Polish Groups

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These notes introduce the reader to Polish groups—topological groups in which the underlying space is Polish. No background in topological groups is assumed, but I will present the material fairly rapidly and leave quite a lot for the reader to check on his or her own.

My main motivation for writing such notes arises from the applications and connections in model theory, where closed subgroups $G \leqslant \operatorname{Sym}\Omega$ are studied, the topology arising (as in the Galois theory of infinite field extensions) from consideration of pointwise stabilizers of finite sets. Such groups are Polish with the *small subgroup property* and these are the main object of study. However, much of the theory carries through to Polish groups in general. If there is any original material at all in these notes it probably concerns the generalization of the small index property—familiar nowadays to model theorists in the case of the small subgroup property—to Polish groups in general.

Topological groups

A *topological group* is a group G with a topology on G for which the operations

$$\cdot : G^2 \to G \quad \text{and} \quad {}^{-1} : G \to G$$

are continuous. (Use the product topology for G^2.) Thus the notion 'topological group' arises by just combining the definition of a group with the topological notion of continuity in the most obvious way. It is quite remarkable how these two seemingly quite different ideas interact with consequences both for the topology on G and for the group theoretic structure of G.

To start with, by composing the continuous maps $h \mapsto (h, g)$ and $(h, g) \mapsto hg$, it is clear that in such a group that the right-translation map $h \mapsto hg$ is a continuous, and (by considering g^{-1}) a homeomorphism $G \to G$. In particular, if $H < G$ is an open subgroup then every coset Hg of H is open. In the same way, the map $h \mapsto h^g = g^{-1}hg$ is also a homeomorphism $G \to G$. So conjugates of open subgroups are open.

Notice further that any open subgroup $H \leqslant G$ is always closed, since H is the complement of the open set $\bigcup_{g \notin H} Hg$. This doesn't apply to open sets in general though, and closed subgroups need not be open. Concerning subgroups, we have the following (the proof is left as a straightforward exercise).

PROPOSITION 1. Let G be a topological group and $H < G$ a subgroup. Then

1. The closure $\overline{H}$ of H is a subgroup of G and $\overline{H}$ is normal in G iff H is.
2. If H is open then H is closed.
3. If H is closed and of finite index, then H is open.
4. If $K < H$ for some open K then H is open.

Neighbourhood bases of open subgroups

One of the easiest ways of specifying a topological structure on a group is by describing a neighbourhood base at 1 of open subgroups.

For one direction, suppose G is a topological group and $\mathscr{N} = \{H < G : H \text{ is open}\}$; then as we have seen, $\mathscr{N}$ is closed under conjugation, and finite intersections of elements of $\mathscr{N}$ are open and also subgroups, hence once again elements of $\mathscr{N}$.

Conversely, we can define a topology on a group G by specifying a collection $\mathscr{N}$ of subgroups of G closed under conjugation and finite intersections. The subgroups in $\mathscr{N}$ will be open, and the topology is defined by saying that the set of cosets Hg for $H \in \mathscr{N}$ and $g \in G$ forms a subbase for a topology on G. (In other words, this topology is defined by taking as a base all finite intersections of cosets gH of elements H of $\mathscr{N}$, and then the open sets are defined to be those sets which are unions of sets in this base.) This topology makes G into a topological group, since if $U = Hg$ where $H \in \mathscr{N}$ then $U^{-1} = g^{-1}H^{-1} = g^{-1}H = H^g g^{-1}$ is a coset of $H^g \in \mathscr{N}$, and

$$\begin{aligned} \{(r,s) \in G^2 : rs \in Hg\} &= \{(r,s) \in G^2 : s \in r^{-1}Hg = H^r r^{-1} g\} \\ &= \bigcup \{(x,y) \in G^2 : x \in H^r r^{-1}, y \in H^r r^{-1} g\} \end{aligned}$$

(where the union is over all cosets $H^r r^{-1}$). This is a basic open set in the product topology for G, so $\cdot$ is continuous.

As examples of such groups, let $G \leqslant \operatorname{Sym}\Omega$ where Ω is countably infinite, and let $\mathscr{N}$ consist of all stabilizers

$$G_{(A)} = \{g \in G : ag = a \text{ all } a \in A\}$$

for finite $A \subseteq \Omega$. Then $G_{(A)} \cap G_{(B)} = G_{(A \cup B)}$ and $G^g_{(A)} = G_{(Ag)}$ so the conditions on $\mathscr{N}$ above are satisfied so this defines a topological group structure on G.

Neighbourhood bases at 1 in general

It is often useful to have a general condition on a neighbourhood base of open sets at 1 giving rise to a topopological group structure on G. The proof of the following is straightforward and omitted.

PROPOSITION 2. Let G be a group and $\mathscr{N} \subseteq \mathscr{P}(G)$ satisfy:

1. $1 \in A$ for all $A \in \mathscr{N}$
2. for all $A, B \in \mathscr{N}$ there is $C \in \mathscr{N}$ with $C \subseteq A \cap B$;
3. for all $A \in \mathscr{N}$ there is $B \in \mathscr{N}$ with $BB^{-1} \subseteq A$;
4. for all $A \in \mathscr{N}$ and $g \in G$ there is $B \in \mathscr{N}$ with $B \subseteq A^g$.

Then there is a unique topology on G making G into a topological group having $\mathscr{N}$ as a base of open neighbourhoods of 1.

Proof. Define a topology on G by specifying that the sets Ag with $A \in \mathscr{N}$ and $g \in G$ forms a subbase of open sets. □

Further topological assumptions

As we have already seen, the homeomorphisms $x \mapsto xg$ are often used to describe properties of the topology in terms of the neighbourhoods of 1. For example,

PROPOSITION 3. A topological group G is Hausdorff if and only if for all $g \neq 1$ there is an open neighbourhood $H \subseteq G$ of 1 not containing g.

Proof. Let $g \neq h$ in G. Then there is open U with $1 \in U$ and $g^{-1}h \notin U$. By continuity there is an open $V \subseteq U$ with $V^{-1}V \subseteq U$ and $1 \in V$. If $v = whg^{-1}$ with $v, w \in V$ then $hg^{-1} = w^{-1}v \in V^{-1}V \subseteq U$, contradiction, so $V \cap Vhg^{-1} = \varnothing$ hence $Vg \cap Vh = \varnothing$. Both Vg and Vh are open since V is, and $g \in Vg$, $h \in Vh$ since $1 \in V$. □

In other words, G is Hausdorff iff the closure of $\{1\}$ is $\{1\}$. If G is not Hausdorff, then the closure of $\{1\}$ is a closed normal subgroup with the indiscrete topology, and we can factor out by it and obtain a Hausdorff topological group $G/\overline{\{1\}}$ (see Proposition 11 below). For this reason topological groups are usually assumed to be at least Hausdorff. In these notes, however, we will usually make much stronger assumptions on the topology on G.

DEFINITION 4. A topological space X is *separable* if X has a countable dense subset. X is *Polish* if it is separable and there is a compatible metric d with respect to which X is a complete metric space.

Polish spaces arise in many ways in mathematics, and are studied in descriptive set theory. Another characterization of Polish spaces, for example, is that they are exactly the spaces homeomorphic to a countable intersection of open subsets of the space $[0,1]^{\mathbb{N}}$, where $[0,1]$ is the unit interval of $\mathbb{R}$ given the usual topology and $[0,1]^{\mathbb{N}}$ is given the product topology. (See Bourbaki for this and other results of this kind.)

DEFINITION 5. A *Polish group* is a topological group G for which the topology on G is Polish.

Here is a useful property of Polish groups (actually just requiring separability), related to the ideas in Proposition 1 above.

Proposition 6. Let G be a Polish group and $H < G$ an open subgroup. Then H has at most countable index in G.

Proof. If not, the set of cosets Hg of H forms an uncountable pairwise-disjoint set of nonempty open subsets of G, contradicting the existence of a countable dense subset of G. $\square$

Polish groups with the small subgroup property

Here, we investigate the examples of subgroups $G \leqslant \mathrm{Sym}\,\Omega$ and groups with topology defined by a neighbourhood base of open *subgroups* further, as they give a family of important examples.

Theorem 7. Let G be a group and $\mathscr{N}$ a set of subgroups of G such that

1. $\mathscr{N}$ is countable.
2. $\mathscr{N}$ is closed under finite intersections.
3. $\mathscr{N}$ is closed under conjugation.
4. Each $H \in \mathscr{N}$ has countable index in G.
5. For all $g \neq 1$ there is $H \in \mathscr{N}$ with $g \notin H$.

Then the unique topology making G into a topological group with neighbourhood base $\mathscr{N}$ at 1 is separable and metrizable. The completion, $\overline{G}$, (i.e., the set of Cauchy sequences of G modulo the obvious equivalence relation) is a Polish group containing G as a subgroup.

Proof. If $H \in \mathscr{N}$, $[G{:}H] \leqslant \aleph_0$, so H has countably many cosets Hg. It follows that G is separable.

Let $\mathscr{N} = \{H_i : i \in \mathbb{N}\}$ and define

$$d(g,h) = 1/2^n$$

where n is maximal such that $H_i g = H_i h$ and $H_i g^{-1} = H_i h^{-1}$ for all $i < n$. If there is no such n put $d(g,h) = 0$.

Obviously $d(g,h) = d(h,g)$. If $g \neq h$ then $gh^{-1} \neq 1$ so there is $H \in \mathscr{N}$ with $Hg \neq Hh$, hence $d(g,h) = 0$ implies $g = h$. If $d(g,h) = 1/2^n$ and $d(h,k) = 1/2^k$ with $n \leqslant k$ then $gH_i = hH_i = kH_i$ for all $i < n$, so $d(g,k) \leqslant 1/2^n$, giving the 'ultrametric' version

$$d(g,k) \leqslant \min(d(g,h), d(h,k))$$

of the triangle inequality.

For this metric, the basic open neighbourhoods of some $h \in G$ are precisely

$$\{g : d(g,h) < 1/2^n\} = \bigcap_{i \leqslant n} (H_i h \cap hH_i)$$

so are open in the usual topology. Conversely it is clear that each $H_i h$ contains a neighbourhood of this form, and so d is a metric for the original topology.

The completion $\overline{G}$ can be given the structure of a topological group (I'll omit the details, but note that $d(g,h) = d(g^{-1}, h^{-1})$ is useful here) with a natural embedding $G \to \overline{G}$ of topological groups. $\square$

Topological groups such as G above for which the open subgroups is a neighbourhood base at 1 will be said to have the *small subgroup property*.

If $G \leqslant \mathrm{Sym}\,\Omega$, where Ω is a countably infinite set, we have already seen how $\mathscr{N} = \{G_{(A)} : A \subset \Omega \text{ is finite}\}$ gives such a topology. But as there are only countably many such sets A the other conditions above hold too. So this topology on G is metrizable and separable. The full symmetric group $\mathrm{Sym}\,\Omega$ is itself complete when viewed as a topological group in this way, and the *stabilizers in* G of finite subsets A of Ω, $G_{(A)} = G \cap \mathrm{Sym}(\Omega)_{(A)}$ are precisely the intersections with G of the same stabilizers in $\mathrm{Sym}(\Omega)$, so the topology on G is the subspace topology inherited from $\mathrm{Sym}\,\Omega$ Thus the completion $\overline{G}$ is precisely the closure of G in $\mathrm{Sym}\,\Omega$.

For example, if $G \leqslant \mathrm{Sym}\,\Omega$ is the alternating group, consisting of all permutations $g \in \mathrm{Sym}\,\Omega$ with finite support (the *support* of g is the set of points moved by g) having even signature, or in other words, the subgroup of $\mathrm{Sym}\,\Omega$ generated by all 3-cycles $(a\ b\ c)$, then (as you should be able to check) the completion of G is $\mathrm{Sym}\,\Omega$ itself.

Slightly more generally, given G and $\mathscr{N}$ as above, let Ω consist of all cosets Hh of elements $H \in \mathscr{N}$. This is countable by the assumptions that $\mathscr{N}$ and each $[G{:}H]$ is countable, and G acts on Ω by right multiplication. This action is faithful, by the Hausdorff property, so $G \leqslant \mathrm{Sym}\,\Omega$. We can view $S = \mathrm{Sym}\,\Omega$ as a topological group just as before, by taking as our base of open subgroups the pointwise stabilizers $S_{(A)}$ of finite subsets of Ω. Once again, this is the subspace topology inherited from S. But S is metrizable and complete, so G is complete if and only if it is closed in S. Thus a Polish group with the small subgroup property is simply a subgroup of $\mathrm{Sym}\,\Omega$ for some suitable action on a countably infinite set Ω.

Automorphism groups and back-and-forth

We also see that any automorphism group $\mathrm{Aut}\,M$, where where M is a relational structure on the countably infinite domain Ω, is a Polish group where the topology is defined using stabilizers of finite sets as above. The only extra observation needed here is that G is closed in $\mathrm{Sym}\,\Omega$.

This is really just a standard method familiar to model theorists: since Ω is countable, permutations of Ω can be constructed by 'back and forth', at each stage defining a finite map $\bar{a} \mapsto \bar{b}$ (where $\bar{a}, \bar{b} \in \Omega^n$ for some $n \in \mathbb{N}$) such that $\bar{b} = \bar{a}g$ for some $g \in G$. At odd-numbered stages one adds an arbitrary a' to the domain, finding a corresponding b' satisfying the induction hypothesis that $\bar{b}b' = \bar{a}a'g$ for some $g \in G$, and at even-numbered stages one adds an arbitrary b'' to the range, finding a suitable a''. Then the closure $\overline{G}$ of G in S, equivalently the completion of G as a metric space, is precisely the set of permutations of Ω obtained by some back and forth construction. Put another way, $\overline{G}$ is the set of permutations of Ω which are 'locally' in G:

$$\overline{G} = \{g \in \mathrm{Sym}\,\Omega : \forall n{\in}\mathbb{N}\ \forall \bar{a}{\in}\Omega^n\ \exists h{\in}G\ \bar{a}g = \bar{a}h\}.$$

Given $G = \mathrm{Aut}\,M$, any 'reasonable' back-and-forth system on G gives rise to a topology on G, sometimes with a different neighbourhood base at 1.

DEFINITION 8. Let M be a structure with a countably infinite domain Ω, and let $G = \mathrm{Aut}\,M$. A *back-and-forth system on* M is a set $\mathscr{F}$ of maps $f{:}\,A \to B$

where $A, B \subseteq \Omega$ (theought of as being 'small') such that $M \models \theta(\bar{a}) \leftrightarrow \theta(f(\bar{a}))$ for all $\bar{a} \in A$ and all [quantifier-free] formulas θ in the language of M such that:

> [Forth] for all $f : A \to B$ in $\mathscr{F}$ and all $a \in \Omega$ there is $f' \supseteq f$ with $f' : A' \to B'$ and $A' \supseteq A \cup \{a\}$; and
>
> [Back] for all $f : A \to B$ in $\mathscr{F}$ and all $b \in \Omega$ there is $f' \supseteq f$ with $f' : A' \to B'$ and $B' \supseteq B \cup \{b\}$.

('Quantifier-free' is in square brackets here because it is optional: a back-and-forth system respecting quantifier-free formulas necessarily respects all formulas.)

Given such a back-and forth system, $\mathscr{F}$, we can define

$$\mathscr{N} = \{G_{(A)} : \exists f : A \to B \text{ in } \mathscr{F}\}.$$

The conditions above on a base of open subgroups $\mathscr{N}$ are precisely what you might wish to see in a 'nice' back-and-forth system.

1. $\mathscr{N}$ is countable, i.e., there are not too many small substructures, and $[G:G_{(A)}] \leqslant \aleph_0$ for all $A \in \mathscr{N}$, i.e., each such A has countably many automorphic images.
2. For all $A, B \in \mathscr{N}$ there is $C \in \mathscr{N}$ such that $G_{(A)} \cap G_{(B)} \supseteq G_{(C)}$, i.e., small substructures have the amalgamation property.
3. If $f : A \to B$ is in $\mathscr{F}$ and $g \in G$ then $f^g : Ag \to Bg$, so the back-and-forth system is preserved by automorphisms.
4. $\mathrm{id}{\restriction_A} : A \to A$ is in $\mathscr{F}$ for each $A \in \mathscr{N}$.

In such cases, $\mathscr{N}$ gives rise to a separable metric topology on G, as we have seen already.

I can't resist another example here: if M is a countable recursively saturated model of arithmetic, Ω being the domain of M, $G = \mathrm{Aut}\, M \leqslant \mathrm{Sym}\, \Omega$, and

$$\mathscr{N} = \{G_{(A)} : A \subset M \text{ is finite in the sense of } M\}$$

(think of this as a 'nonstandard' version of the stabilizers-of-finite-subsets topology) then $\mathscr{N}$ gives G a different topology, and the completion of G is a Polish group.

Polish groups without small subgroups

Lest the reader think that all examples of Polish groups come from permutation groups in this way, let me note that $\mathbb{R}$ as an additive group is obviously Polish, as are matrix groups with entries from $\mathbb{R}$, and you will be able to find many other examples like these.

It is useful to be able to generalize the conditions above on a neighbourhood base at 1 for a Polish group with the small subgroup property to other such Polish groups. First, we need a notion generalizing the index of a *set* $A \subseteq G$, and not necessarily a subgroup.

DEFINITION 9. If $A \subseteq G$, where G is a group, define the index of A in G by

$$[G{:}A] = \inf\{|T| : T \subseteq G, AT = G\}$$

i.e., the limit of the cardinalities of all possible 'transversals' T for which $AT = \{at : a \in A, t \in T\} = G$.

PROPOSITION 10. If G is a [complete] separable metrizable topological group then there is a set $\mathscr{N}$ of open neighbourhoods of 1 such that

1. $\mathscr{N}$ is countable
2. for each $A \in \mathscr{N}$, $[G{:}A] \leqslant \aleph_0$
3. for each $A \in \mathscr{N}$ there is $B \in \mathscr{N}$ with $B^{-1}B \subseteq A$
4. for each $g \in G$ and $A \in \mathscr{N}$ there is $B \in \mathscr{N}$ with $B^g \subseteq A$
5. for each $A, B \in \mathscr{N}$ there is $C \in \mathscr{N}$ with $C \subseteq A \cap B$
6. for each $g \neq 1$ in G there is $A \in \mathscr{N}$ with $g \notin A$
7. [if G is complete] if (g_n) is a sequence of elements of G such that for all $A \in \mathscr{N}$ there is $n_0 \in \mathbb{N}$ with $g_n^{-1}g_m, g_m g_n^{-1} \in A$ for all $n, m \geqslant n_0$ then there is $g \in G$ with: for all $A \in \mathscr{N}$ there is $n_0 \in \mathbb{N}$ with $g_n^{-1}g, gg_n^{-1} \in A$ for all $n \geqslant n_0$.

Conversely, given such an $\mathscr{N}$ there is a unique topology on G making G into a separable metrizable topological group with neighbourhood base $\mathscr{N}$ at 1, and if the last condition holds this metric is complete.

Proof. In one direction, the conditions are obvious: just take for $\mathscr{N}$ the set of all $U_\delta = \{y : d(y, 1) < \delta\}$ for rational $\delta > 0$. Since separability is likely to be the least familiar here, I do this verification. Let $S \subseteq G$ be countable and dense. Then for all $x \in G$, $U_\delta^{-1}x$ is open and comtains x. So there is $s \in S \cap U_\delta^{-1}x$, i.e., $s = u^{-1}x$, giving $x = rs \in U_\delta s$. Hence $[G{:}U_\delta] \leqslant |S| \leqslant \aleph_0$.

Conversely, given such $\mathscr{N}$, define a subbase for a topology on G to be

$$\{Ag : g \in G, A \in \mathscr{N}\}.$$

Now, given $g \in G$ and $A \in \mathscr{N}$ consider the neighbourhood Ag of g. By the properties of $\mathscr{N}$ given there is $B \in \mathscr{N}$ with $gB^{-1}g^{-1} \subseteq A$, so $(Bg^{-1})^{-1} = gB^{-1} \subseteq Ag$, so $^{-1}$ is continuous. Also, if $hk = g$ there are $C, D \in mathcalN$ with $C^{-1}C \subseteq A$ and $D^{-1} \subseteq C$, $D^{h^{-1}} \subseteq C$. Then

$$(Dh)(Dk) \subseteq Dhh^{-1}Chk = DChk \subseteq C^{-1}Cg \subseteq Ag$$

so $\cdot$ is continuous. Thus G is a topological group.

The topology on G is separable, for take an enumeration $\{A_i : i \in \mathbb{N}\}$ and inductively define countable sets $S_n \subseteq G$ with $S_{n+1} \supseteq S_n$ and $(\bigcap_{i<n} A_i)S_n = G$, using the fact that each such intersection $\bigcap_{i<n} A_i$ contains some A_j and hence has countable index. Then $S = \bigcup_n S_n$ is countable and dense in G, for if $g \in G$ and $A \in \mathscr{N}$, then $A^{-1} \supseteq \bigcap_{i<n} A_i$ for some n and $g \in (\bigcap_{i<n} A_i)S_n$. Hence $g = a^{-1}s$ for some $a \in A$ and $s \in S$, hence $s \in S \cap Ag$, as required.

To define a metric on G we choose a sequence B_i of open neighbourhoods of 1 such that

1. $B_0 = G$,

2. $B_i \subseteq A_i$, all i,

3. $B_i \subseteq B_j$, all $i \geqslant j$,

4. $B_{i+1}^3 \subseteq B_i$, all i,

5. $B_i^{-1} = B_i$, all i.

The existence of such B_i is a simple induction, using the properties of $\mathscr{N}$ given. More specifically, given B_i, choose $U \in \mathscr{N}$ with $U^3 \subseteq B_i \cap A_{i+1}$ and $B_{i+1} = U \cap U^{-1}$.

We have to construct a suitable metric d from these B_is. For $x, y \in G$, define $A(x, y)$ to be the set

$$A(x, y) = \left\{ \sum_{k=1}^{n} 2^{-j(k)} : xy^{-1} = \prod_{k=1}^{n} z_k, \text{ where } z_k \in B_{j(k)} \text{ for } 1 \leqslant k \leqslant n \right\}$$

and we define $d(x, y) = \inf A(x, y)$. Note that as $z_1 = xy^{-1} \in B_0$, $A(x, y)$ is always nonempty, and it is obviously bounded below by 0 so this inf exists. Moreover $d(x, y) \geqslant 0$ for all $x, y \in G$, and as $B_i^{-1} = B_i$ we have $d(x, y) = d(y, x)$ for all $x, y \in G$.

Also $1 = xx^{-1} \in B_i$ for all i, so $d(x, x) = 0$ for all $x \in G$. If $xy^{-1} = \prod_{k=1}^{n} z_k$ and $yz^{-1} = \prod_{k=n+1}^{m} z_k$ then $xz^{-1} = \prod_{k=1}^{m} z_k$, and it follows that $d(x, z) \leqslant d(x, y) + d(y, z)$. We also note that $d(x, y) = d(xz, yz)$ for all $x, y, z \in G$, for $xz(yz)^{-1} = xzz^{-1}y^{-1} = xy^{-1}$ so d is right-invariant.

It remains to show that $d(x, y) = 0$ implies $x = y$ and that this metric induces the topology we started with.

We prove by induction that if $d(x, 1) \leqslant 2^{-r-1}$ then $x \in B_r$. Our induction hypothesis is the assertion

> $P(n)$: for each r, if $x = \prod_{k=1}^{n} z_k$ with $z_k \in B_{j(k)}$ and $\sum_{k=1}^{n} 2^{-j(k)} \leqslant 2^{-r}$ then $x \in B_r$.

For $n = 1$ this follows immediately from $B_r \supseteq B_i$ for all $i \geqslant r$. Now assume $P(n)$ holds, and that $x = \prod_{k=1}^{n+1} z_k$, $z_k \in B_{j(k)}$, and $\sum_{k=1}^{n+1} 2^{-j(k)} \leqslant 2^{-r}$. Since there are at least two terms in $\sum_{k=1}^{n+1} 2^{-j(k)}$, we see that each $2^{-j(k)} < 2^{-r}$, and hence $2^{-j(k)} \leqslant 2^{-r-1}$ for each k. The argument now splits into two cases.

Case 1. $\sum_{k=1}^{n} 2^{-j(k)} \leqslant 2^{-r-1}$. Then by $P(n)$, $z = \prod_{k=1}^{n} z_k \in B_{r+1}$. Also $2^{-j(n+1)} \leqslant 2^{-r-1}$ hence $z_{n+1} \in B_{r+1}$. So it follows that $x = zz_{n+1} \in B_r$ by our choice of sets B_i.

Case 2. $\sum_{k=1}^{n} 2^{-j(k)} > 2^{-r-1}$. Take $m \leqslant n-1$ such that $\sum_{k=1}^{m} 2^{-j(k)} \leqslant 2^{-r-1}$ and $\sum_{k=1}^{m+1} 2^{-j(k)} > 2^{-r-1}$. Then $2^{j(m+1)} \leqslant 2^{-r-1}$ and

$$\sum_{k=m+2}^{n+1} 2^{-j(k)} = \sum_{k=1}^{n+1} 2^{-j(k)} - \sum_{k=1}^{m+1} 2^{-j(k)} < 2^{-r} - 2^{-r-1} = 2^{-r-1}.$$

So by $P(n)$, $z = \prod_{k=1}^{m} z_k \in B_{r+1}$, $z_{m+1} \in B_{r+1}$, and $w = \prod_{k=m+2}^{n+1} z_k \in B_{r+1}$, and $x = zz_{m+1}w \in B_{r+1}^3 \subseteq B_r$, as required.

The final pieces to check are now straightforward. First, if $d(x, y) = 0$ then $xy^{-1} \in B_r$ for all r, by what was just proved, hence $x = y$ by the Hausdorff property. Finally, the topology induced by the metric is indeed the same as the original, for $B_r \subseteq \{y : d(y, 1) < 2^{-r}\}$ is trivially true by definition, and conversely $\{y : d(y, 1) < 2^{-r}\} \subseteq B_r$ is true by the above inductive argument. To complete the proof, the last condition in the list is obviously equivalent to the statement that the group with this topology is complete (as a uniform space, and hence as a metric space). □

Quotients

Let G be a topological group and suppose $H < G$ is a subgroup. Then G/H denotes the set of right cosets Hg of H in G and is given the quotient topology, open sets being precisely those $U \subseteq G/H$ for which

$$\bigcup U = \{hg : h \in H, g \in G \text{ and } gH \in U\}$$

is open in G. There is a natural action of G on G/H defined by

$$(Hg, h) \mapsto H(gh).$$

This is a map from $G/H \times G$ to G, and we give $G/H \times G$ the product topology. Finally, of course, if H is normal in G then G/H is given the familiar quotient group structure.

Proposition 11. Let G be a topological group

1. If H is closed in G then the action $(Hg, h) \mapsto H(gh)$ is continuous.
2. G/H is Hausdorff iff H is closed in G.
3. If both H and G/H are Hausdorff then G is Hausdorff.
4. G/H is discrete iff H is open.
5. If H is closed and normal in G, then the quotient topology on G/H makes G/H into a topological group.

Proposition 12. If G is a Polish group and H is a closed subgroup, then G/H is a Polish space. If in addition H is normal in G then G/H is a Polish group.

Baire category

Let X be a Polish space (not necessarily a group). Recall that a subset $A \subseteq X$ is dense in X if its closure $\overline{A}$ is X. A is said to be *nowhere dense* in X if its closure has empty interior.

So A is nowhere dense iff $\overline{A}$ is nowhere dense iff the complement of $\overline{A}$ (which is open) meets every open set, i.e., iff this complement is dense in X.

A is said to be *meagre* in X if there is a countable family of nowhere dense A_n such that $A = \bigcup_n A_n$. Equivalently (taking the closures of the A_ns) A is meagre in X if there are countably many closed B_ns such that $A \subseteq \bigcup_n B_n$ and the complement of each B_n is dense in X. Complements of meagre sets are called *comeagre*, and these are precisely the sets A which contain a countable intersection of open dense sets U_n, i.e., $A \supseteq \bigcap_n U_n$.

For example, in $\mathbb{R}$ with the usual topology, each singleton set $\{a\}$ is closed and has dense complement. Therefore every countable set of reals is meagre. The *Baire category theorem* says every meagre set of reals has nonempty complement.

THEOREM 13. (Baire category theorem) In a Polish space X, any comeagre set is dense in X.

Proof. Suppose U_n is dense and open for all $n \in \mathbb{N}$. Choose any $x_0 \in X$ and any $\epsilon_0 > 0$, and inductively construct a sequence $x_n \in \bigcap_{i<n} U_i$ such that

$$\{y : d(x_n, y) < \epsilon_n\} \subseteq \bigcap_{i<n} U_i,$$

as follows: take any x_{n+1} in

$$U_n \cap \{y : d(x_n, y) < \epsilon_n\}$$

(such x_{n+1} exists as U_n is dense) and then choose $\epsilon_{n+1} < \epsilon_n/2$ such that

$$\{y : d(x_{n+1}, y) < \epsilon_{n+1}\} \subseteq \bigcap_{i<n+1} U_i,$$

this ϵ_{n+1} exists since the U_ns are all open. This inductive construction defines a Cauchy sequence (x_n) which (by completeness) tends to some limit x. You can check that $x \in \bigcap_n U_n$ with $d(x, x_0) < 2\epsilon_0$. □

A useful modification of the previous theorem is as follows.

THEOREM 14. Suppose X is a Polish space with no isolated points. Then every comeagre set has cardinality $2^{\aleph_0}$.

Proof. By separability and metrizability, $2^{\aleph_0}$ is the maximum cardinality of any Polish space X (and hence of any subset). To show this maximum is attained by a comeagre set, modify the proof of the previous theorem, constructing a tree of points x_σ instead of a single sequence x_n. Given x_σ, $\mathrm{len}(\sigma) = n$ and ϵ_n with

$$\{y : d(x_\sigma, y) < \epsilon_n\} \subseteq \bigcap i < nU_i$$

find distinct $x_{\sigma 0}$ and $x_{\sigma 1}$ in $U_n \cap \{y : d(x_\sigma, y) < \epsilon_n\}$ (using the condition on no isolated points) and put $\epsilon_{n+1} \leqslant \epsilon_n/2, d(x_{\sigma 0}, x_{\sigma 1})/2$ with

$$\{y : d(x_{\sigma i}, y) < \epsilon_{n+1}\} \subseteq \bigcap i < n + 1U_i$$

for $i = 0, 1$. Then for each path f through the complete binary tree, $(x_{f\restriction n})$ defines a Cauchy sequence with distinct limit x_f in $\bigcap_i U_i$, as required. □

In general you should think of comeagre sets as 'very large' and meagre sets as 'very small'. With this in mind, it makes sense to consider those sets which are *nearly* open, i.e., differ from an open set by a meagre set. Recall that the *symmmetric difference of two sets*, $U \triangle V$, is $(U \setminus V) \cup (V \setminus U)$. We say that a set A has the *property of Baire* if there is an open set U such that $U \triangle A$ is meagre. It turns out that a set has the property of Baire iff there is a closed set V such that $V \triangle A$ is meagre. (Hint: consider the closure of U.)

Sets with the property of Baire are analogous to measurable sets in some sense, and this analogy suggests many useful notions. For example, a function $f: X \to Y$ is *Baire measurable* if $f^{-1}U$ has the property of Baire for all open $U \subseteq Y$. Then for example, if X, Y are Polish and $f: X \to Y$ is Baire measurable, then there is a dense subset $U = \bigcap_{n \in \mathbb{N}} U_n$ of X, where each U_n is open and f restricted to U is continuous.

Some applications

We give here some well-known and useful applications of Baire's theorem to Polish groups. The first result is typical of these, and implies for example that a subgroup with the property of Baire is either meagre or open.

THEOREM 15. Let G be a Polish group, and $A \subseteq G$ a nonmeagre subset with the Property of Baire. Then $A^{-1}A$ contains an open neighbourhood of 1.

Proof. Take U open with $A \triangle U$ meagre. Then $U \neq \varnothing$, otherwise A is meagre. Take $g \in U$ and consider $F: (x, y) \mapsto gx^{-1}y$. As F is continuous $F^{-1}U$ is open and clearly contains $(1, 1)$ so there is an open neighbourhood V of 1, with $gV^{-1}V \subseteq U$. We claim that $V \subseteq A^{-1}A$.

Let $v \in V$ and note that $A \triangle U$ is meagre, so

$$(A \triangle U) \cup (A \triangle U)v = (A \triangle U) \cup (Av \triangle Uv)$$

is meagre. As gVv and gV are both open, by the Baire theorem there is

$$w \in (gVv \cap gV) \setminus ((A \triangle U) \cup (Av \triangle Uv)).$$

But $w \in gV = g1^{-1}V \subseteq U$ so $w \in U$ hence $w \in A$, and $w \in gVv = g1^{-1}Vv \subseteq Uv$ so $w \in Uv$ hence $w \in Av$, i.e., $w \in A \cap Av$. So $w = r = sv$ for some $r, s \in A$, hence $v = s^{-1}r \in A^{-1}A$. $\square$

This result has some nice applications to homomorphisms of Polish groups.

THEOREM 16. Let $\phi: G \to H$ be a homomorphism of Polish groups. If ϕ is Baire measurable, then ϕ is continuous.

Proof. Let $U \subseteq H$ be an open neighbourhood of 1. By continuity of $(x, y) \mapsto x^{-1}y$ there is an open $V \subseteq H$ with $1 \in V$ and $V^{-1}V \subseteq U$. By separability there are h_n $(n \in \mathbb{N})$ such that $\{h_n : n \in \mathbb{N}\}$ is dense in H. Hence $\bigcup h_n V = H$, and so $\bigcup \phi^{-1}[h_n V] = G$. It follows that some $\phi^{-1}[h_n V]$ is nonmeagre in G. By Baire measurability, $A = \phi^{-1}[h_n V]$ has the property of Baire.

Now by the previous theorem there is an open $W \subseteq G$ with $1 \in W$ and $W \subseteq A^{-1}A$. Then $\phi(W) \subseteq \phi(A^{-1}A) \subseteq \phi(A)^{-1}\phi(A) = V^{-1}{h_n}^{-1}h_n V = V^{-1}V \subseteq U$, so $\phi^{-1}[A]$ contains an open neighbourhood of 1.

The rest is just standard manoeuvres with translations $x \mapsto xg$ in topological groups. If $\phi(g) = h$ with $g \in G$, $h \in U \subseteq H$ and U open, we wish to find an open $V \subseteq \phi^{-1}[U]$ containing g. Let $U' = Uh^{-1}$ so $1 \in U'$ and by the above find $1 \in V' \subseteq G$ open with $V' \subseteq \phi^{-1}(U')$. Then $V'g \subseteq \phi^{-1}[U']g$ or $\phi(V'g) \subseteq U'\phi(g) = U'h = U$ as required. $\square$

For the next result, we need the following basic fact about Polish spaces.

Fact 17. If $\phi: X \to Y$ is a continuous map on Polish spaces X, Y, and if $U \subseteq X$ is open then $\phi(U)$ has the property of Baire.

Theorem 18. Let $\phi: G \to H$ be a continuous homomorphism of Polish groups. If $\phi(G)$ is not meagre in H, then ϕ is open (i.e., sends open sets to open sets). In particular, if ϕ is onto, then it induces an isomorphism $G/\ker\phi \cong H$ of Polish groups (i.e., an isomorphism of groups which is also a homeomorphism of topological spaces).

Proof. Similar to the previous theorem. If $U \subseteq G$ is an open neighbourhood of 1 take $1 \in V \subseteq G$ with V open and $V^{-1}V \subseteq U$. Now take $\{h_n : n \in \mathbb{N}\} \subseteq G$ dense. Then $\bigcup_n \phi(h_n V) = \phi(G)$ is nonmeagre, so at least one $\phi(h_n V)$ is nonmeagre. This gives an open neighbourhood W of 1 in H with $\phi(U) \supseteq W$. The rest is the standard manoeuvres with translations. $\square$

The small index property

We mention another corollary of Theorem 15: if H is a subgroup of a Polish group G and H has the property of Baire then, as already noted, H is meagre or open. If not meagre, Proposition 6 tells us that H has index $[G{:}H] \leqslant \aleph_0$. Hodges, Hodkinson, Lascar and Shelah gave an elegant tree argument giving a converse to this.

Theorem 19. In a Polish group G, a meagre subgroup $H < G$ has index $2^{\aleph_0}$.

This theorem is a key ingredient for proving the *small index property* for some well-understood Polish groups.

Definition 20. A Polish group G has the *small index property* if whenever $H < G$ is a subgroup of countable index then H is open in G.

The small index property is a converse to Proposition 6 and, provided the group G has a neighbourhood base at 1 of *subgroups* (a rather common situation, as we saw earlier), gives a way in which the topology can be uniquely determined from the abstract group structure of G.

Proposition 21. Suppose G is a topological group with a Polish topology given by a neighbourhood base of subgroups for which the small index property holds. Then this is the unique such topology on G.

(Hint: given two such, use Theorem 18 and the identity homomorphism.)

Not all Polish groups have the small index property though. Using the axiom of choice, examples of Polish groups with a neighbourhood base at 1 of subgroups in which the small index property fails can be given.

Generics

The other key ingredient in proving the small index property in certain cases is the idea of generic elements. An element g of a Polish group G is *generic* if its conjugacy class $\{h^{-1}gh : h \in G\}$ is comeagre in G. (This notion generalizes to G^n in the obvious way.) Obviously there can only be one conjugacy class of generics. Not all Polish groups have generics, but many do.

EXAMPLE 22. In $S = \mathrm{Sym}\,\Omega$ with the Polish topology from stabilizers of finite sets, there is a conjugacy class of generics, which may conveniently be described in terms of cycle types. This class is the set of g with infinitely many n-cycles for each $n = 1, 2, 3, \ldots$ and no infinite cycles.

To see this you have to convince yourself that

$$U_n = \{g : g \text{ has } n \text{ 1-, 2-, 3-,} \ldots, n\text{-cycles}\}$$

and

$$\begin{aligned} V_a &= \{g : a \text{ is not in an infinite cycle of } g\} \\ &= \bigcup_n \{g : g^n : a \mapsto a\} \end{aligned}$$

are open and dense for each $n \in \mathbb{N}$ and each $a \in \Omega$, so the conjugacy class is $\bigcap_n U_n \cap \bigcap_a V_a$, hence comeagre.

Here is a very simple application of generics.

THEOREM 23. Suppose G is a Polish group with generic elements. Then G is perfect. In fact every $g \in G$ is a commutator $k^{-1}a^{-1}ka$ for some $a, k \in G$

Proof. Let C be the conjugacy class of generics, and let $g \in G$. Both C and Cg are comeagre, so by Baire's theorem there is $h \in C \cap Cg$. So $h = kg$ for some $h, k \in C$. But C is a conjugacy class, so $h = a^{-1}ka$ for some a, hence $g = k^{-1}a^{-1}ka$. □

Actions

If G is a topological group, and X is a topological space an *action* of G on X is a map $(g, x) \mapsto gx$ from $G \times X \to X$ satisfying the usual axioms, $1x = x$ and $(gh)x = g(hx)$. If this map is continuous (for the product topology) it is said to be a continuous action. Suppose we have such an action, then according to standard terminology, for $x \in X$ the *stabilizer* of x is the subgroup $G_x = \{g \in G : gx = x\}$, the *orbit* of x is $Gx = \{gx : g \in G\}$. The set of orbits, $\{Gx : x \in X\}$ partitions X. We denote this partition by X/G and the corresponding equivalence relation by $\sim_G$. The stabilizer G_x and the orbit Gx are related: there is a canonical bijection $G/G_x \to Gx$ defined by , thus $|Gx| = [G : G_x]$.

PROPOSITION 24. If G is a topological group acting on a topological set X, then for all $x \in X$ the stabilizer G_x is closed and the canonical map $gG_x \mapsto gx$ is continuous.

I remark (but I don't know the proof) that for a Polish group G acting on a metrizable space X, the action $(g, x) \mapsto gx$ is continuous iff each of the maps $x \mapsto hx$ from $X \to X$ and $g \mapsto gx$ from $G \to X$ are separately continuous for each $h \in G$ and $y \in X$.

We shall investigate this situation further in the case when both G and X are Polish, but first we need to digress and discuss Borel sets.

Borel sets

Let X be a Polish space. The *Borel sets* of X are the elements of the smallest collection $\mathscr{B}(X) \subseteq \mathscr{P}(X)$ of subsets of X containing all open (and closed) sets, and closed under countable unions and countable intersections.

Thus so-called G_δ sets, i.e., sets of the form $\bigcap_{n\in\mathbb{N}} U_i$ with each U_i open are Borel, as are the so-called F_σ sets of the form $\bigcup_{n\in\mathbb{N}} F_i$, each F_i closed, and so on. In fact, the 'and so on' here goes on for longer than you might expect—to the first uncountable ordinal ω_1 to be precise.

Definition 25. (The Borel hierarchy) Given a Polish space X, $\Sigma^0_1(X)$ denotes the set of open subsets of X, $\Pi^0_1(X)$ denotes the set of closed subsets of X, and for each ordinal ξ,

$$\Sigma^0_\xi(X) = \{\bigcup_{n\in\mathbb{N}} A_n : A_n \in \Pi^0_{\xi_n} \text{ for } \xi_n < \xi\}$$

and $\Pi^0_\xi(X)$ is the set of complements of elements of Σ^0_ξ.

It turns out that $\mathscr{B}(X) = \bigcup_{\xi<\omega_1} \Sigma^0_\xi(X) = \bigcup_{\xi<\omega_1} \Pi^0_\xi(X)$ We also define the *analytic* sets of X to be the images $f(A)$ of Borel $A \subseteq Y$, where $f\colon Y \to X$ is a continuous map of Polish spaces X, Y, and the *coanalytic* sets, which are the complements of analytic sets. A function $f\colon X \to Y$ is a *Borel function* if the inverse image of any Borel set in Y is Borel in X.

We quote without proof some basic results about these sets.

Theorem 26. (The Lusin separation theorem) If X is Polish and $A, B \subseteq X$ are disjoint and analytic then there is a Borel set C with $A \subseteq C$ and $B \cap C = \varnothing$.

Theorem 27. (Suslin) Let X be Polish. Then A is Borel in X iff it is both analytic and coanalytic in X.

Theorem 28. (Lusin–Souslin) Let X, Y be Polish spaces and $f\colon X \to Y$ a Borel function. Then if $A \subseteq X$ is Borel and $f \restriction A$ is injective, then $f(A)$ is Borel.

Continuous group actions

Suppose G is a Polish group acting continuously on a Polish space X. Then by Proposition 12 G/G_x is a Polish space and the canonical map takes this continuously and injectively onto the orbit Gx, by Proposition 24. Therefore, by Theorem 28 each orbit Gx is Borel in X. (But it turns out that $\sim_G$ need not be Borel.) It is natural to ask about the level of Gx in the Borel hierarchy, and whether or not the canonical map is a homeomorphism.

Theorem 29. (Effros) Let G be a Polish group acting continuously on a Polish space X. Then for each $x \in X$ the following are equivalent.

1. $gG_x \mapsto gx$ is a homeomorphism $G/G_x \to Gx$.
2. Gx is not meagre.
3. Gx is G_δ in X.

THEOREM 30. (Effros) Let G be a Polish group acting continuously on a Polish space X. Then the following are equivalent.

1. X/G is T_0.
2. $\sim_G$ is G_δ.
3. every orbit Gx is G_δ in X.

We conclude this section by stating the *Topological Vaught Conjecture* (due to Miller) which generalizes the Vaught conjecture in model theory.

CONJECTURE 31. Let G be a Polish group acting continuously on a Polish space X. Then either X/G is countable or else there is a subset $C \subseteq X$ homeomorphic to the Cantor cube $\{0,1\}^{\mathbb{N}}$ consisting of pairwise $\sim_G$-inequivalent elements.

Work in progress

These notes are clearly incomplete in many respects. I intend to expand on them as and when I have time. I am particularly aware that some parts of these notes are incomplete in the proofs (many of which are missing, usually because I do not know them), and bibliography. I would welcome any suggestions concerning them.

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Numerous individuals have made helpful remarks concerning these notes, without these people these notes would not be here. In writing them, I drew on two main sources: the survey paper by Kechris [2] and lecture notes by Tom Körner from an undergradute course on topological groups given in Cambridge University. The books on topology by Kuratowski and Bourbaki [4, 1] were also useful.

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