The model theory of generic cuts

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We survey some properties of the recently discovered notion of *generic cuts* [6, 8], in an attempt to put these cuts in a proper model-theoretic context. The main new results are an existentially closure property (in Section 4) and some model completeness properties (in Section 5) for generic cuts. Some known results will be proven in alternative ways here.

Part of this work is from the second author's doctoral dissertation, written under the supervision of the first author.

1 Preliminaries

For background in model theory, see Hodges [4]. Unless otherwise stated, we follow the notation in the books by Kaye [5] and by Kossak–Schmerl [11]. We fix some notation and repeat a few more relevant definitions here.

The usual language for arithmetic $\{0, 1, +, \times, <\}$ is denoted by $\mathscr{L}_\mathrm{A}$. Let $\mathscr{L}_\mathrm{Sk}$ denote the Skolemized language for arithmetic, i.e., $\mathscr{L}_\mathrm{Sk}$ contains, in addition to the symbols in $\mathscr{L}_\mathrm{A}$, a function symbol $f_\theta(\bar{x})$ for each formula $\theta(\bar{x}, y) \in \mathscr{L}_\mathrm{A}$, intending to mean the least y satisfying $\theta(\bar{x}, y)$. By 'definable', we always mean 'definable with parameters'. We write $\mathsf{Q}x\ \ldots$ for 'there are cofinally many x such that $\ldots$'.

A *cut* of a model of arithmetic is a nonempty proper initial segment that has no maximum element. We write $I \subseteq_\mathrm{e} M$ for 'I is a cut of M'.

Let $\bar{c} \in M \models \mathrm{PA}$. We denote by $\mathrm{Aut}(M)$ the automorphism group of M, and by $\mathrm{Aut}(M, \bar{c})$ the pointwise stabilizer of $\bar{c}$ in $\mathrm{Aut}(M)$. If $X \subseteq M$ and f is a function with domain M, then X^f denotes the image of X under f, i.e.,

$$X^f = \{f(x) : x \in X\}.$$

Similarly, we sometimes write x^f for $f(x)$. Two cuts $I, J \subseteq_\mathrm{e} M$ are *conjugate* over $\bar{c}$ if there is $g \in \mathrm{Aut}(M)$ such that $I^g = J$.

2 Generic cuts

Generic cuts were discovered by the first author during an axiomatic study on *indicators*. The motivation was to understand structures of the form (M, I), where I is a cut of a model of arithmetic M, in a model-theoretic context. We review some basic definitions related to generic cuts here.

Generic cuts are constructed using a notion of forcing, i.e. a Banach–Mazur game in an appropriate topological space. The open sets are generated by certain *intervals* in a model of arithmetic. If $I \subseteq_{\mathrm{e}} M \models \mathrm{PA}$ and $[a, b] \subseteq M$, then we write $I \in [a, b]$ to mean $a \in I < b$. We will regard an interval both as a set of numbers and as a set of cuts — the context will tell the difference. The consistency condition for our forcing comes from an *indicator*.

Definition. Let $M \models \mathrm{PA}$ be nonstandard. An *indicator* on M is a function $Y \colon M^2 \to M$ that satisfies the following properties.

(i) Y is *piecewise definable*, i.e.,

$$\{\langle x, y, Y(x, y)\rangle : x, y < B\}$$

is definable in M for every $B \in M$.

(ii) For all $x \in M$, there exists $y \in M$ such that $Y(x, y) > \mathbb{N}$.

(iii) $M \models \forall x, y\ (x \geqslant y \to Y(x, y) = 0)$.

(iv) $M \models \forall x, y, x', y'\ (x \leqslant x' \wedge y' \leqslant y \to Y(x', y') \leqslant Y(x, y))$.

(v) $y > (x + 1)^2$ for all $x, y \in M$ with $Y(x, y) > \mathbb{N}$.

(vi) $M \models \forall x, y, z\ \big(Y(x, y) \geqslant z \to \exists x', y'\ (Y(x', y) = z \wedge Y(x, y') = z)\big)$.

(vii) If $x, y \in M$ such that $Y(x, y) > \mathbb{N}$ and $z \in [x, y]$, then either $Y(x, z) > \mathbb{N}$ or $Y(z, x) > \mathbb{N}$.

We will often treat an indicator as if it were definable. Formally, one has to replace the function with the code of some large enough initial part of it.

Definition. Let $M \models \mathrm{PA}$ be nonstandard, and Y be an indicator on M.

- A Y*-interval* is an interval $[a, b] \subseteq M$ such that $Y(a, b) > \mathbb{N}$. Double square brackets $[\![\cdot, \cdot]\!]$ will be reserved for denoting Y-intervals.
- A Y*-cut* is a cut $I \subseteq_{\mathrm{e}} M$ that satisfies $Y(x, y) > \mathbb{N}$ for all intervals $[x, y]$ containing I.

Suppose we have an indicator Y on a nonstandard model $M \models \mathrm{PA}$. We think of Y-intervals as being 'large' enough to 'catch' at least one Y-cut. It is straightforward to verify that the collection of Y-intervals generates a topology on the class of all Y-cuts, and if M is countable, this topological space is homeomorphic to the Cantor set 2^{ω}. Instead of defining generic cuts in terms of forcing, we define them using this topology and the automorphisms of M.

Definition. Let $M \models \mathrm{PA}$ be nonstandard and Y be an indicator on M. A cut $I \subseteq_{\mathrm{e}} M$ is Y*-generic* if it is contained in a class of Y-cuts $\mathscr{G}$ that has the following properties.

(a) $\mathscr{G}$ is *closed under* $\mathrm{Aut}(M)$, i.e., $I^g \in \mathscr{G}$ for all $I \in \mathscr{G}$ and all $g \in \mathrm{Aut}(M)$.

(b) $\mathscr{G}$ is *dense* in the space of Y-cuts, i.e., every Y-interval contains a cut in $\mathscr{G}$.

(c) For each $I \in \mathscr{G}$ and each $\bar{c} \in M$, there is an interval $[a, b] \subseteq M$ containing I in which all cuts in $\mathscr{G}$ are conjugate over $\bar{c}$.

Theorem 2.1. Let M be a countable arithmetically saturated model of PA, and Y be an indicator on M.

(a) There is a unique class $\mathscr{G}$ of Y-cuts that satisfies (a)–(c) in the definition of Y-generic cuts.

(b) The class of Y-generic cuts is the smallest comeagre set in the space of Y-cuts.

The proof of this [8] goes via the combinatorial notion of *pregeneric intervals*, which we do not want to delve into here. (Pregenericity will, nevertheless, make a brief appearance in Theorem 3.6.) Actually, generic cuts were defined in terms of these intervals in our previous paper [8].

Notice part (a) of this theorem implies that around any generic cut, there is an interval in which all generic cuts are conjugate. This is the key property of generic cuts that we will use over and over again.

The name 'generic cuts' comes from part (b) of this theorem. It says that generic cuts satisfy all the properties that are possessed by almost all cuts, when 'almost all' means 'comeagre'.

3 Saturation

We work with a fixed countable arithmetically saturated model $M \models \mathrm{PA}$ and an indicator Y on M throughout this section.

Before we show how saturated generic cuts are, we need to describe the language involved more explicitly.

Definition. Define $\mathscr{L}_{\mathrm{Sk}}^{\mathrm{cut}}$ to be the language obtained from $\mathscr{L}_{\mathrm{Sk}}$ by adding one new unary predicate symbol $\mathbb{I}$, which is intended to be interpreted as a cut. So with some abuse of notation, we sometimes write $t \notin \mathbb{I}$ as $t > \mathbb{I}$. $\mathrm{PA}^{\mathrm{cut}}$ is the $\mathscr{L}_{\mathrm{Sk}}^{\mathrm{cut}}$ theory that consists of PA, the definitions for the Skolem functions, and an axiom saying $\mathbb{I}$ is a cut.

We divide the $\mathscr{L}_{\mathrm{Sk}}^{\mathrm{cut}}$ formulas into levels as in usual model theory.

Definition. The formula classes $\forall_n$ and $\exists_n$ in the language $\mathscr{L}_{\mathrm{Sk}}^{\mathrm{cut}}$ will be referred to as Π_n^{cut} and Σ_n^{cut} respectively, for all $n \in \mathbb{N}$.

Note that $\mathrm{PA}^{\mathrm{cut}}$ belongs to Π_1^{cut} in this hierarchy.

We start with a notion defined in Kirby's thesis [9].

Definition. The Y*-index* of a cut $I \subseteq_{\mathrm{e}} M$ is defined to be

$$\{n \in M : (M, I) \models \forall x{\in}\mathbb{I}\ \forall y{>}\mathbb{I}\ Y(x, y) > n\}.$$

The index of a cut is clearly an initial segment of the model.

Proposition 3.1. If I is a Y-generic cut in M, then the Y-index of I is $\mathbb{N}$.

Proof. Every natural number is in the Y-index of I because I is a Y-cut. Take any nonstandard $\nu \in M$. Recall that Y is piecewise definable. Choose any $B \in M \setminus I$, and let $\hat{Y} \in M$ be the code of $\{\langle x, y, Y(x,y)\rangle : x, y < B\}$. Using the genericity of I, let $[a,b]$ be an interval containing I in which all Y-generic cuts are conjugate over $\langle \hat{Y}, \nu\rangle$. Without loss, assume $b < B$. Let $[u,v] \subseteq [a,b]$ such that $Y(u,v) = \nu - 1$, and J be a Y-generic cut in $[u,v]$. By the choice of $[a,b]$, we know that I and J are conjugate over $\langle \hat{Y}, \nu\rangle$. Since

$$(M, J) \models \exists x{\in}\mathbb{I}\ \exists y{>}\mathbb{I}\ \hat{Y}(x,y) \leqslant \nu,$$

the same formula is true in (M, I) too. So ν is not in the Y-index of I. □

Remark. There is a notion that is similar to the index, called the *cofinality*, of a cut. It turns out that the cofinality of a generic cut depends on the indicator chosen. See Theorem 4.13 in Kaye [6] for the details.

This proposition implies a non-saturation property of generic cuts. Recall the following definition: if $\mathscr{L}$ is a recursive language and Γ is a class of $\mathscr{L}$ formulas, then an $\mathscr{L}$ structure $\mathfrak{M}$ is Γ-*recursively saturated* if and only if all recursive types that just consist of formulas in Γ are realized in $\mathfrak{M}$. All our types can only contain finitely many parameters.

Corollary 3.2. If I is a Y-generic cut in M, then (M, I) is not Π_1^{cut}-recursively saturated.

Proof. Consider the type

$$p(v) = \{v > n : n \in \mathbb{N}\} \cup \{\forall x{\in}\mathbb{I}\ \forall y{>}\mathbb{I}\ \hat{Y}(x,y) > v\},$$

where $\hat{Y}$ is a code for some suitable initial part of Y. □

We have a straightforward corollary to this, which should not be surprising.

Corollary 3.3. Let Y be a indicator that is uniformly parameter-free definable in PA. Then there is no consistent $\mathscr{L}_{\mathrm{Sk}}^{\mathrm{cut}}$ theory T such that I is Y-generic in M whenever $(M, I) \models T$.

Proof. If T is such a theory, then there is a countable Π_1^{cut}-recursively saturated model of T. This contradicts Corollary 3.2. □

We will show that the failure of Π_1^{cut}-recursive saturation is best possible for generic cuts. The proof uses a lemma that is worth stating on its own.

Lemma 3.4. Let $[\![a,b]\!]$ be a Y-interval and $p(v)$ be a recursive set of Σ_1^{cut} formulas that involves only finitely many parameters from M. Then the following are equivalent.

(a) There exists a Y-interval $[\![r,s]\!] \subseteq [\![a,b]\!]$ for which we can find an element that realizes $p(v)$ in all (M, I) where I is a Y-cut in $[\![r,s]\!]$.

(b) There is a Y-cut $I \in [\![a,b]\!]$ such that $p(v)$ is realized in (M, I).

(c) There is a Y-cut $I \in [\![a,b]\!]$ such that $p(v)$ is finitely satisfied in (M, I).

Proof. It is clear that (a) $\Rightarrow$ (b) and (b) $\Rightarrow$ (c). So it suffices to show (c) $\Rightarrow$ (a).

Let I be a Y-cut in $[\![a,b]\!]$ such that $p(v)$ is finitely satisfied in (M,I). After some rewriting and coding, we may assume

$$p(v) = \{\exists \bar{x}{\in}\mathbb{I}\ \exists \bar{y}{>}\mathbb{I}\ \theta_i(v,\bar{x},\bar{y},\bar{c}) : i \in \mathbb{N}\},$$

where $\bar{c} \in M$ and $\theta_0, \theta_1, \ldots \in \mathscr{L}_{\mathrm{A}}$. Consider the recursive set

$$\begin{aligned} q(v,r,s) = {} & \{Y(r,s) > n : n \in \mathbb{N}\} \cup \{a \leqslant r \wedge s \leqslant b\} \\ & \cup \{\exists \bar{x}{<}r\ \exists \bar{y}{>}s\ \theta_i(v,\bar{x},\bar{y},\bar{c}) : i \in \mathbb{N}\} \end{aligned}$$

of $\mathscr{L}_{\mathrm{A}}$ formulas. We are done if this is realized in M. So by recursive saturation, it suffices to show $q(v,r,s)$ is finitely satisfied in M. We can prove

$$M \models \exists v\ \exists [r,s]{\subseteq}[\![a,b]\!]\ \Big(Y(r,s) > n \wedge \bigwedge_{i<n} \exists \bar{x}{<}r\ \exists \bar{y}{>}s\ \theta_i(v,\bar{x},\bar{y},\bar{c})\Big)$$

for every $n \in \mathbb{N}$, because I is a Y-cut in $[\![a,b]\!]$ and $p(v)$ is finitely satisfied in (M,I). $\square$

Proposition 3.5. If I is a Y-generic cut in M, then (M,I) is Σ_1^{cut}-recursively saturated.

Proof. Let $p(v)$ be a Σ_1^{cut}-recursive type that is finitely satisfied in (M,I), and $\bar{c} \in M$ be the parameters that appear in $p(v)$. Using genericity, pick an interval $[a,b] \subseteq M$ containing I in which all Y-generic cuts are conjugate over $\bar{c}$. By the previous lemma, we can find a Y-interval $[\![r,s]\!] \subseteq [\![a,b]\!]$ in which all Y-cuts J make $p(v)$ realized in (M,J). Inside such an interval, there is a Y-generic cut that is conjugate to I over $\bar{c}$. So $p(v)$ is realized in (M,I) too. $\square$

Strangely, generic cuts seem to possess more than what Σ_1^{cut}-recursive saturation can offer. For example, the following apparently needs Π_1^{cut}-recursive saturation. The extra bit comes from the strength of $\mathbb{N}$.

Theorem 3.6. Let I be a Y-generic cut in M. Then for every $\bar{c} \in M$, there exists an interval $[a,b]$ containing I such that for every $\mathscr{L}_{\mathrm{A}}$ formula $\chi(x,y,\bar{z})$,

$$(M,I) \models \exists x{\in}\mathbb{I}\ \exists y{>}\mathbb{I}\ \chi(x,y,\bar{c}) \rightarrow \exists x{<}a\ \exists y{>}b\ \chi(x,y,\bar{c}).$$

Proof. Let $[u,v]$ be an interval around I in which all Y-generic cuts are conjugate over $\bar{c}$. Using recursive saturation, find $f \in M$ coding a function $\mathbb{N} \rightarrow M$ such that

$$f(\chi) = (\max n)\big(\exists [r,s]{\subseteq}[u,v]\ (Y(r,s) \geqslant n \wedge \exists x{<}r\ \exists y{>}s\ \chi(x,y,\bar{c}))\big),$$

for all $\mathscr{L}_{\mathrm{A}}$ formulas $\chi \in \mathbb{N}$. Let $d \in M \setminus \mathbb{N}$ such that

$$f(\chi) > \mathbb{N} \Leftrightarrow f(\chi) > d,$$

for all $\chi \in \mathbb{N}$. Such d exists because $\mathbb{N}$ is strong in M. We claim that for every $\chi \in \mathbb{N}$, we have $f(\chi) > d$ if and only if

$$(M,I) \models \exists x{\in}\mathbb{I}\ \exists y{>}\mathbb{I}\ \chi(x,y,\bar{c}).$$

This suffices because we can then write our type as

$$p(a,b) = \{a \in \mathbb{I} \wedge \mathbb{I} < b\} \cup \{f(\chi) > d \to \exists x{<}a\ \exists y{>}b\ \chi(x,y,\bar{c}) : \chi \in \mathscr{L}_{\mathrm{A}}\},$$

which is recursive and Σ_1^{cut}.

Let $\chi \in \mathbb{N}$. If $f(\chi) \leqslant d$, then $f(\chi) \in \mathbb{N}$, and so

$$(M,I) \models \forall x{\in}\mathbb{I}\ \forall y{>}\mathbb{I}\ \neg\chi(x,y,\bar{c})$$

since I is a Y-cut. Conversely, suppose $f(\chi) > d$. Let $[r,s] \subseteq [u,v]$ such that $Y(r,s) = f(\chi) > \mathbb{N}$ and $M \models \exists x{<}r\ \exists y{>}s\ \chi(x,y,\bar{c})$. By our choice of $[u,v]$, the cut I is conjugate over $\bar{c}$ to some Y-generic cut $J \in [r,s]$. For any such J, we have

$$(M,J) \models \exists x{\in}\mathbb{I}\ \exists y{>}\mathbb{I}\ \chi(x,y,\bar{c}).$$

Therefore, this formula is also true in (M,I). □

We present two applications of this theorem, the first of which is easy.

Corollary 3.7. Let I be a Y-generic cut in M. Then for all $\bar{c} \in M$, there exists $a \in I$ such that for every $\mathscr{L}_{\mathrm{A}}$ formula $\eta(v,\bar{z})$,

$$(M,I) \models \exists x{\in}\mathbb{I}\ \forall v{\in}\mathbb{I}\ \big(v > x \to \eta(v,\bar{c})\big) \to \exists x{<}a\ \forall v{\in}\mathbb{I}\ \big(v > x \to \eta(v,\bar{c})\big).$$

Proof. Let $\chi(x,y,\bar{c})$ be $\forall v{\in}[x,y]\ \eta(v,\bar{c})$. □

The second one seems slightly more tricky.

Definition. Let $\mathfrak{M}$ be a structure in some first-order language $\mathscr{L}$, and $\bar{c}, r \in \mathfrak{M}$. The *existential type* of r over $\bar{c}$, denoted by $\mathrm{etp}_{\mathfrak{M}}(r/\bar{c})$, is defined to be

$$\{\varphi(w,\bar{c}) \in \exists_1 : \mathfrak{M} \models \varphi(r,\bar{c})\}.$$

Corollary 3.8. Let I be a Y-generic cut in M. Then the existential type of any tuple $\bar{c}$ in (M,I) is *coded*, i.e., there exists $t \in M$ such that for all $\varphi(\bar{w}) \in \Sigma_1^{\mathrm{cut}}$,

$$(M,I) \models \ulcorner\varphi\urcorner \in t \leftrightarrow \varphi(\bar{c}).$$

We are regarding each element of M as an M-finite set here. See our previous paper [7] for the details of this interpretation.

Proof. Before all, notice that every Σ_1^{cut} formula is equivalent modulo $\mathrm{PA}^{\mathrm{cut}}$ to a finite disjunction of formulas of the form

$$\exists x{\in}\mathbb{I}\ \exists y{>}\mathbb{I}\ \chi(x,y,\bar{z}),$$

where $\chi \in \mathscr{L}_{\mathrm{A}}$. So these are the only formulas that we need to consider. By Theorem 3.6, we can find an interval $[a,b]$ containing I such that

$$(M,I) \models \exists x{\in}\mathbb{I}\ \exists y{>}\mathbb{I}\ \chi(x,y,\bar{c}) \leftrightarrow \exists x{<}a\ \exists y{>}b\ \chi(x,y,\bar{c}),$$

for every $\mathscr{L}_{\mathrm{A}}$ formula $\chi(x,y,\bar{z})$. So recursive saturation implies that

$$p(t) = \{\ulcorner\exists x{\in}\mathbb{I}\ \exists y{>}\mathbb{I}\ \chi(x,y,\bar{z})\urcorner \in t \leftrightarrow \exists x{\in}\mathbb{I}\ \exists y{>}\mathbb{I}\ \chi(x,y,\bar{c}) : \chi \in \mathscr{L}_{\mathrm{A}}\}$$

is realized in M. □

4 Existential closure

In this section, we show that generic cuts are existentially closed in a suitable category. Existentially closed models of arithmetic were studied by Goldrei, Macintyre, Simmons [2, 12], Hirschfeld, Wheeler [3], and others in the 1970s. A more recent reference is Adamowicz–Bigorajska [1].

Recall that an *existentially closed* model of a theory T is a model $\mathfrak{M} \models T$ such that whenever $\mathfrak{K} \models T$ that extends $\mathfrak{M}$, if $\sigma(\bar{z}) \in \exists_1$ and $\bar{c} \in \mathfrak{M}$, then $\mathfrak{K} \models \sigma(\bar{c})$ implies $\mathfrak{M} \models \sigma(\bar{c})$. This property requires us to consider indicators across several models.

Definition. Let $\mathbf{K}$ be a class of models of PA. An *indicator* over $\mathbf{K}$ is a recursive sequence $(Y(x,y) = n)_{n\in\mathbb{N}}$ of $\mathscr{L}_\mathrm{A}$ formulas which can be extended to $(Y(x,y) = n)_{n\in M}$ to obtain an indicator on M for every $M \in \mathbf{K}$. For simplicity, we will often write an indicator over $\mathbf{K}$ as Y instead of $(Y(x,y) = n)_{n\in\mathbb{N}}$. For an indicator Y over some class of models of PA, define $\mathrm{PA}^{\mathrm{cut}}_Y = \mathrm{PA}^{\mathrm{cut}} \cup \{\forall x{\in}\mathbb{I}\ \forall y{>}\mathbb{I}\ Y(x,y) \geqslant n : n \in \mathbb{N}\}$.

Note that $\mathrm{PA}^{\mathrm{cut}}_Y$ is Π^{cut}_1.

We need one technical lemma before the theorem.

Lemma 4.1. Fix a countable arithmetically saturated model $M \models \mathrm{PA}$, an indicator Y on M, and a Y-generic cut I in M. If Θ is an $\mathscr{L}_\mathrm{A}$ definable function under which I is closed, then there is $n \in \mathbb{N}$ such that for all large enough $x \in I$, we have $Y(x, \Theta(x)) < n$.

Proof. Find an interval $[a, b]$ around I in which all Y-generic cuts are conjugate over the parameters $\bar{c}$ needed to define Θ. Suppose I is closed under Θ, but for all $n \in \mathbb{N}$, there are cofinally many $x \in I$ such that $Y(x, \Theta(x)) \geqslant n$. Then

$$M \models \exists x{\in}[a, b]\ (\Theta(x) \in [a, b] \wedge Y(x, \Theta(x)) \geqslant n),$$

for all $n \in \mathbb{N}$. Using recursive saturation, find nonstandard $\nu \in M$ and $x \in [a, b]$ such that

$$M \models \Theta(x) \in [a, b] \wedge Y(x, \Theta(x)) \geqslant \nu.$$

Note $[x, \Theta(x)]$ is a Y-interval. So it contains a Y-generic cut, say J. However J is not closed under Θ, and hence I cannot be conjugate to J over $\bar{c}$. This contradicts our choice of $[a, b]$. □

Theorem 4.2. Fix an indicator Y over some class of models of PA. If M is a countable arithmetically saturated model of PA, and I is a Y-generic cut in M, then (M, I) is an existentially closed model of $\mathrm{PA}^{\mathrm{cut}}_Y$.

Proof. Let $(K, J) \models \mathrm{PA}^{\mathrm{cut}}_Y$ that extends (M, I). Suppose $(K, J) \models \sigma(c)$, where $c \in M$ and $\sigma(z) \in \Sigma^{\mathrm{cut}}_1$. Without loss of generality, assume $c > I$ and $\sigma(z)$ is of the form

$$\exists x\ \exists y\ (\theta(x, y, z) \wedge x \in \mathbb{I} \wedge y \notin \mathbb{I}),$$

where $\theta \in \mathscr{L}_\mathrm{A}$. Define

$$\Theta(x) = \begin{cases} (\max y)(\theta(x, y, c)), & \text{if it exists;} \\ c, & \text{if } \mathsf{Q}y\ \theta(x, y, c); \\ 0, & \text{if } \neg\exists y\ \theta(x, y, c). \end{cases}$$

The fact that $(K, J) \models \sigma(c)$ implies that J is not closed under Θ.

Suppose first that we have $a \in I$ such that $\Theta(x) > J$ for some $x < a$ in J. Then

$$(K, J) \models (\min b)(\forall x{<}a\ (\Theta(x) \leqslant b)) > \mathbb{I}.$$

This transfers to (M, I), and so $(M, I) \models \sigma(c)$.

Suppose next that such an a cannot be found. Then I is closed under Θ. By the previous lemma, there is $n \in \mathbb{N}$ such that for all large enough $x \in I$,

$$(M, I) \models Y(x, \Theta(x)) < n.$$

This transfers to (K, J). However, such a statement cannot be true in (K, J) because J is a Y-cut under which Θ is not closed. $\square$

The standard techniques in model theory apply. Suppose Y is an indicator over some class of models of PA. By the previous theorem, if $(M, I) \models \mathrm{PA}^{\mathrm{cut}}$ and I is Y-generic, then the existential type of an element in (M, I) is maximal. By Corollary 3.3, the theory $\mathrm{PA}_Y^{\mathrm{cut}}$ does not have a model companion, and so the class of existentially closed models of $\mathrm{PA}_Y^{\mathrm{cut}}$ is not first-order axiomatizable.

A strengthening of existential closure is *existential universality*, which has been around in the literature for some time. Recall that an *existentially universal* model of a theory T is a model $\mathfrak{M} \models T$ such that whenever $\mathfrak{K} \models T$ that extends $\mathfrak{M}$, if $p(v, \bar{z})$ is a set of $\exists_1$ formulas and $\bar{c} \in \mathfrak{M}$, then $\mathfrak{K} \models \exists v \bigwedge\!\!\!\bigwedge p(v, \bar{c})$ implies $\mathfrak{M} \models \exists v \bigwedge\!\!\!\bigwedge p(v, \bar{c})$. No countable nonstandard model $M \models \mathrm{PA}$ is existentially universal: just consider

$$p(v) = \{i \in v : i \in S\} \cup \{i \notin v : i \in \mathbb{N} \setminus S\},$$

where $S \subseteq \mathbb{N}$ that is not in $\mathrm{SSy}(M)$. For the same reason, no countable model of $\mathrm{PA}^{\mathrm{cut}}$ is existentially universal. To avoid this, we relax the definition a bit.

Definition. Let T be a theory in a recursive language $\mathscr{L}$. Then a *recursively existentially universal* model of T is a model $\mathfrak{M} \models T$ such that whenever $\mathfrak{K} \models T$ that extends $\mathfrak{M}$, if $p(v, \bar{z})$ is a recursive set of $\exists_1$ formulas and $\bar{c} \in \mathfrak{M}$, then $\mathfrak{K} \models \exists v \bigwedge\!\!\!\bigwedge p(v, \bar{c})$ implies $\mathfrak{M} \models \exists v \bigwedge\!\!\!\bigwedge p(v, \bar{c})$.

Corollary 4.3. Fix an indicator Y over some class of models of PA. If M is a countable arithmetically saturated model of PA, and I is a Y-generic cut in M, then (M, I) is a recursively existentially universal model of $\mathrm{PA}_Y^{\mathrm{cut}}$.

Proof. Take a tuple $\bar{c} \in M$ and a recursive set of Σ_1^{cut} formulas $p(v, \bar{z})$ that is realized in some extension of (M, I) satisfying $\mathrm{PA}_Y^{\mathrm{cut}}$. Then $p(v, \bar{c})$ is finitely satisfied in (M, I) by existential closure. So Σ_1^{cut}-recursive saturation guarantees that $p(v, \bar{c})$ is realized in (M, I). $\square$

5 Model completeness

Throughout this section, we work with a fixed countable arithmetically saturated model $M \models \mathrm{PA}$, an indicator Y on M, and a Y-generic cut I in M.

In our previous paper [8], we established a weak quantifier elimination result. We reformulate this theorem here in more model-theoretic terms, and list some consequences related to model completeness.

Theorem 5.1. Let $\bar{c} \in M$ and J be a Y-generic cut in M. If $\mathrm{etp}_{(M,I)}(\bar{c}) = \mathrm{etp}_{(M,J)}(\bar{c})$, then $(M, I, \bar{c}) \cong (M, J, \bar{c})$.

Proof sketch. Without loss, assume $I < J$. Using genericity, pick intervals $[a, b]$ and $[u, v]$ containing I and J respectively in which all Y-generic cuts are conjugate over $\bar{c}$. Let $x \in I$ that is above a. We claim that there is $h \in \mathrm{Aut}(M, \bar{c})$ such that $x^h \in [u, v]$, which will give us what we want. By recursive saturation, it suffices to show that the $\mathscr{L}_{\mathrm{A}}$ type of x over $\bar{c}$ is finitely satisfied in $[u, v]$. Let $\theta(x, \bar{w}) \in \mathscr{L}_{\mathrm{A}}$ such that $M \models \theta(x, \bar{c})$. Then $(M, I) \models \mathsf{Q}y{\in}\mathbb{I}\ \theta(y, \bar{c})$ by Corollary 3.7. Notice $\mathsf{Q}y{\in}\mathbb{I}\ \theta(y, \bar{w})$ is Π_1^{cut}, and $\neg\mathsf{Q}y{\in}\mathbb{I}\ \theta(y, \bar{w}) \notin \mathrm{etp}_{(M,I)}(\bar{c})$. Thus $\neg\mathsf{Q}y{\in}\mathbb{I}\ \theta(y, \bar{w}) \notin \mathrm{etp}_{(M,J)}(\bar{c})$, which means $(M, J) \models \mathsf{Q}y{\in}\mathbb{I}\ \theta(y, \bar{c})$. In particular, there is $y \in J$ above u such that $M \models \theta(y, \bar{c})$. $\square$

Corollary 5.2. For any $\bar{c}, r, s \in M$, if $\mathrm{etp}_{(M,I)}(r/\bar{c}) = \mathrm{etp}_{(M,I)}(s/\bar{c})$, then $(M, I, \bar{c}, r) \cong (M, I, \bar{c}, s)$.

Proof. Suppose $\mathrm{etp}_{(M,I)}(r/\bar{c}) = \mathrm{etp}_{(M,I)}(s/\bar{c})$. Using recursive saturation, let $g \in \mathrm{Aut}(M, \bar{c})$ such that $s^g = r$. Setting $J = I^g$ gives $(M, I, \bar{c}, s) \cong (M, J, \bar{c}, r)$, and so by hypothesis,

$$\mathrm{etp}_{(M,I)}(\bar{c}, r) = \mathrm{etp}_{(M,I)}(\bar{c}, s) = \mathrm{etp}_{(M,J)}(\bar{c}, r).$$

The previous theorem then does the rest. $\square$

Definable elements in models of $\mathrm{PA}^{\mathrm{cut}}$ were studied in Kossak–Bamber [10]. Corollary 5.2 gives us more information about these definable elements in the case of generic cuts.

Corollary 5.3. All $\mathscr{L}_{\mathrm{Sk}}^{\mathrm{cut}}$ definable elements of (M, I) are Σ_1^{cut} definable with the same parameters.

Proof. Let $a, \bar{c} \in M$ and $\theta(w, \bar{z}) \in \mathscr{L}_{\mathrm{Sk}}^{\mathrm{cut}}$ such that a is the unique element w that satisfies $\theta(w, \bar{c})$ in (M, I). Consider $p(w) = \mathrm{etp}_{(M,I)}(a/\bar{c})$. If $a' \in M$ that satisfies all of $p(w)$ in (M, I), then $(M, I, \bar{c}, a) \cong (M, I, \bar{c}, a')$ by Corollary 5.2, and so $a = a'$ because they both satisfy $\theta(w, \bar{c})$. It follows that

$$q(w) = p(w) \cup \{w \neq a\}$$

is not realized in (M, I). The set $q(w)$ can be rewritten as a recursive set of Σ_1^{cut} formulas using Corollary 3.8. So by Σ_1^{cut}-recursive saturation, this set is not finitely satisfied in (M, I). Therefore, there is $\varphi(w, \bar{c}) \in p(w)$ such that

$$(M, I) \models \forall w\ (\varphi(w, \bar{c}) \rightarrow w = a),$$

as required. $\square$

Remark. Typical Σ_1^{cut} definable elements are

$$(\max w \in \mathbb{I})(\theta(w, \bar{c})) \quad \text{and} \quad (\min w > \mathbb{I})(\theta(w, \bar{c})),$$

where $\theta(w, \bar{c})$ is in $\mathscr{L}_{\mathrm{A}}$.

Question 5.4. Can such translation from $\mathscr{L}_{\mathrm{Sk}}^{\mathrm{cut}}$ to Σ_1^{cut} formulas be uniform in the parameters?

Recall that a definition of model completeness says: a theory T is *model complete* if and only if every $\forall_1$ formula is equivalent modulo T to a $\exists_1$ formula.

Corollary 5.5. For every Π_1^{cut} formula $\theta(w,\bar{z})$ and every $\bar{c} \in M$, there exists a set $\Phi(w,\bar{z})$ of Σ_1^{cut} formulas such that

$$(M,I) \models \forall w \left(\theta(w,\bar{c}) \leftrightarrow \bigvee\!\!\!\bigvee \Phi(w,\bar{c})\right).$$

Proof. Let $A = \{w \in M : M \models \theta(w,\bar{c})\}$. For the moment, work with a fixed $a \in A$. Set $p_a(w) = \mathrm{etp}_{(M,I)}(a/\bar{c})$. By Corollary 5.2,

$$(M,I) \models \forall w \left(\bigwedge\!\!\!\bigwedge p_a(w) \to \theta(w,\bar{c})\right).$$

Therefore, the set $p_a(w) \cup \{\neg\theta(w,\bar{c})\}$ is not realized in (M,I). As in the previous proof, we use Σ_1^{cut}-recursive saturation to pick $\varphi_a(w,\bar{c}) \in p_a(w)$ such that

$$(M,I) \models \forall w\ (\varphi_a(w,\bar{c}) \to \theta(w,\bar{c})).$$

For every $a \in A$, we can find such a formula $\varphi_a(w,\bar{c})$. It can be verified that $\Phi(w,\bar{z}) = \{\varphi_a(w,\bar{z}) : a \in A\}$ does what we want. □

Question 5.6. Is the theorem above true for all $\mathscr{L}_{\mathrm{Sk}}^{\mathrm{cut}}$ formulas $\theta(w,\bar{z})$?

In the theorem above, we cannot guarantee that the set $\Phi(w,\bar{z})$ is finite. This can be proved using Corollary 3.2 and Proposition 3.5. Therefore, $\mathrm{Th}(M,I)$ is not model complete. Nevertheless, let us recall another definition of model completeness: a theory T is *model complete* if and only if $\mathfrak{M} \subseteq \mathfrak{K}$ implies $\mathfrak{M} \prec \mathfrak{K}$ for all models $\mathfrak{M}, \mathfrak{K} \models T$.

Corollary 5.7. Let J be a Y-generic cut in M. Then all embeddings $(M,I) \to (M,J)$ are elementary embeddings.

Proof. Let $h\colon (M,I) \to (M,J)$ be an embedding, and $c \in M$. Using recursive saturation, pick $g \in \mathrm{Aut}(M)$ such that $c^{hg} = c$. Since (M,I) is existentially closed, we have

$$\mathrm{etp}_{(M,I)}(c) = \mathrm{etp}_{(M,J)}(c^h) = \mathrm{etp}_{(M,J^g)}(c).$$

Therefore, $(M,I,c) \cong (M,J^g,c)$ by Theorem 5.1. This implies $(M,I,c) \cong (M,J,c^h)$, which shows what we want. □

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