k^{-1},a_0,a_1,\ldots,a_{n-1}}$ is egressive. We take $w' = wh^{-1}k^{-1}$. $\square$

The last result here is a improved version of a result due to Macpherson [4]. (Macpherson proved that if $G \leqslant \operatorname{Sym}\Omega$ is closed and oligomorphic then there is a dense free subgroup $F \leqslant G$, which is free of rank $\aleph_0$.)

THEOREM 4.4. Let $G \leqslant \operatorname{Sym}\Omega$ be a closed permutation group with an egressive orbit. Then for each κ with $\aleph_0 \leqslant \kappa \leqslant 2^{\aleph_0}$ there is a dense subgroup $F_\kappa \leqslant G$, where F_κ is free of rank κ.

Proof. The construction is by infinitely many back-and-forth arguments. At each stage i we will have started finitely many such back and forth systems going, i.e., we will have finitely many finite maps

$$\alpha_j\colon \bar{c}_j \mapsto \bar{d}_j \qquad (*)$$

each extendible to a permutation in G, where the indices j are all sequences of integers of the form $(n_0, n_1, \ldots, n_{i-1})$, with $0 \leqslant n_0 \leqslant i-1$ and $n_1, n_2, \ldots n_{i-1} \in \{0, 1\}$. To pass from stage i to stage $i+1$ we will extend each of the previously constructed α_j in two different ways to $\alpha_{j\frown 0}$ and $\alpha_{j\frown 1}$, and also add 2^{i+1} new finite maps $\alpha_{i,n_1,\ldots,n_i}$. At the end of the construction we will have α_js indexed by nodes on countably many complete binary trees. In the back-and-forth part of the construction we will arrange that: (1) for every $n \in \mathbb{N}$ and every $f\colon \{1, 2, \ldots\} \to \{0, 1\}$ there is a unique $\beta_{n,f} \in G$ extending each $\alpha_{n,f(0),\ldots,f(i)}$; (2) all these $\beta_{n,f}$ freely generate a subgroup of G (which has rank $2^{\aleph_0}$); and (3) if we take any subset

$$S \subseteq \{\beta_{n,f} : n \in \mathbb{N},\ f\colon \{1, 2, \ldots\} \to \{0, 1\}\}$$

such that for all $n \in \mathbb{N}$ there is some $\beta_{n,f} \in S$, then S is dense in G. This suffices, since there are such sets S of cardinality κ for each cardinal κ with $\aleph_0 \leqslant \kappa \leqslant 2^{\aleph_0}$.

Most of the parts of the construction at stage i are standard. To arrange part (1) in the last paragraph we simply need to ensure u_i is in both the domain and range of the finite maps α_j, where u_i is the ith element in some predefined enumeration of Ω. This is the usual 'back-and-forth' construction and can always be fulfilled since by our inductive assumption each of the α_js extend to some permutation of Ω in G and since G is closed. To arrange part (3), at stage i we introduce a new $\alpha_i : \bar{v}_i \to \bar{w}_i$, where $(\bar{v}_i, \bar{w}_i)$ is the ith element in some predetermined enumeration of

$$\{(\bar{v}, \bar{w}) : \bar{v}g = \bar{w} \in \Omega^n \text{ for some } g \in G \text{ and } n \in \mathbb{N}\}.$$

All these finite maps are then extended to indices $(n_0, n_1, \ldots, n_i)$, with $0 \leqslant n_0 \leqslant i$ and $n_1, n_2, \ldots n_i \in \{0, 1\}$ in any way we like.

The other part of the construction is to ensure that some nontrivial reduced word in these α_j's is forced to determine a nontrivial permutation of Ω. To do this, we consider an arbitrary nontrivial reduced word w (the ith word in a pre-chosen enumeration) in the α_j. Note in particular that as there are only countably many incides $j = (n_0, n_1, \ldots, n_{i-1})$ in the whole construction all such words can indeed be enumerated; we will arrange that for *any* extension of the α_js to permutations in G the permutation corresponding to w will not be the identity—it will follow from this that the $\beta_{n,f}$s mentioned in the last paragraph freely generate a subgroup of G.

Let

$$w = \beta_0 \beta_1 \cdots \beta_{k-2} \beta_{k-1}$$

in reduced form, where each β_m is $\alpha_j^{\pm 1}$ some j. 'Reduced form' means two $\alpha_j^{\pm 1}$ can only appear next to one another if they have the same exponent, $+1$ or -1. Let $\bar{a}$ be a tuple formed from the finitely many elements of Ω that appear in $(*)$. We construct $x_0, \ldots, x_k \in \Omega$ inductively (in a way similar to the proof in Proposition 4.3) so that all of the data

$$\beta_m : x_m \mapsto x_{m+1}$$

is consistent with the finite maps $(*)$ for the α_js already fixed and so that

$$x_m G_{\bar{a}, x_0, x_1, \ldots, x_{m-1}} \text{ is egressive} \qquad \text{for each } m. \tag{$\dagger$}$$

Note that this condition obviously implies that the x_m are all distinct.

We may pick for x_0 any element of Ω such that $x_0 G_{\bar{a}}$ is egressive.

Now suppose $x_0, \ldots, x_m$ are given as above; without serious loss of generality suppose β_m is α_1, and we have already fixed

$$\alpha_1 : \bar{c}_1, \bar{u} \mapsto \bar{d}_1, \bar{v} \tag{$\ddagger$}$$

where $\bar{u}, \bar{v}$ are elements of $\{x_0, x_1, \ldots, x_m\}$ corresponding to previous β's equal to $\alpha_1^{\pm 1}$. Let

$$h : \bar{c}_1, \bar{u} \mapsto \bar{d}_1, \bar{v}$$

be any element of G. We want $y \in x_m G_{\bar{c}_1, \bar{u}}$ such that

$$y G_{\bar{a}h^{-1}, x_0 h^{-1}, \ldots, x_m h^{-1}} \text{ is egressive},$$

for then the element $x_{m+1} = yh$ of Ω has an egressive $G_{\bar{a},x_0,\dots,x_m}$-orbit, and also for some $h' \in G_{\bar{c}_1,\bar{u}}$, $x_j h' = y$ so

$$h'h: \bar{c}_1, \bar{u}, x_m \mapsto \bar{d}_1, \bar{v}, x_{m+1}.$$

If $x_m G_{\bar{c}_1,\bar{u}}$ is egressive we can always find such a y. But the orbit $x_m G_{\bar{a},x_0,\dots,x_{m-1}}$ is by assumption egressive, and $\{\bar{c}_1, \bar{u}\} \subseteq \{\bar{a}, x_0, \dots, x_m\}$, so the orbit $x_m G_{\bar{c}_1,\bar{u}}$ is egressive just in case $x_m \notin \{\bar{u}\}$. Now, $x_m \in \{\bar{u}\}$ if and only if $m > 0$ and $\beta_{m-1}: x_{m-1} \mapsto x_m$ is the map α_1^{-1}, in which case w would not be in reduced form, which is impossible. So $x_m G_{\bar{c}_1,\bar{u}}$ is egressive, as required.

Since egressive orbits are infinite, we have infinitely many possible choices for the final x_k, and for at least one of these choices we have

$$x_0 w = x_0 \beta_0 \beta_1 \cdots \beta_{k-1} = x_k \neq x_0.$$

We add all the data

$$\beta_m: \bar{r}_m \mapsto \bar{s}_m$$

(where $\bar{r}_m, \bar{s}_m \in \{\bar{x}\}$) determined during this construction to our finite maps $(*)$ and continue to the next stage $i+1$. □

5 Reflective permutation groups

In this section the ideas of the previous two sections are developed to give the notion of a reflective permutation group or structure.

First observe that, if $G = \mathrm{Aut}\, M$ where M is a recursively saturated structure with domain Ω, then for all $\bar{a} \in \Omega$ the orbit $xG_{\bar{a}}$ is egressive if and only if it is infinite. (This is easy to see using resplendency and a compactness argument.) I formalize this as a definition.

Definition 5.1. Given $G \leqslant \mathrm{Sym}\,\Omega$, an infinite $G_{\bar{a}}$-orbit $\mathscr{O} = bG_{\bar{a}}$ is *minimal* if for some $c \in \Omega$ there is no infinite suborbit $b'G_{\bar{a},c}$ for $b' \in \mathscr{O}$. Obviously, a minimal $G_{\bar{a}}$-orbit cannot be egressive.

The group G is *reflective* if the following conditions hold.

R1) $\mathrm{acl}(\varnothing) \neq \Omega$, i.e., there is an infinite G-orbit $aG \subseteq \Omega$.

R2) G has no minimal infinite suborbits, i.e., whenever $a, \bar{b} \in \Omega$ with $aG_{\bar{b}}$ infinite (i.e., $a \notin \mathrm{acl}(\bar{b})$) and $c \in \Omega$ there is $a' \in aG_{\bar{b}}$ such that $a'G_{\bar{b},c}$ is infinite.

A relational structure M on Ω will also be said to be reflective if its automorphism group $\mathrm{Aut}\, M$ satisfies R1, R2 above.

The word 'reflective' is used here because elementary embeddings of M into itself are reminiscent of reflections of M inside itself, and because of the analogy with reflection principles in set theory. Also, reflective models and their automorphism groups are apparently rather rich, but in general 'not-so-radiant' as resplendent models, as we shall see.

Proposition 5.2. Property R2 holds of a group $G \leqslant \mathrm{Sym}\,\Omega$ if and only if for all $a, \bar{b} \in \Omega$, $a \notin \mathrm{acl}(\bar{b})$ implies there is $f \in \overline{\overline{G}}$ with $\bar{b}f = \bar{b}$ and $a \notin \mathrm{im}\, f$.

Proof. Given R2, $\Omega = \{x_i : i \in \mathbb{N}\}$, and given $a \notin \operatorname{acl}(\bar{b})$, obtain a descending sequence $\mathscr{O}_i = a_i G_{\bar{b},x_0,x_1,\ldots x_{i-1}}$ using R2, and then apply results in the last section to the group $G_{\bar{b}}$. Conversely, given a descending sequence $aG_{\bar{b}} \supseteq \cdots \supseteq \mathscr{O}_i = a_i G_{\bar{b},x_0,x_1,\ldots x_{i-1}} \supseteq \cdots$ of infinite orbits and given $c = x_n$, consider a_{n+1} which is in $aG_{\bar{b}}$ and not algebraic over $\bar{b}, c$. $\square$

Slight improvements of results in the previous section hold for reflective permutation groups. For example, if $G \leqslant \operatorname{Sym}\Omega$ is reflective and $a, \bar{b} \in \Omega$ with $aG_{\bar{b}}$ infinite, then there is $g \in \overline{G_{\bar{b}}}$ with the g-orbit of a, $\{ag^n : n \in \mathbb{Z}\}$, infinite. In particular, for reflective $G \leqslant \operatorname{Sym}\Omega$, the notion of algebraic closure of a finite set A corresponds to

$$\{x \in \Omega : \exists g \in G_{(A)} : \{xg^n : n \in \mathbb{Z}\} \text{ is infinite}\}$$

I omit the details.

Of more interest is the strengthening of Theorem 4.4. In the context of reflective groups this becomes,

THEOREM 5.3. Let $G \leqslant \operatorname{Sym}\Omega$ be a closed reflective permutation group. Then there is a dense free subgroup $F \leqslant G$ of rank $\aleph_0$ (or indeed, of any other rank κ with $\aleph_0 \leqslant \kappa \leqslant 2^{\aleph_0}$) such that for all $w \neq 1$ from F, all $n \in \mathbb{N}$, all $\bar{a} \in \Omega^n$, and all $b \in \Omega \setminus \operatorname{acl}(\bar{a})$ there is $c \in bG_{\bar{a}}$ with $cw \neq c$.

Proof. Exactly as before, but for each nontrivial reduced word w and each $\bar{a}, b$ as above arrange also that $cw \neq c$ for some such c. To do this, use the fact that $bG_{\bar{a}}$ is an egressive $G_{\bar{a}}$ orbit, hence for any finite collection of parameters (the parameters named at that particular point in the construction), $\bar{d}$, there is $c \in bG_{\bar{a}}$ such that $c \notin \operatorname{acl}(\bar{d})$. The previous proof shows how to arrange that $cw \neq c$. $\square$

The rest of this section is devoted to some straightforward examples.

EXAMPLE 5.4. If Ω is the disjoint union of countably many infinite Ω_i $(i \in \mathbb{N})$ and $G = \prod_i \operatorname{Sym}\Omega_i$ with its natural intransitive action on Ω, then G is clearly reflective, but not oligomorphic.

In fact, in this case rather more is true: $G = \operatorname{Aut} M$ for some recursively saturated structure M on Ω. Here, take P_i to be a unary predicate representing Ω_i for all $i \geqslant 1$, i.e., for all i except 0. The complete theory of the structure M is given by the statements that the P_i are all infinite and pairwise disjoint. (This theory has elimination of quantifiers and is readily shown to be complete.) Thus there is a countable recursively saturated model K which (by considering certain obvious types) must have infinitely many elements not in any P_i. In fact, K is $\aleph_0$-saturated. Clearly K is isomorphic to our M, and hence M is recursively saturated.

On the other hand, examples of non-reflective structures with egressive orbits can be found by just taking the cartesian product of a reflective group $G_1 \leqslant \operatorname{Sym}\Omega_1$ with a nonreflective $G_2 \leqslant \operatorname{Sym}\Omega_2$ and the obvious intransitive action on the disjoint union $\Omega_1 \cup \Omega_2$. For a transitive example, take $\Omega = (\mathbb{Z} \times \mathbb{Z}) \times \Omega_1$ where Ω_1 is infinite, and the natural imprimitive action of the full wreath product $G = \operatorname{Aut}(\mathbb{Z} \times \mathbb{Z}, <) \operatorname{Wr} \operatorname{Sym}\Omega_1$ on Ω. Then each orbit xG is egressive, but if

x, y are in the same $\mathbb{Z} \times \mathbb{Z}$ block, xG_y is never egressive, even though it may be infinite.

The next example shows that reflective groups need not be the automorphism group of any recursively saturated structure.

EXAMPLE 5.5. Let $\mathscr{L}$ be the language with unary predicates $P_1, \ldots, P_n, \ldots$, and Q, and a single binary predicate R. Let T be the $\mathscr{L}$-theory stating that:

1. there are precisely $i+1$ elements satisfying P_i;
2. $P_1, \ldots, P_n, \ldots$ and Q are pairwise disjoint;
3. $\forall x, y\, (R(x, y) \to \neg Q(x) \wedge Q(y))$;
4. $\forall x, y\, (Q(x) \wedge Q(y) \to (x = y \leftrightarrow \forall z\, (R(z, x) \leftrightarrow R(z, y))))$;
5. $\bigwedge_i \neg Q(r_i) \to \exists x \left(\bigwedge_i R(r_i, x) \wedge \bigwedge_j \neg R(s_j, x) \right)$ for all disjoint finite $\{r_1, \ldots, r_n\}$ and $\{s_1, \ldots, s_m\}$.

Think of R as a 'membership' relation, like $\in$ in set theory. Then item 4 is a sort of extensionality axiom. The theory T has quantifier-elimination and is complete. Take a recursively saturated model M^* of T with domain Ω^*, and choose a substructure M with domain Ω consisting of all elements satisfying P_i in M^* for some i, and for each $S \subseteq \bigcup_i P_i$ exactly one $x \in \Omega^*$ such that for all $y \in \bigcup_i P_i$

$$M^* \models R(y, x) \Leftrightarrow y \in S.$$

Then $M \models T$, as the reader may verify.

Note that any $\pi \in \prod_i \operatorname{Sym} P_i$ extends uniquely to an automorphism of M (and also extends—but this time not uniquely—to one of M^*). Thus there are infinite $\operatorname{Aut} M$-orbits of Ω, and in fact $G = \operatorname{Aut} M \leqslant \operatorname{Sym} \Omega$ is reflective. But, in M, $\operatorname{acl}(\varnothing)$ contains each P_i, so by the extensionality axiom there is no nontrivial $g \in G$ fixing $\operatorname{acl}(\varnothing)$ pointwise.

To see $G = \operatorname{Aut} M$ is reflective, note that each $\bar{a} \in M$ induces a equivalence relation $E_i(\bar{a})$ of P_i (defined by $x \sim y$ iff $R(x, b) \leftrightarrow R(y, b)$ for each $b \in \{\bar{a}\} \setminus \bigcup_i P_i$ and $x = c \leftrightarrow y = c$ for each $c \in \{\bar{a}\} \cap \bigcup_i P_i$) such that $g \in G$ fixes $\bar{a}$ pointwise iff it preserves each $E_i(\bar{a})$. Therefore an orbit $xG_{\bar{a}}$ is infinite if and only if $x \notin \bigcup_i P_i$ and $E_i(\bar{a}, x)$ is finer than $E_i(\bar{i})$ for infinitely many i. It is then obvious that there are infinite orbits $xG_{\bar{a}}$ for all $\bar{a}$, and that, given b, any such can always be refined to some infinite $x' G_{\bar{a}, b}$.

Compare this example with with the fact that, if M is recursively saturated, there is always some nontrivial $g \in \operatorname{Aut} M$ fixing all algebraic points. It is an easy matter to modify the definition of 'reflective' to eliminate examples such as that just given.

DEFINITION 5.6. A permutation group $G \leqslant \operatorname{Sym} \Omega$ is *strongly reflective* if:

R1$^+$) $G_{(\operatorname{acl}(\varnothing))} \neq \{1\}$, i.e., there are $x \in \Omega$ and $g \in G$ such that $yg = y$ for all $y \in \operatorname{acl}(\varnothing)$ and $xg \neq x$; and

R2$^+$) whenever $a, \bar{b} \in \Omega$ with $aG_{(\operatorname{acl}(\bar{b}))} \neq \{1\}$ and $c \in \Omega$ there is $a' \in aG_{(\operatorname{acl}(\bar{b}))}$ such that $a' G_{(\operatorname{acl}(\bar{b}, c))} \neq \{1\}$.

My feeling is that this definition is 'not very far' from the original definition 5.1, since if G preserves a linear order then every $x \in \operatorname{acl}(\bar{a})$ will be definable over $\bar{a}$, and hence if G is reflective it is strongly reflective. The next section is devoted to an example of just such a G, which nevertheless fails to be the automorphism group of any recursively saturated structure.

I note here that if M is any countable recursively saturated structure and $x \in M$ is not algebraic (i.e., is not in $\operatorname{acl}(\varnothing)$) then there is a nontrivial $g \in \operatorname{Aut} M$ fixing $\operatorname{acl}(\varnothing)$ pointwise and moving x. To see this, consider the set of sentences in the extended language with x and g stating that g is an automorphism, $xg \neq x$, and

$$\exists^{\leqslant n} y\, \theta(y) \to \forall y\, (\theta(y) \to yg = y)$$

for all pairs θ, n.

6 An example

The object of this section is to present a rather extreme example of a reflective group $G \leqslant \operatorname{Sym}\Omega$ which is not the automorphism group of any recursively saturated relational structure on Ω. In this example, the relation '$x \in \operatorname{acl} y$' will turn out to be a linear order $x \leqslant y$ on Ω which is preserved by G, and hence G (considered as a permutation group) is not the automorphism group of any recursively saturated structure on Ω since in any countable recursively saturated structure there are a, b with $a \notin \operatorname{acl}(b)$ and $b \notin \operatorname{acl}(a)$. This observation will be extended in the next section by showing that G does not have any nontrivial abelian subgroups except infinite cyclic ones, and thus the canonical structure M_G is in fact rather 'unsaturated' as measured in the sense of subgroups that embed in G.

We start by making some elementary observations.

Lemma 6.1. Let $G \leqslant \operatorname{Sym}\Omega$, in which $x \in \operatorname{acl} y$ defines a linear order $x \leqslant y$. Then:

1) for all $\bar{a} \in \Omega$, $\operatorname{acl}(\bar{a}) = \operatorname{acl}(\max(\bar{a}))$;

2) the order $\leqslant$ is respected by the action of the group;

3) given $a, b \in \Omega$, the orbit aG_b is either infinite or the singleton $\{a\}$;

4) if $g, h \in G$ and $a, b \in \Omega$ with $a \leqslant b$ and $bg = bh$ then $ag = ah$.

Proof. 1) If $a_i \leqslant a_j$ then $a_i \in \operatorname{acl}(a_j)$ hence $\bar{a} \in \operatorname{acl}(\max(\bar{a}))$.

2) If $a \leqslant b$ then aG_b is finite so, for all $g \in G$, $agg^{-1}G_bg = ag\,G_{bg}$ is finite, hence $ag \leqslant bg$.

3) If $g \in G_b$ has $ag \neq a$ then $ag < a$ or $a < ag$ and in either case $\{ag^n : n \in \mathbb{Z}\}$ is infinite.

4) If $bg = bh$ then $gh^{-1} \in G_b$ so $agh^{-1} = a$. □

A particular case of (4) when $h = 1$ shows that the fix-point set of any such $g \in G \leqslant \operatorname{Sym}\Omega$ is an initial segment of Ω.

The following general proposition shows how reflective examples arise.

PROPOSITION 6.2. Suppose $F \leqslant \mathrm{Aut}(R, \leqslant)$, where $\leqslant$ is a linear order on R, such that:

1) for all $f \in F$, the fix-point set

$$\mathrm{fix}\, f = \{x \in R : xf = x\}$$

is an initial segment of R; and

2) for all a, b in R with $a > b$ the orbit aF_b is cofinal in R.

Then F acts faithfully on each nontrivial F-orbit Ω, F (considered as a subgroup of $\mathrm{Sym}\,\Omega$) is reflective, and $x \in \mathrm{acl}(y) \Leftrightarrow x \leqslant y$ for all $x, y \in \Omega$.

Conversely, if $F \leqslant \mathrm{Sym}\, R$ is reflective, R is infinite, and $x \in \mathrm{acl}(y)$ defines a linear order $x \leqslant y$ on R then $\leqslant$ is preserved by F and conditions (1)–(3) above hold. defines

Proof. Note that, since Ω is not a singleton it cannot have end-points. If $u < v \in \Omega$ then the F_u-orbit of v contains arbitrarily large elements, by (2), and hence so does Ω. Hence, if $f \in F$ is nontrivial and $b \notin \mathrm{fix}\, f$, there is $a \in \Omega$ with $a > b$ so $af \neq a$ since $\mathrm{fix}\, f$ is an initial segment. Thus F is faithful on Ω.

If $x \leqslant y$ and $yf = y$ then $y \in \mathrm{fix}\, f$ so $x \in \mathrm{fix}\, f$. Thus $x \in \mathrm{acl}(y)$. If $x > y$ then by 2 there is $f \in F$ with $x > \mathrm{fix}(f) > y$ so $\{xf^n : n \in \mathbb{Z}\}$ is infinite hence $x \notin \mathrm{acl}(y)$. Thus $x \in \mathrm{acl}(y) \Leftrightarrow x \leqslant y$ for all $x, y \in \Omega$. Note in particular that $\mathrm{acl}(\bar{a}) = \mathrm{acl}(\max(\bar{a}))$.

If $a > b$ (i.e., aF_b is infinite) and $c \in \Omega$ is arbitrary, then by (2) there is $g \in F_b$ with $ag > \max(b, c)$ so $ag \notin \mathrm{acl}(b, c)$. Hence F is reflective.

For the converse, use Lemma 6.1 for (1). For (2), note that given $x < y$ in R and $z \in R$ there is $g \in F_x$ with $yg \notin \mathrm{acl}(z)$ by reflectivity, so $yg > z$ as required. □

Obviously, for $F \leqslant \mathrm{Aut}(R, \leqslant)$ satisfying the conditions in this proposition, if Ω s an F-orbit then $(\Omega, \leqslant)$ is a dense linear order without endpoints, so if F (and hence Ω) is countable it has the same order-type as $(\mathbb{Q}, \leqslant)$. Note too that the closure of any reflective $F \leqslant \mathrm{Sym}\,\Omega$ is also reflective.

THEOREM 6.3. There is an action of F, the free group of rank ω, on a linearly ordered set satisfying the hypotheses of Proposition 6.2.

Proof. I shall present a suitable order-preserving action of F on the set of reals. The argument here is more detailed than is needed to prove Theorem 6.3; the extra details will, however, be used to prove a stronger result (Theorem 7.2) in the next section.

The main result used here is due to S. White [6], who has shown that the subgroup of $\mathrm{Aut}(\mathbb{R}, <)$ generated by $S\colon x \mapsto x+1$ and $P\colon x \mapsto x^3$ is free on these two generators. The form of his result most useful here is as follows.

FACT 6.4. (White) For any nontrivial reduced word in S, P and for any non-algebraic $x \in \mathbb{R}$, $xw \neq x$.

(Note that for a nontrivial word w, the relation $xw = x$ over $\mathbb{R}$ is a nontrivial algebraic relation involving x, so can only be satisfied by finitely many algebraic

$x \in \mathbb{R}$; hence the statement of the theorem is essentially the same as saying $\langle S, P\rangle$ is free.)

We use $\mathbb{A}$ to denote the set of algebraic $x \in \mathbb{R}$. Both S and P send $\mathbb{A}$ onto $\mathbb{A}$.

Noting that P has 1 as a fixed point enables us to define the following family of order-isomorphisms of the reals:

$$f_i \colon x \mapsto \begin{cases} x & \text{if } x \leqslant i+1 \\ (x-i)^{3^{i+1}} + i & \text{if } x > i+1 \end{cases}$$

for $i \in \mathbb{N}$. So f_i is $S^{-i}P^{i+1}S^i$ on $(i+1, \infty)$ and the identity elsewhere. Since

$$\langle S^{-i}P^{i+1}S^i : i \in \mathbb{N}\rangle$$

is free on the generators shown, White's theorem shows that

$$\langle f_i : i \in \mathbb{N}\rangle$$

is also a free group of order-automorphisms of $\mathbb{R}$ with these generators.

If w is a reduced word on $f_0, f_1, f_2, \ldots$, and $p \in \mathbb{R}$ is sufficiently large, then to evaluate pw it will never be necessary to use the 'if $x \leqslant i+1$' clause in the definition of f_i or f_i^{-1}. If this is the case, I will say that w is *fulfilled at* p; if w is fulfilled at all p in an interval $I \subseteq \mathbb{R}$, I will say that w is *fulfilled on* I. The set $\{p \in \mathbb{R} : w \text{ is fulfilled at } p\}$ is always an interval (q, ∞) where $q \in \mathbb{R} \cup \{-\infty\}$ and $q = -\infty$ iff $w = 1$.

If $\{p \in \mathbb{R} : w \text{ is fulfilled at } p\} = (q, \infty)$ is not the whole of $\mathbb{R}$, then q is the largest real number at which w is not fulfilled. Obviously, there is another reduced word w' (obtained by deleting those letters from w for which the '$x \leqslant i+1$' clause is used in evaluating qw and then reduced) such that $qw = qw'$. In fact, in the evaluation of qw, the '$x \leqslant i+1$' clause will only be used for arguments $x = i+1$ (for otherwise w will turn out to be *not* fulfilled at some $p > q$, since the f_i are all order-automorphisms). Given w', we may then define $q' \in \mathbb{R} \cup \{-\infty\}$ by $(q', \infty) = \{p \in \mathbb{R} : w' \text{ is fulfilled at } p\}$, where $q' < q$. This process may be continued as often as required to obtain $q_0 = -\infty < q_1 < q_2 < \cdots < q_n = \infty$ and reduced words $1 = w_1, w_2, \ldots, w_n = w$ such that $w_i = w_{i+1}$ on (q_{i-1}, q_i) and w_i is fulfilled on (q_{i-1}, q_i). Since each interval (q_{i-1}, q_i) has nonempty interior, it contains nonalgebraic points, so w_i is fixed by at most finitely many points here—unless $i = 1$ in which case $w_i = 1$ fixes all points on this interval.

The point q_1 in the preceding discussion will be called the critical point of w, and the description just given shows the following.

Lemma 6.5. For every $w \in \langle f_i : i \in \mathbb{N}\rangle$ with $w \neq 1$ there is a *critical point*,

$$\operatorname{crit} w = \sup\{x \in \mathbb{R} : yw = y \text{ for all } y \leqslant x\} \in \mathbb{R},$$

such that $xw = x$ for all $x \leqslant \operatorname{crit} w$ and for at most finitely many (necessarily algebraic) $x > \operatorname{crit} w$.

For example, $\operatorname{crit} f_i = i+1$. If we define $g_{n,m} = f_0^m f_n f_0^{-m}$ then by straightforward calculations we have

LEMMA 6.6. (i) $\operatorname{crit} g_{n,m} = (n+1)^{1/3^m}$.

(ii) for all nonalgebraic $x > \operatorname{crit} g_{n,m}$

$$x g_{n,m}^{i} \to +\infty \text{ as } i \to \infty$$

and

$$x g_{n,m}^{-i} \to \operatorname{crit} g_{n,m} \text{ as } i \to \infty.$$

We will also require the following lemma on the critical points of $g_{n,m}$.

LEMMA 6.7. For each $n_0 \geqslant 1$ in $\mathbb{N}$, the set

$$\{n^{1/3^m} : n, m \in \mathbb{N}, n, m \geqslant n_0\}$$

is dense in $[1, \infty)$.

Proof. Let $q > p \geqslant 1$ and note that

$$q^{3^m} - p^{3^m} = (q-p)(q^{3^m-1} + q^{3^m-2}p + \cdots + p^{3^m-1}) > (q-p)3^m$$

since each $q^i p^j$ is greater than 1. Hence there is $m \geqslant n_0$ with $q^{3^m} \geqslant n_0$ and $q^{3^m} - p^{3^m} > 1$. Put $n = \left[q^{3^m}\right]$ (where $[]$ is 'integer part'), so

$$q^{3^m} \geqslant n > p^{3^m}$$

so $q \geqslant n^{1/3^m} > p$ as required. □

LEMMA 6.8. There are integers $n_j, m_j \geqslant 1$ $(j \in \mathbb{N})$ such that $\langle g_{n_j,m_j} : j \in \mathbb{N}\rangle$ is free on the generators shown, and $\{\operatorname{crit} g_{n_j,m_j} : j \in \mathbb{N}\}$ is dense in $[1, \infty)$.

Proof. Enumerate all pairs (p_i, q_i) of rationals with $1 \leqslant p_i < q_i$. Given

$$n_0, \ldots, n_j, m_0, \ldots, m_j$$

already determined, choose $n_{j+1}, m_{j+1} > \max(n_0, \ldots, n_j, m_0, \ldots, m_j)$ such that

$$\operatorname{crit} g_{n_{j+1},m_{j+1}} \in (p_{j+1}, q_{j+1})$$

using the previous lemma. Since $n_j \mapsto m_j$ defines a bijection from $\{n_j : j \in \mathbb{N}\}$ to $\{m_j : j \in \mathbb{N}\}$, the group generated by the g_{n_j,m_j} is free as required. □

Now, to complete the proof of Theorem 6.3, let $F = \langle g_{n_j,m_j} : j \in \mathbb{N}\rangle$ as given in the previous lemma, acting as order-isomorphisms of $(1, \infty) \setminus \mathbb{A}$. The properties required in Proposition 6.2 are all clearly satisfied. □

7 Examples with no noncyclic abelian subgroups

This section presents a proof that every abelian subgroup of the group G constructed in the proof of Theorem 6.3 is cyclic. Thus G is not isomorphic to any $\operatorname{Aut} M$ for countable recursively saturated M *as an abstract group*. We use Lemma 6.1 freely without mention.

DEFINITION 7.1. Given $G \leqslant \mathrm{Sym}\,\Omega$ on which $x \in \mathrm{acl}\,y$ is a linear order, $x \leqslant y$, and given $g \in G$, we define the 'critical point' of g to be precisely the fix-point set of g:

$$\mathrm{crit}\,g = \{x \in \Omega : xg = x\}.$$

Since this is a Dedekind cut, it is identified with the corresponding element of the Dedekind completion of $\Omega \cup \{+\infty, -\infty\}$.

Before we can state and prove the theorem here precisely, some terminology concerning words in free groups is required.

Much of the proof of the main theorem of this section, Theorem 7.2, involves calculations with expressions of the form xw where w is a word in the generators f_i of the free group F of rank $\aleph_0$ acting on a set Ω with $x \in \Omega$. The expression xw is considered as a sequence of letters $x, w_1, x_2, \ldots, w_n$, where each w_j is of the form $f_i^{\pm 1}$. (In all cases, a 'word' is not necessarily reduced, unless explicitly stated otherwise, and given two words u, v, the concatenation of u and v is denoted by uv—no reduction is implied.) An *instance* of a word v (given by $v_0, v_1, \ldots, v_r$) in the expression xw is a particular subsequence $w_i, w_{i+1}, \ldots, w_{i+r}$ equal to v, with the position i at which it occurs also taken into account. Thus v may have none, one or more instances in xw. However, in all the cases for which we need this terminology there will be one obvious 'displayed' instance that we might refer to. Thus with some slight abuse of terminology we will talk about '*the* instance of v in $xuvw$', meaning the one immediately after xu, even though there might also be others.

Suppose the free group $F \leqslant \mathrm{Sym}\,\Omega$ has the property that the relation on Ω^2 defined by 'xF_y is finite' is a linear order $x \leqslant y$. Then, given $x \in \Omega$ and a word $w = w_1, x_2, \ldots, w_n$, we say that w is *fulfilled at* x if $xw_1 \ldots w_i \notin \mathrm{crit}\,w_{i+1}$, for all $i = 0, 1, \ldots, n-1$. If this happens, we also say that the (unique!) instance of w is fulfilled in the expression xw. More generally, given words $u_1, u_2, \ldots u_k$, the instance of u_i in $xu_1u_2 \ldots u_i \ldots u_k$ is *fulfilled* if u_i is fulfilled at $au_1u_2 \ldots u_{i-1}$. (Note that this means $w = u_1u_2 \ldots u_k$ is fulfilled in the expression aw if and only if each u_i is fulfilled in the expression $au_1u_2 \ldots u_k$.) In expressions of this sort, when it is necessary to indicate which subwords u_i are known to be fulfilled we do this by underlining this subword as in $au_1u_2 \ldots u_{i-1}\underline{u_i}u_{i+1} \ldots u_k$. The absence of any underline will not signify anything, and in particular does not say whether or not the word in question is fulfilled.

We have already implicity referred to the standard notion of a *subword u of v*. For words u, v, we will say that u is a *subsequence of* v, in symbols $u \subseteq v$, if the word u is obtained from v by deleting letters from v. More formally, if $u = \alpha_1\alpha_2 \ldots \alpha_k$ and $v = \beta_1\beta_2 \ldots \beta_l$ where all α_is, β_js are letters, then $u \subseteq v$ holds if there are $1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant l$ such that $\alpha_j = \beta_{i_j}$ for all $j \leqslant k$.

By convention, distinct words are distinct even if they have the same action, so we need a definition that says when one word can be substituted by another in an expression of the form xw. Given words r, s, u, v, we say that *u equals v at an instance $xrus$ of a word u* if $xru = xrv$. Similarly, *u equals v at an instance $xru^{-1}s$ of u^{-1}* means $xru^{-1} = xrv^{-1}$.

THEOREM 7.2. Let $G \leqslant \mathrm{Sym}\,\Omega$ be closed, where Ω is countably infinite, and suppose that $x \in \mathrm{acl}\,y$ defines a linear order $x \leqslant y$ on Ω. Suppose also that there is a dense subgroup $F \leqslant G$ with generators $\{f_i : i \in \mathbb{N}\}$ such that, for all reduced words w in the generators of F, whenever x is in Ω with w fulfilled at x then $xw \neq x$. Then every abelian subgroup of G is cyclic.

I remark that the closure $G = \overline{F}$ of the group in Theorem 6.3 is such a group.

Proof. Suppose $g, h \in G$ commute. Then for each $x \in \Omega$ there are reduced words $u, v \in F$ such that $yu = yg$, $yv = yh$, where

$$y = \max(x, xg, xh, xg^{-1}, xh^{-1}, xgh, xg^{-1}h^{-1}).$$

It follows that

$$\begin{aligned} xuv &= xvu \\ xu^{-1}v &= xvu^{-1} \\ xuv^{-1} &= xv^{-1}u \\ xu^{-1}v^{-1} &= xv^{-1}u^{-1} \end{aligned}$$

as $zu = zg$ and $zv = zh$ for all $z \leqslant y$. We first aim to obtain from these equations information about u, v, and then will combine all such information for all $x \in \Omega$, showing that g, h generate a cyclic group.

The first key observation is that without loss of generality we may assume that at *least one* occurrence of u or u^{-1} in the above equations is fulfilled, and similarly at least one occurence of v or v^{-1} is fulfilled. For otherwise, replace the words u, v in the set of equations above with subsequences u', v' equal to the corresponding u, v in all instances as follows. First, write down all xwu and $xw'u^{-1}$ that occur in the expressions above, for words $w, w' \in F$, and choose the largest such value, z say, of all such xw, $xw'u^{-1}$. (Remember: u^{-1} is fulfilled at xw' if and only if u is fulfilled at $xw'u^{-1}$.) Next choose a subsequence $u' \subseteq u$ fulfilled at z. This will not necessarily be fulfilled anywhere else, but u and u' agree on all places that occur in the original expressions because z was as large as possible. The process is repeated for v. Obviously, we may also assume that u, v are reduced words.

The next lemma together with the observation in the last paragraph enables us to obtain reductions between the words in u, v in the four equations.

LEMMA 7.3. Suppose u, w are words in F, w being reduced, and suppose $aw = au$ for some $a \in \Omega$, with w fulfilled at a. Then $w \subseteq u$.

Proof. Use induction on the length of u. The case of $u = 1$ is just the property of F given in the statement of the theorem. If $w = 1$ the result is trivial, so we may assume $w = w'c$ and $u = u'd$ for letters c, d. If $c = d$ then $aw' = au'$ and we may use the induction hypothesis. Otherwise, $c \neq d$. If d is fulfilled in $au'd$ then d^{-1} is fulfilled in $w = w'cd^{-1}$ so by induction $w'cd^{-1} \subseteq u' \subseteq u$ since the word on the left is reduced. Otherwise $au'd = au'$ hence $w'c \subseteq u' \subseteq u$ by induction again. $\square$

To apply this lemma we must use the linear order to distinguish between four separate cases.

Lemma 7.4. Suppose

$$\begin{aligned} xuv &= xvu \\ xu^{-1}v &= xvu^{-1} \\ xuv^{-1} &= xv^{-1}u \\ xu^{-1}v^{-1} &= xv^{-1}u^{-1} \end{aligned}$$

where u, v are nontrivial reduced words and at least one occurrence of each of u, v is fulfilled. Then at least one of the words uv, $u^{-1}v$, uv^{-1} or $u^{-1}v^{-1}$ in the generators for F reduces.

Proof. We identify four separate cases depending on whether or not $x \leqslant xu$, and whether or not $x \leqslant xv$. By symmetry, we might as well suppose $x \leqslant xu$ and $x \leqslant xv$. But then $xuv = xvu$ with 'v' on the left and 'u' on the right fulfilled. Rearranging, we have $(xu)vu^{-1} = (xu)u^{-1}v$ with vu^{-1} on the left fulfilled. By the previous lemma, if vu^{-1} does not reduce then $vu^{-1} \subseteq u^{-1}v$. But these two words have the same length, and as one is a subsequence of the other we deduce that $vu^{-1} = u^{-1}v$ as words. Therefore the word uv (and also vu) reduces. □

I will also need another very similar (but rather more complicated) lemma.

Lemma 7.5. Suppose $x \in \Omega$, $u, v, a, b \in F$ are reduced words and

$$\begin{aligned} xb^{-1}uvb &= xa^{-1}vua \\ xa^{-1}u^{-1}ba^{-1}v &= xa^{-1}vba^{-1}u^{-1} \\ xb^{-1}uab^{-1}v^{-1} &= xb^{-1}v^{-1}ab^{-1}u \\ xa^{-1}u^{-1}v^{-1}a &= xb^{-1}v^{-1}u^{-1}b \end{aligned}$$

with at least one occurrence of each of u, v, a, b fulfilled. Then at one of the words

$$\begin{gathered} b^{-1}uvba^{-1}u^{-1}v^{-1}a \\ a^{-1}u^{-1}ba^{-1}vuab^{-1}v^{-1}a \\ b^{-1}uab^{-1}v^{-1}u^{-1}ba^{-1}vb \\ a^{-1}u^{-1}v^{-1}ab^{-1}uvb \end{gathered}$$

reduces.

I defer the proof: it is similar to the proof of the previous lemma, but with many more cases.

The next lemma is the main inductive argument here.

Lemma 7.6. Suppose u, v are reduced words, $x \in \Omega$, and

$$\begin{aligned} xuv &= xvu \\ xu^{-1}v &= xvu^{-1} \\ xuv^{-1} &= xv^{-1}u \\ xu^{-1}v^{-1} &= xv^{-1}u^{-1}. \end{aligned}$$

Then there is a word w in F and $i, j \in \mathbb{Z}$ such that u equals w^i at each each instance of u above and v equals w^j at each instance of v above.

Proof. Use induction on the sum of the lengths of u and v, the result being obviously trivial when u or v is equal to 1. (Note that the induction hypothesis here starts with a quantifier 'for all $x \in \Omega \ldots$' over x.)

By Lemma 7.4, one of uv, $u^{-1}v$, uv^{-1} or $u^{-1}v^{-1}$ reduces and by symmetry we assume uv reduces. This given, we take reduced words a, b in F such that $u = a^{-1}u_0b$ and $v = b^{-1}v_0a$ and $\mathrm{len}(a)+\mathrm{len}(b) \geqslant 1$ is *maximal.* (In other words, we are assuming that neither u_0v_0 nor v_0u_0 reduce.) This gives

$$\begin{aligned}
xa^{-1}u_0v_0a &= xb^{-1}v_0u_0b \\
xb^{-1}{u_0}^{-1}ab^{-1}v_0a &= xb^{-1}v_0ab^{-1}{u_0}^{-1}a \\
xa^{-1}u_0ba^{-1}{v_0}^{-1}b &= xa^{-1}{v_0}^{-1}ba^{-1}u_0b \\
xb^{-1}{u_0}^{-1}{v_0}^{-1}b &= xa^{-1}{v_0}^{-1}{u_0}^{-1}a.
\end{aligned}$$

To start with, consider the case when neither u_0 nor v_0 is 1. Rearranging the above equations gives

$$\begin{aligned}
(xa^{-1})(u_0ba^{-1})(ab^{-1}v_0) &= (xa^{-1})(ab^{-1}v_0)(u_0ba^{-1}) \\
(xa^{-1})(u_0ba^{-1})^{-1}(ab^{-1}v_0) &= (xa^{-1})(ab^{-1}v_0)(u_0ba^{-1})^{-1} \\
(xa^{-1})(u_0ba^{-1})(ab^{-1}v_0)^{-1} &= (xa^{-1})(ab^{-1}v_0)^{-1}(u_0ba^{-1}) \\
(xa^{-1})(u_0ba^{-1})^{-1}(ab^{-1}v_0)^{-1} &= (xa^{-1})(ab^{-1}v_0)^{-1}(u_0ba^{-1})^{-1}.
\end{aligned}$$

Note that u_0b and $b^{-1}v_0$ are reduced (being subwords of u, v respectively), however it may be that ba^{-1} reduces. If this happens we obtain $x' = xa^{-1}$ and reductions $u' = u_0ba^{-1}$ and $v' = ab^{-1}v_0$ of lengths less than those of u, v, with

$$\begin{aligned}
x'u'v' &= x'v'u' \\
x'u'^{-1}v' &= x'v'u'^{-1} \\
x'u'v'^{-1} &= x'v'^{-1}u' \\
x'u'^{-1}v'^{-1} &= x'v'^{-1}u'^{-1}.
\end{aligned}$$

and we may use our induction hypothesis, giving w and i, j with $u' = w^i$ and $v' = w^j$ at all instances in the equations above. Then substituting back $x' = xa^{-1}$ etc., we have $u = (a^{-1}wa)^i$ and $v = (a^{-1}wa)^j$ at all instances in the original four equations.

If on the other hand u_0ba^{-1} and $ab^{-1}v_0$ are already reduced then we have

$$\begin{aligned}
xa^{-1}u_0v_0a &= xb^{-1}v_0u_0b \\
xb^{-1}{u_0}^{-1}ab^{-1}v_0 &= xb^{-1}v_0ab^{-1}{u_0}^{-1} \\
xa^{-1}u_0ba^{-1}{v_0}^{-1} &= xa^{-1}{v_0}^{-1}ba^{-1}u_0 \\
xb^{-1}{u_0}^{-1}{v_0}^{-1}b &= xa^{-1}{v_0}^{-1}{u_0}^{-1}a.
\end{aligned}$$

But $u = a^{-1}u_0b$ is reduced, $v = b^{-1}v_0a$ is reduced, u_0v_0 and v_0u_0 are reduced (by maximality of a, b), and $b^{-1}a, a^{-1}b$ are reduced by supposition. Thus each of

$$\begin{gathered}
a^{-1}u_0v_0ab^{-1}{u_0}^{-1}{v_0}^{-1}b \\
b^{-1}{u_0}^{-1}ab^{-1}v_0u_0ba^{-1}{v_0}^{-1}b \\
a^{-1}u_0ba^{-1}{v_0}^{-1}{u_0}^{-1}ab^{-1}v_0a \\
b^{-1}{u_0}^{-1}{v_0}^{-1}ba^{-1}u_0v_0a
\end{gathered}$$

is reduced, which contradicts Lemma 7.5.

It remains to do the cases when u_0 or v_0 is trivial. Without loss, suppose $u_0 = 1$. Then

$$\begin{array}{rcl} xa^{-1}v_0a &=& xb^{-1}v_0b \\ xb^{-1}ab^{-1}v_0a &=& xb^{-1}v_0ab^{-1}a \\ xa^{-1}ba^{-1}v_0{}^{-1}b &=& xa^{-1}v_0{}^{-1}ba^{-1}b \\ xb^{-1}v_0{}^{-1}b &=& xa^{-1}v_0{}^{-1}a. \end{array}$$

This gives

$$\begin{array}{rcl} (xa^{-1})v_0ab^{-1} &=& (xa^{-1})ab^{-1}v_0 \\ (xb^{-1})ba^{-1}v_0 &=& (xb^{-1})v_0ba^{-1} \\ (xb^{-1})ab^{-1}v_0 &=& (xb^{-1})v_0ab^{-1} \\ (xa^{-1})ba^{-1}v_0{}^{-1} &=& (xa^{-1})v_0{}^{-1}ba^{-1} \\ (xa^{-1})ab^{-1}v_0{}^{-1} &=& (xa^{-1})v_0{}^{-1}ab^{-1} \\ (xb^{-1})v_0{}^{-1}ba^{-1} &=& (xb^{-1})ba^{-1}v_0{}^{-1} \end{array}$$

or, in other words, if $w_0 = ab^{-1}$ then: (A) $v_0w_0 = w_0v_0$, $v_0^{-1}w_0^{-1} = w_0^{-1}v_0^{-1}$, and $v_0^{-1}w_0 = w_0v_0^{-1}$ at xa^{-1}; and (B) $v_0w_0 = w_0v_0$, $v_0^{-1}w_0^{-1} = w_0^{-1}v_0^{-1}$, and $v_0w_0^{-1} = w_0^{-1}v_0$ at xb^{-1}. The argument now splits into two similar cases. If $xa^{-1} \leqslant xb^{-1}$ then $v_0w_0^{-1} = w_0^{-1}v_0$ at xa^{-1}, since $xa^{-1} \leqslant xb^{-1}$ and these are equal at xb^{-1}, so the induction hypothesis can be applied at $x_0 = xa^{-1}$ to v_0 and w_0. Otherwise $xa^{-1} > xb^{-1}$ and $v_0^{-1}w_0 = w_0v_0^{-1}$ at xb^{-1} so the induction hypothesis can be applied here to v_0 and w_0. In the case $xa^{-1} \leqslant xb^{-1}$ we obtain w, i, j such that $w^i = w_0$ at

$$xa^{-1}, \quad xa^{-1}v_0, \quad xa^{-1}v_0^{-1}w_0^{-1}, \quad xa^{-1}w_0^{-1}$$

and $w^j = v_0$ at

$$xa^{-1}, \quad xa^{-1}w_0, \quad xa^{-1}v_0^{-1}, \quad xa^{-1}w_0^{-1}v_0^{-1}, \quad xa^{-1}w_0v_0^{-1}.$$

But then $w^i = w_0$ and $w^j = v_0$ at the ‘missing points’ from the equation

$$xb^{-1}v_0w_0^{-1} = xb^{-1}w_0^{-1}v_0$$

for:

1. $xb^{-1} = xa^{-1}w_0$ so $v_0 = w^j$ at xb^{-1};
2. $xb^{-1}w_0^{-1} = xa^{-1}$ so $v_0 = w^j$ at $xb^{-1}w_0^{-1}$;
3. $xb^{-1}v_0w_0^{-1} = xb^{-1}w_0^{-1}v_0 = xa^{-1}v_0$ so $w_0 = w^i$ at $xb^{-1}v_0w_0^{-1}$; and
4. $xb^{-1}w_0^{-1} = xa^{-1}$ so $w_0 = w^i$ at $xb^{-1}w_0^{-1}$.

This gives $v = (a^{-1}wa)^{i+j}$ and $u = (a^{-1}wa)^{-i}$ in all instances of the original four equations, as the reader may check. The case $xa^{-1} > xb^{-1}$ is similar. $\square$

Now, returning to the proof of Theorem 7.2, we have $g, h \in G$ with $gh = hg$ and have to show that they generate a cyclic group. Suppose $g \neq 1$ and $h \neq 1$, and let $x_0 \in \Omega$ be such that $x_0 g \neq x_0 \neq x_0 h$. Then by Lemma 7.6 there is $w_0 \in F$ and $i, j \in \mathbb{Z}$ with $x_0 g = x_0 w_0^i$ and $x_0 h = x_0 w_0^j$. Take $x_1 > \max(x_0, x_0 g, x_0 h)$ and repeat the above argument obtaining $w_1 \in F$ and $k, l \in \mathbb{Z}$ with $x_1 g = x_1 w_1^k$ and $x_1 h = x_1 w_1^l$. Since $x_0 < x_1$ it follows that $x_0 w_1^k = x_0 w_0^i$. As $x_0 w_0^i \neq x_0$ and all permutations in G preserve $\leqslant$, the only way this can happen is if $k = i$ and $x_0 w_0 = x_0 w_1$. In the same way (using $x_0 h \neq x_0$) we see that $j = l$.

If we take an unbounded sequence

$$x_0 < x_1 < x_2 < \cdots$$

in Ω, this process can be continued to give a sequence of permutations $w_n \in G$ such that $y w_{n+1} = y w_n$, $y w_n^i = yg$ and $y w_n^j = yh$ for all $y \leqslant x_n$. But then (w_n) is obviously a Cauchy sequence so has a limit $\lim_{n\to\infty} w_n = w$ in G. Clearly $g = w^i$ and $h = w^j$. Let k be the highest common factor of i, j, so $\langle g, h \rangle = \langle w^k \rangle \subseteq G$, and hence g, h generate a cyclic subgroup of G, as required. $\square$

Finally, the proof of Lemma 7.5 is similar to that of Lemma 7.4 but has many more cases and involves much more calculation.

Proof of Lemma 7.5. First, note that the equations given are symmetrical under the following permutations of the letters involved.

α: $(a\ b)(a^{-1}\ b^{-1})(u\ u^{-1})(v\ v^{-1})$

β: $(a\ b)(a^{-1}\ b^{-1})(u\ v)(u^{-1}\ v^{-1})$

There are four main cases, depending on whether $xb^{-1}ua \geqslant x$ (equivalently $xa^{-1}u^{-1}b \leqslant x$) and $xa^{-1}vb \geqslant x$ (equivalently $xb^{-1}v^{-1}a \leqslant x$). Notice that α reverses the order here, and β preserves the order, so I only need do two cases: 1. when $xb^{-1}ua \geqslant x$ and $xa^{-1}vb \geqslant x$; and 2. when $xb^{-1}ua \geqslant x$ and $xa^{-1}vb < x$.

Case 1. I show that

$$xb^{-1}u\underline{vb} = xa^{-1}v\underline{ua}$$

(i.e., that the underlined words here are fulfilled), which suffices by Lemma 7.3.

To show u is fulfilled on the right hand side here, I must show that all other applications of u in the original four equations are to smaller elements of Ω than the one here. In other words, I must show

$$xa^{-1}v \geqslant xb^{-1}, xa^{-1}u^{-1}, xa^{-1}vba^{-1}u^{-1}, xb^{-1}v^{-1}ab^{-1}, xb^{-1}v^{-1}u^{-1}.$$

But (using the four equations given, the two inequalities in our assumption for this case, and the fact that G preserves $\leqslant$) we have

$$\begin{array}{l} xa^{-1}v \geqslant xb^{-1} \geqslant xa^{-1}u^{-1} \geqslant xb^{-1}v^{-1}u^{-1} \\ xb^{-1} \geqslant xb^{-1}v^{-1}ab^{-1} \\ xa^{-1}v = xb^{-1}uvba^{-1}u^{-1} \geqslant xa^{-1}vba^{-1}u^{-1}. \end{array}$$

The case of v being fulfilled is symmetrical, by using $(a\ b)(a^{-1}\ b^{-1})$ and noting that the above argument only used the first equation of the original four.

To see b is fulfilled in $xb^{-1}uvb$, we need

$$xb^{-1}uv \geqslant xb^{-1}, xa^{-1}u^{-1}, xa^{-1}v, xb^{-1}uab^{-1}, xb^{-1}v^{-1}ab^{-1}, xb^{-1}v^{-1}u^{-1}.$$

But

$$\begin{array}{l} xb^{-1}uv \geqslant xa^{-1}v \geqslant xb^{-1} \geqslant xb^{-1}v^{-1}ab^{-1} \geqslant xa^{-1}u^{-1}v^{-1}ab^{-1} = xb^{-1}v^{-1}u^{-1} \\ xb^{-1} \geqslant xa^{-1}u^{-1} \\ xb^{-1}uv = xa^{-1}vuab^{-1} \geqslant xb^{-1}uab^{-1}. \end{array}$$

The case of a is symmetrical, by $(a\ b)(a^{-1}\ b^{-1})$ again.

Case 2. This subdivides into two subcases, depending on whether $xb^{-1}u \geqslant xb^{-1}v^{-1}$ or not. Using the symmetry β, we may assume the former holds. Then we claim that

$$xb^{-1}u\underline{ab^{-1}v^{-1}} = xb^{-1}v^{-1}ab^{-1}\underline{u}.$$

For u, we need

$$xb^{-1}v^{-1}ab^{-1} \geqslant xb^{-1}, xa^{-1}v, xa^{-1}u^{-1}, xa^{-1}vba^{-1}u^{-1}, xb^{-1}v^{-1}u^{-1}.$$

This is as follows:

$$\begin{array}{l} xb^{-1}v^{-1}ab^{-1} = xb^{-1}uab^{-1}v^{-1}u^{-1} \geqslant xb^{-1}v^{-1}u^{-1} \\ xb^{-1}v^{-1}ab^{-1} \geqslant xb^{-1} \geqslant xa^{-1}u^{-1} \geqslant xa^{-1}vba^{-1}u^{-1} \\ xb^{-1} \geqslant xa^{-1}v. \end{array}$$

For a, we have

$$xb^{-1}u \geqslant xb^{-1}v^{-1}, xa^{-1}, xa^{-1}vu, xa^{-1}u^{-1}ba^{-1}, xa^{-1}vba^{-1}, xa^{-1}u^{-1}v^{-1}$$

as follows:

$$\begin{array}{l} xb^{-1}u \geqslant xb^{-1}v^{-1} \geqslant xa^{-1}u^{-1}v^{-1} \\ xb^{-1}v^{-1} \geqslant xa^{-1} \geqslant xa^{-1}u^{-1}ba^{-1} \\ xb^{-1}u \geqslant xa^{-1}vu \\ xb^{-1}u = xb^{-1}v^{-1}ab^{-1}uvba^{-1} \geqslant xb^{-1}uvba^{-1} \geqslant xa^{-1}vba^{-1}. \end{array}$$

For b, we have

$$xb^{-1}uab^{-1} \geqslant xb^{-1}, xb^{-1}v^{-1}ab^{-1}, xb^{-1}uv, xa^{-1}u^{-1}, xa^{-1}v, xb^{-1}v^{-1}u^{-1}$$

as follows:

$$\begin{array}{l} xb^{-1}uab^{-1} \geqslant xb^{-1}v^{-1}ab^{-1} \geqslant xb^{-1} \geqslant xa^{-1}v \\ xb^{-1} \geqslant xa^{-1}u^{-1} \\ xb^{-1} \geqslant xb^{-1}v^{-1}u^{-1} \\ xb^{-1}uab^{-1} = xb^{-1}v^{-1}ab^{-1}uv \geqslant xb^{-1}uv \geqslant xa^{-1}v. \end{array}$$

Finally, for v, we have

$$xb^{-1}uab^{-1}v^{-1} \geqslant xb^{-1}v^{-1}, xb^{-1}u, xa^{-1}, xa^{-1}u^{-1}ba^{-1}, xa^{-1}u^{-1}v^{-1}$$

as follows:

$$\begin{array}{l} xb^{-1}uab^{-1}v^{-1} \geqslant xb^{-1}v^{-1}ab^{-1}v^{-1} \geqslant xb^{-1}v^{-1} \geqslant xa^{-1} \geqslant xa^{-1}u^{-1}ba^{-1} \\ xb^{-1}v^{-1} \geqslant xa^{-1}u^{-1}v^{-1} \\ xb^{-1}uab^{-1}v^{-1} = xb^{-1}v^{-1}ab^{-1}u \geqslant xb^{-1}u \geqslant xa^{-1}. \end{array}$$

□

8 Questions

(This section needs complete revision.)

QUESTION 8.1. Characterize (in terms of permutation-group theoretic notions) those closed permutation group $G \leqslant \operatorname{Sym}\Omega$ which are the automorphism groups $\operatorname{Aut} M$ of *some* recursively saturated structure M on domain Ω.

I was unable to modify the definition of 'reflective' so as to get noncyclic abelian subgroups to embed. The next question is just this, and is necessarily rather vague.

QUESTION 8.2. Find natural conditions on a closed permutation group $G \leqslant \operatorname{Sym}(\Omega)$ that imply that G has a free-abelian subgroup of rank ω.

The conditions here should be true in all $\operatorname{Aut} M$, where M is countable recursively saturated, and they should have a direct proof in the case when G is closed and oligomorphic.

The next item lists various properties of the automorphism group of a countable recursively saturated structure.

FACT 8.3. Let $G = \operatorname{Aut} M$, where M is countable, recursively saturated, and has domain Ω. Then

1. There are infinite $I \subseteq M$ such that for all $n \in \mathbb{N}$ and all n-subsets $A, B \subseteq I$ there is $g \in G$ with $Ag = B$.
2. There are $f \in \operatorname{Inj} M_G$ and $g \in G \setminus \{1\}$ such that $g \restriction \operatorname{im} f = 1$.
3. (Schmerl [?]) There is $I \subseteq M$, a dense linear order $<$ (without endpoints) on I, and a homomorphism $\theta\colon \operatorname{Aut}(I,<) \to G_{\{I\}}$ such that $\theta(\pi) \restriction I = \pi$ for all $\pi \in \operatorname{Aut}(I,<)$.

QUESTION 8.4. What properties of a closed permutation group $G \leqslant \operatorname{Sym}\Omega$ imply the properties listed in Fact 8.3?

There are other interesting properties not directly related to recursive saturation that might be investigated too:

QUESTION 8.5. Consider the stronger condition on reflective groups $G \leqslant \operatorname{Sym}\Omega$ implying the existence of algebraically independent elements (but which is in fact false in general for recursively saturated models, and is instead rather reminiscent of stability),

> For all $\bar{a} \in \Omega^{<\omega}$ and $x \in \Omega$ there are no infinite sequences $x_1, x_2, \ldots \in xG_{\bar{a}}$ such that $x_i \in \operatorname{acl}(x_{i+1}, \bar{a})$ but $x_{i+1} \notin \operatorname{acl}(x_i, \bar{a})$ for all i.

Investigate the properties of such groups G.

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