Arithmetic and the Incompleteness Theorems

Richard Kaye

School of Mathematics and Statistics The University of Birmingham Birmingham B15 2TT

U.K.

http://www.mat.bham.ac.uk/R.W.Kaye/

12th August 2000

Contents

1	Gödel's incompleteness theorems and first-order arithmetic		2
	1.1	Gödel's incompleteness theorems $\ \ldots \ \ldots \ \ldots \ \ldots \ \ldots$	3
	1.2	The development of first-order arithmetic	15
2 Models of arithmetic		21	
	2.1	Constructions of models of arithmetic	21
	2.2	Towards a structure theory for models of arithmetic $\dots \dots$	24
		Initial segments and Mathematical independence results	

NOTE: This is a preliminary draft of a paper on the Gödel incompleteness theorems intended for the History of Logic project (edited by Dirk van Dalen et al). Margin notes indicate some places where the final version may differ from the current version. They are indicated by 'Ed:' (for the attention of the editors of the project) or 'RWK:' (for my attention and questions). I would welcome any comments on the paper, especially on the parts of it annotated with the 'RWK:' style of margin note, or relating to the questions at the end.

If you wish to quote, refer to, or use any material from this paper please consult me first, via my home page.

1 Gödel's incompleteness theorems and first-order arithmetic

The story of Gödel's incompleteness theorems is justifiably one of the highlights of mathematical logic in the twentieth century. The fact that they answered the most important questions in the foundations of mathematics at the time of their publication give them sufficient importance in their own right. But as well as this, Gödel's theorems gave impetus to the newly forming branches of logic of proof theory and recursion theory, and added the important technique of diagonalization to the logician's toolbox.

Much of the story of how Gödel came to discover his incompleteness theorems is well-documented, for example by Wang [63, 65, 66, 67], and other articles in this volume discuss the history of the ideas and problems that directly lead to Gödel's incompleteness theorems. Rather than re-telling this story in detail, I have chosen to concentrate on the mathematical and logical ideas in the theorems and then trace their evolution into the study of arithmetic, the study of models of first-order arithmetic in particular. (For the proof theory of first-order arithmetic, see Sieg's article in the current volume.) Of course, Gödel's ideas and methods are used throughout mathematical logic, in particular in proof theory and computability theory, and these areas are covered by other articles in the current volume.

This article therefore starts by describing the incompleteness theorems and the key ideas therein, concentrating on Gödel's famous 1931 paper [20]. The remaining parts of Section 1 describe some of the subsequent results concerning first-order arithmetic in general. Section 2 traces a somewhat (and surprisingly) neglected story in the history of twentieth century logic, that of the development of the theory of models of first-order arithmetic. In it I focus particularly on aspects of the model theory of arithmetic that relate to Gödel's theorems and

Ed: please feel free to rewrite and/or move this introduction as required by the context of the whole volume

Ed: crossreferences required. those that lead up to the Paris-Harrington independence results.

1.1 Gödel's incompleteness theorems

By 1930, studies in the foundations of mathematics might be regarded to have been in a rather mature state, there having been over fifty years of work since the calculus had been framed in its modern rigorous form by Cauchy, Weierstrass and others, and forty two years since the publication of Dedekind's analysis of the concept of real number and natural number [10]. That is not to deny that the subject had been through crises: the most obvious of these was the inconsistencies in Frege's *Grundgesetze* found by Russell [46], and the obvious concern that Cantor's set theory might also be found to be similarly inconsistent. But in the years that followed, confidence had returned, mainly through the steadying hands of Hilbert and Russell and the programmes that the formalists and logicists had proposed to make the foundations of mathematics sound again. Then in 1931, a paper by Kurt Gödel was published that must have rocked confidence in the idea of the 'knowable' in mathematics once again.

Gödel published an abstract of his results at the end of 1930 [19], and the main paper containing proofs of the incompleteness theorems appeared in a paper entitled 'On formally undecidable propositions of *Principia Mathematica* and related systems I' in the *Monatshefte für Mathematik und Physik* the following year [20]. In terms of content, Gödel's paper contains a staggering number of hugely important ideas. (1) He presents the notion of a (primitive) recursive fuction, and in particular provides the means for the representation of these functions in formal systems of arithmetic. (2) The idea of coding finite sequences in first-order arithmetic is introduced, and the details of how to do this, based on the Chinese remainder theorem, are developed. (3) Gödel-numbering of formulas and proofs is introduced and used. (4) The extension of Cantor's diagonalization technique to formal systems and first-order systems of arithmetic in particular is presented for the first time. (5) The idea of ω -consistency is introduced and consistent non- ω -consistent systems are exhibited.

At the time of writing, the most comprehensive systems for mathemetics were Principia Mathematica (PM) and Zermelo–Fraenkel set theory (ZF), both of which were sufficiently developed at the time to be able to formalize all of the methods of proof currently in use in mathematics and which might prima facie be thought to be complete. Gödel's results show that in fact they are in fact incomplete for 'relatively simple' statements, i.e., statements of first-order arithmetic (the first-order theory of the natural numbers with =, + and \cdot alone), and his results apply to a wide class of arithmetically sound theories, incuding

RWK: Why doesn't Gödel refer to Cantor?

extensions of PM and ZF by finitely many axioms. (At the time of writing the general notion of computability was not available to Gödel, so he could not express the result in its full generality, but Gödel makes it quite plain in a note added in August 1963 to the English translation by van Heijinoort [60] that Turing's work enables the idea of a 'formal system' to be given a precise and adequate definition that enables general versions of the incompleteness theorems.)

The question of the completeness of these systems had been around for some time, through the work of the formalist school of Hilbert and the logicist school of Russell. Gödel was fully aware of the logicists' work, and it is interesting to speculate whether Gödel's paper was intended as a subtle and indirect attack on the logicists who claimed that mathematical and metamathematical truth reduces to type-theoretical logic. Certainly he bases his ground system P discussed in the 1931 paper on the somewhat more cumbersome logic of *Principia Mathematica*, perhaps to make an indirect point. Although Gödel does not make any comment either for or against the logicists, it seems clear to me that he realized his results would put an end to their philosophical programme for the foundations of mathematics. On the other hand, the relationship between Gödel and the formalist school of Hilbert is much more complex and will be discussed later.

Gödel's paper itself is a model of clarity, and section 1 of it contains a sketch of the proof of the first incompleteness theorem. It will be useful to go through some of the details of this here.

The main technical device used in the paper is what is now called Gödelnumbering, being a way of identifying formulas and proofs by finite sequences of natural numbers, which can in turn be coded up into a single natural number. Using this identification, Gödel can sketch the idea behind his diagonalization technique: he calls a formula with a single free natural-number variable a class sign and, for a class sign α , denotes the result of instantiating that variable with the natural number n by $[\alpha; n]$. He notes that via the numbering of formulas, the relation x = [y; z] is definable in his system P. Because of this, there is a class sign K defined by

$$\neg \operatorname{Bew}[R(x); x]$$

where R(x) expresses the number for the xth class sign, and Bew y expresses the proposition that the formula numbered by y is provable in P. But this class sign K has number R(k) for some natural number k and we can consider the statement [R(k);k]. This statement [R(k);k] can be interpreted as stating its

RWK: N.b., Gödel defends type theory in several places in other later papers, SO can this be possible?

own unprovability, and so will turn out to be true but not provable in P.

In Gödel's own words of the late 1970s (as arranged and reported by Wang [67, p82–3]—the additions in square brackets are his) Gödel's discovery of the incompleteness results arose from a direct attack on Hilbert's question of the consistency of classical analysis.

In summer 1930 I began to study the consistency problem of classical analysis. It is mysterious why Hilbert wanted to prove directly the consistency of analysis by finitary methods. I saw two distinguishable problems: to prove the consistency of number theory by finitary number theory and to prove the consistency of analysis by number theory. By dividing the difficulties, each part can be overcome more easily. Since the domain of finitary number theory was not well defined, I began by tackling the second half: to prove the consistency of analysis relative to full number theory. It is easier to prove the relative consistency of analysis. Then one only has to prove by finitary methods the consistency of number theory. But for the former one has to assume number theory to be true (not just the consistency of a formal system for it).

I represented real numbers by predicates in number theory [which express properties of natural numbers and found that I had to use the concept of truth [for number theory] to verify the axioms of analysis. By an enumeration of symbols, sentences, and proofs of the given system, I quickly discovered that the concept of arithmetic truth cannot be defined in arithmetic. If it were possible to define truth in the system itself, we would have something like the liar paradox, showing the system to be inconsistent. [Compare Gödel's letter of 12.10.31 to Ernst Zermelo, in which the easy proof of this is given [64, p90-91].] This aspect of the situation is explicitly discussed in my Princeton lectures of 1934 [22], where the liar paradox is mentioned as a heuristic principle, after the proof of the incompleteness results has been given. The liar paradox itself refers to an empirical situation which is not formalizable in mathematics. In my original paper [published in 1931] there is [in addition] an allusion to Richard's paradox, which is purely linguistic and refers to no empirical fact.

Note that this argument [about truth not being definable in the system itself] can be formalized to show the existence of undecidable propositions without giving any individual instances. [If there were

no undecidable propositions, all (and only) true propositions would he provable in the system. But then we would have a contradiction.] In contrast to truth, provability in a given formal system is an explicit combinatorial property of certain sentences of the system, which is formally specifiable by suitable elementary means. In summer 1930 I reached the conclusion that in any reasonable formal system in which provability in it can he expressed as a property of certain sentences, there must he propositions which are undecidable in it. [This preliminary result was, according to Carnap's diary, announced to Carnap, Feigl, and Waismann at Cafe Reichsrat on 26.8.30. For a more formal explication of the last three paragraphs compare Wang [63, p21–23].]

It was the anti-Platonic prejudice which prevented people from getting my results. This fact is a clear proof that the prejudice is a mistake.

As just mentioned, Gödel clearly points out the analogies with Richard's paradox and the 'liar'. Indeed, as he observes, [R(k); k] is an expression stating its own unprovability, but no circularity is present in the construction of this statement. This is a major breakthrough: statements such as the liar's, and others were well known but here Gödel manages to construct a statement saying 'this statement is not provable' by a slightly roundabout route that avoids the circularity that at first sight would be required. He writes: 'Only subsequently (and so to speak by chance) does it turn out that this formula is precisely the one by which the proposition itself was expressed.' In this respect, Gödel's work is a major step forward from the work of Finsler [13], as discussed by van Heijenoort [60], since Finsler also discussed statements which (he claimed) assert their own unprovability, but—lacking Gödel's technical devices—Finsler could only attempt to express such statements in natural language without any formal syntax or precise notion of proof.

Gödel discusses the method of diagonalization at further length later in the paper, pointing out that it is entirely constructive; indeed, he says that 'it would be very easy (although somewhat cumbersome) to actually write down' the independent statement [R(k);k]. He also points out that the truth of [R(k);k] is easy to see from metamathematical considerations, but these considerations are not available from within the system. (This will in due course lead to the second incompleteness theorem.)

It is interesting to note that Gödel does not discuss the uniqueness of formulas G such that P proves G is equivalent to $\neg \operatorname{Bew} G$. In fact, any such G is

RWK: Note no comparison with Cantor

equivalent (in sufficiently strong systems, such as P) to the consistency of the system in hand, and the more difficult direction of this equivalence is already proved by Gödel in his proof of the second incompleteness theorem.

In the precise proof of the basic form of the first incompleteness theorem Gödel formally introduces his system P based on the logic of Principia Mathematica and Peano's second-order induction axioms. This system is a many-sorted system with type levels: 1 (for individuals, that is natural numbers—Gödel considers 0 to be a natural number, incidently); 2 (sets of individuals); 3 (sets of sets of naturals); and so on for every explicit natural number n. As well as the usual logic, comprehension and extensionality axioms, Gödel has the set membership predicate, a 'zero' element, and the successor function $x \mapsto x+1$. To this logic he adds the Peano axioms (axioms stating that the successor function is one-to-one, 0 is not in the image of the successor function, and the induction axiom). Thus Gödel's system P is essentially equivalent to what might be called ' ω th-order arithmetic'. Although this system has the inconvenience of the type levels, this choice of system has its convenient features. For example, equality, addition and multiplication are all derived terms and Peano's axioms can be used almost verbatim because of this.¹

The next step is to arithmetize the syntactical notions. For this, Gödel introduces the class of primitive recursive functions (he calls them 'recursive') as the least class containing constant functions, the successor function, and closed under substitution and primitive recursion, and the associated notion of a primitive recursive predicate. This class of functions had previously been highlighted by Dedekind [10], and had been used and developed by Skolem [50], Hilbert [35] and Ackermann [2]; the modern terminology referring to them as primitive recursive was introduced by Kleene [38]. Interestingly, in Gödel's presentation the identity function and projection functions $\pi_n^i(x_1, x_2, \dots, x_n) = x_i$ are not mentioned explicitly, and the idea of 'substitution' is left rather vague, but these projection functions can be obtained simply enough by primitive recursion from the zero function and successor. Similarly, Gödel's arithmetization of the formal rules of derivability in his system P are much more precise than the rather vague intutitive informal ideas presented earlier—for Gödel, in this paper, the arithmetization is the detailed definition of syntax and there is no intermediate system involving symbols on paper.) In Gödel's own words [67, p84],

The proof of the (first) incompleteness theorem in my original

 $^{^{1}}$ As we will see, Gödel's arithmetization of syntax was a comparitively late development, and the choice of P was quite possibly made simply to be strong enough for him to be able express syntax in before he had the β function and Chinese remainder theorem idea.

paper is awkward because I wanted to make it completely formalized. The basic idea is given more clearly in my Princeton lectures [of 1934]. I wrote Herbrand two letters, the second of which he did not receive. He had a good brief presentation of my theorems.

In a slight remark, Gödel comments that 'Bew(x) [for 'x is provable'] the only one of the [syntactic notions] of which we cannot assert that it is [primitive] recursive'. The obvious questions that this remark begs are not even mentioned, let alone discussed. We can wonder whether he considered conjecturing or proving it is not primitive recursive. Of course, such a proof would not be short—it would involve diagonalizing out of the class of primitive recursive predicates, and would require the β function technique for coding sequences—but still lies within the techniques that Gödel had been developing at the time. This point seems interesting, when we observe (with hindsight) that the sort of existential quantification here applied to primitive recursive predicates gives the recursively enumerable sets, so Gödel is in fact rather close to the subject of computability that would arise from subsequent work by Turing, Church, Post and Kleene in response to this paper. But the best possible result, that Bew(x) is an example of an r.e., non-recursive, predicate, was still out of reach in 1931. Similarly, the result that provability in the predicate calculus (even with a rather simple language) is non-recursive was also out of reach, but would follow quite quickly from Gödel's work of 1931 and later characterizations of computability in 1936 and 1937.

The notion that relates the arithmetization back to the system P is that of representability. Gödel gives this in generality (even though for his purposes only a few special cases are required) and his Theorem V that all primitive recursive predicates are represented in P is given. This theorem gets cursory treatment, though the important point that it involves an induction on the primitive recursive functions, and construction of proofs in P is presented. Implicitly, the argument uses the higher types available in P rather than the Chinese remainder theorem technique that appears later, presumably because this was the way Gödel first obtained the result. Moreover, the formal details of Gödel's proof as given are somewhat more complicated than actually necessary, and simplifications were later made in this respect by Kleene [40]). This provides at last all the formal details required to present the proof of the incompleteness theorem as sketched earlier. In fact, his treatment goes slightly further, and he proves his Theorem VI, that if κ is a ω -consistent primitive recursive system extending P, then there is a statement R_{κ} such that neither R_{κ} nor its negation are derivable

from κ .²

In relation to this 'classical' arithmetization of syntax as used by Gödel, it is interesting to note that many modern didactic presentations of Gödel's theorems still use the class of primitive recursive functions for the arithmetization of syntax, even though many much smaller classes, such as the polynomial-time computable functions, suffice, and are more useful in the context of the modern concern with complexity theory. It is as if Gödel's clear presentation and the convenience of using primitive recursive functions has cemented in many people's minds the erroneous idea that the primitive recursive functions are necessary or canonical for the proof of Gödel's theorems. But in the 1930s, it may not have seemed at all obvious that Gödel's adoption of primitive recursive functions was not a necessary feature of the proofs of the imcompleteness theorems.

From the point of view of foundational questions, it is important to observe, as Gödel does, that the statement R_{κ} is of the form $\forall x \ \theta_{\kappa}(x)$, where $\theta_{\kappa}(x)$ is a primitive recursive predicate. In more modern terminology, the statement R_{κ} is Π_{1}^{0} or Π_{1} .

Gödel also remarks that his proof is entirely constructive: not only can the statement R_{κ} be constructed from the data given, but any proof in the system of R_{κ} or $\neg R_{\kappa}$ could be effectively converted to providing proofs of $\theta(n)$ for each n and of $\exists x \ \neg \theta(x)$ in κ . Thus, constructively, a proof of either R_{κ} or $\neg R_{\kappa}$ in κ can be converted to a example of ω -inconsistency of the system κ . He also points out that if κ is ω -consistent, then the statement R_{κ} can be used to give a consistent ω -inconsistent system $\kappa + \neg R_{\kappa}$. These remarks appear to be aimed at the intuitionists, but of much more general philosophical interest. That Gödel held them to be of the utmost importance is clear from his remarks to Wang [67, p84] (quoted here with Wang's commentries in square brackets):

Ulam wrote a book [Adventures of a Mathematician, 1976] and I was mentioned in it at several places. Ulam says that perhaps I was never sure whether I had merely detected another paradox like Burli-Forti's. This is absolutely false. Ulam doesn't understand my result, which is proved by using only finitary arithmetic. As a

²The system κ is said to be ω -consistent if for no predicate $\theta(x)$ in the language of κ is it the case that κ proves $\neg \theta(n)$ for all natural numbers n and also κ proves $\exists x \ \theta(x)$. ω -consistency is a slightly stronger notion that ordinary consistency. Gödel needs ω -consistency to show that the statement R_{κ} equivalent to 'for all x, x is not a proof in κ of R_{κ} ' is independent of κ . This argument goes as follows: if κ proves R_{κ} then there is a proof n of R_{κ} , so κ also proves $\neg R_{\kappa}$ and hence is not consistent; on the other hand, if κ proves $\neg R_{\kappa}$ then κ proves 'there is x such that x is a proof in κ of R_{κ} ' so by ω -consistency of κ there is some concrete p such that κ together with the statement 'p is a proof in κ of R_{κ} , contradicting the consistency of κ .

matter of fact it is much more. [I take this sentence to mean that the proof is not only precise but perfectly clear.] How can Wittgenstein consider it [Gödel's result] as a paradox if he had understood it?

All the results up to this point were worked out and announced in an small meeting in September 1930. As for the improvements that follow, Wang [67, pages 82–83] reports the following recollections by Gödel in the late 1970s.

I took part in a little conference at Königsberg in autumn 1930. Carnap and [John] von Neumann were there. The meeting had no "discussion." I just made a remark and mentioned my [incompleteness] result. [The meeting was the second *Tagung für Erkenntnislehre der exakten Wissenschaften*, at which Gödel presented his proof of the completeness of predicate logic, obtained in 1929, on 6 September, and mentioned incidentally his new result during the discussion session the next day.]

At that time, I had only an incompleteness theorem for combinatorial questions (not for number theory), in the form as described later in the introduction of my [famous] paper. I did not yet have the surprising result giving undecidable propositions about polynomials [by using the Chinese remainder theorem].

I had just an undecidable combinatorial proposition. I only represented primitive symbols by integers and proofs by sequences of sequences of integers. The undecidable proposition can be given in fragments of type theory (and of course in stronger systems), though not directly in number theory. I had a private talk with von Neumann, who called it a most interesting result and was enthusiastic. To von Neumann's question whether the proposition could be expressed in number theory I replied: of course they can be mapped into integers but there would he new relations [different from the familiar ones in number theory]. He believed that it could be transformed into a proposition about integers. This suggested a simplification, but he contributed nothing to the proof because the idea that it can be transformed into integers is trivial. I should, however, have mentioned the suggestion; otherwise too much credit would have gone into it. If today, I would have mentioned it. The result that the proposition can be transformed into one about polynomials was very unexpected and done entirely by myself. This is related to my early interest in number theory, stimulated by Furtwangler's lectures.

To show that every primitive recursive predicate is represented in arithmetic, Gödel uses the sequence of numbers

$$f_k^{(n,d)}$$
 = remainder on dividing n by $1 + (k+1)d$ $(k=0,1,2,3,...)$

associated with numbers $n, d \in \mathbb{N}$. The important lemma here is that for any sequence (f_k) of natural numbers there are $n, d \in \mathbb{N}$ such that $f_i^{(n,d)} = f_i$ for all $i \leq k$, and is proved using the Chinese remainder theorem. Gödel mentions (without proof) that his arithmetization of primitive recursive predicates can be formalized in P, and hence the undecidable statements of the previous section are arithmetic. This device is now known as Gödel's β function, after the notation he used in his Princeton lectures [22]

The final major result (and the most important one for as foundational studies at the time and Hilbert's programme) is the deduction of the second inprompleteness theorem as a consequence of the first. Gödel remarked [67, p84]

Shortly after the Königsberg meeting, I discovered the improved undecidable proposition and the second theorem [about consistency proofs]. Then I received a letter from von Neumann noting independently the indemonstrability of consistency as a consequence of my first theorem. Hilbert and von Neumann had previously conjectured the decidability of number theory. To write down the results took a long time. [This undoubtedly refers to his famous paper. The 'long time' certainly included the period between 7 September (when the initial result was announced) and 17 November 1930 (when the paper was received for publication). It is also possible that he had spent a long time writing an early version before the September meeting.]

(Wang's comments in square brackets.)

The second theorem is, in a sense, the formalization of the first theorem in the system κ , for as previously observed, the consistency of κ is sufficient to deduce the truth of the independent statement R_{κ} . Formalizing this, $P \vdash \operatorname{Con}(\kappa) \to R_{\kappa}$, so κ (being an extension of P) cannot prove the statement $\operatorname{Con}(\kappa)$.

Gödel once again observes that his proof is again constructive, that is, a proof of the consistency of κ inside κ can be effectively converted to a proof of 0 = 1 (or some other inconsistency) inside κ . Interestingly, Gödel writes, 'I wish

³Slicker and more general forms of the second incompleteness theorem are now known, but the method just presented is the one used by Gödel.

to note expressly that Theorem XI [the second incompleteness theorem] does not contradict Hilbert's formalistic viewpoint. For this viewpoint presupposes only the existence of a consistency proof in which nothing but finitary means of proof is used, and it is conceivable that there exist finitary proofs that *cannot* [Gödel's italics] be expressed in the formalism of P.'

Gödel seems to have had much more sympathy with the formalist school than the logicists', and the initial motivation for his work certainly came from attempts to answer Hilbert's questions. But even so this remark strikes one as being rather cautious, especially from today's perspective which sees Gödel's paper as the point where the formalists were shown to be far too optimistic in their approaches to the foundations of mathematics and incorrect in many of their conjectures. Gödel was well known in his later life for avoiding academic conflict whereever possible, and this may just be another example of that. Alternatively, we can speculate that he made this remark because at the time he didn't really appreciate the power of the ordinary (single-typed) predicate calculus, and especially its ability to represent the very clear idea of computaable operations that arguably include the finitary ones. We may even wonder whether Gödel had ideas of other kinds of convincing finitary proofs (he hints as much later on, but what he has in mind seems rather unclear). It is certainly true that Gödel did return back to reconsider Hilbert's programme on several occasions, especially in his lectures and unpublished work [29, 31, 24], so we can be sure that Gödel's remark here is not the result of a simple error or misunderstanding.

Whatever the outcome of these speculations, Gödel's theorems did have a very major impact on the Hilbert programme. To understand this, I will review Hilbert's programme from a somewhat modern point of view here. (Hilbert's programme is put into its proper historical context in an article elsewhere in this volume.)

Hilbert's Programme was a framework for making sound the foundations of mathematics, the nature of 'number' and the role of infinite objects in mathematics. Hilbert recognized that certain ('finitary') calculations with concrete numbers, or with symbols on a page, could be carried out and verified. Certain kinds of mathematical statements describing a calculation and its verification were therefore irrefutable, and the underlying idea behind a 'finitary' proof is that it should not make any appeal to an 'actual' or 'completed' infinity. Although there is still much debate as to what statements Hilbert counted in this realm and the now-straightforward and basic calculations of the complexity of statements in the arithmetic (Σ_n/Π_n) hierarchy were not commonplace at

Ed: crossreference needed the time, I will simplify the discussion slightly and identify the collection of statements that (if true) can be verified by such a calculation with what are now known as the Σ_1 (or Σ_1^0) statements of arithmetic. (Certainly, with hind-sight, we know that all of the calculations that can be carried out on a machine such as a Turing machine fall within Σ_1 , and this is all that is required for the present discussion.) Thus a statement is Σ_1 if it states that certain concrete numbers and arithmetic calculations could be found to verify its truth. (Note that a Σ_1 statement does not in general itself provide those numbers or calculations, thus the truth of such a statement may not be immediately obvious.) A statement that expresses the falsity of a Σ_1 statement is called a Π_1 (or Π_1^0) statement. Hilbert noted that a great many interesting and difficult mathematical problems concern proving that a Π_1 statement is true, and that in many cases this cannot (obviously) be done by direct calculations, except for those calculations which are actually proofs in some formal system S that we accept as being sound.

Hilbert's second observation concerned systems that formalise proof in arithmetic. He observed that even the very simplest systems are able to formalize concrete arithmetic calculations, and hence are able to prove all true Σ_1 statements. This means that, as far as Π_1 statements are concerned, the soundness of a system S only depends on its consistency, for if S were able to prove a false Π_1 statement $\neg \theta$, then it would also be able to prove the true Σ_1 statement θ and hence would be inconsistent. More importantly, Hilbert observed that the inconsistency of such a system is itself a Σ_1 statement. Thus if S is an inconsistent system, it will actually be able to prove its own inconsistency. Now suppose we have two such systems: one, S, that we know for sure is consistent; and a second one, T, which is 'stronger' and involves some abstract notions of infinite objects. If T is inconsistent, then its inconsistency is a Σ_1 truth, so provable in S. Thus, provided we know the consistency of T (for example, by proving this consistency in the system S which we know is sound for such statements), we know that T itself is sound for Π_1 statements. We still cannot be sure that it is sound for statements about the 'ideal' infinite objects available in T, but this need not matter, since we would know the soundness of T for the 'real' Π_1 and Σ_1 statements. By finding a suitable sequence of systems $S_0 = S, S_1 = T, S_2, S_3, \ldots$, Hilbert hoped to put mathematics on a firm footing. Consistency proofs for fragments of arithmetic were indeed found, by Ackermann [1], von Neumann [61], and Herbrand [33]; indeed, even after Gödel's theorem, consistency proofs for arithmetic were still important, especially in the realm of proof theory, and two particularly notable proofs of the consistency of PA were by Gentzen [16] using transfinite induction up to ϵ_0 and Gödel [24] using primitive recursive functionals.

The difficulty in deciding what the consequences of Gödel's incompleteness theorems for Hilbert's programme is in determining just what constitutes *finitary* methods and *finitary* proofs, and Gödel was well aware of the points here. Certainly, a restrictive reading of Hilbert's would seem to suggest that finitary methods really do all lie in the realm of primitive recursion, and thus Hilbert's programme in this restricted sense is doomed to failure by Gödel's second incompleteness theorem. But, as Gödel said much later [?, footnote b]

There is nothing in the term 'finitary' which would suggest a restriction to concrete knowledge. Only Hilbert's special interpretation of it makes this restriction.

One other possible viewpoint in 1931 would be to consider that 'finitary processes' might denote 'computable processes' (in the sense of human computers following algorithms as discussed by Turing and others later) but nevertheless still outside the realm of primitive recursion. As I have mentioned, the precise connections (or lack of them) between primitive recursion and Gödel's incompleteness theorems was not fully understood at this point in time. But more particularly, the acceptance of Church's thesis, especially after Turing's [58] account of computability, and more general results on representability—from the Kleene normal form for example—would rule out this form of Hilbert's programme just as well.

Gödel himself seems to think that finitary could include processes beyond the recursive, and hints at extending type levels through the transfinite to settle his undecidable propositions. It is common folklore that Gödel intended to complete the proof of the second incompleteness theorem and extend the generality and applicability of both incompleteness theorems (see Gödel's last paragraph of his 1931 paper and van Heijenoort's footnote 68a to his English translation of the paper [60, 27], which was seen and approved by Gödel). A similar remark is made by Rosser [45] in 1939, who writes

The hidden assumptions [on a system for which the Gödel incompleteness theorem apply] have never been put down explicitly in a form intelligible to the average reader. It is my understanding from conversations with Gödel that an exact formulation of these assumptions was to constitute part of the second part of the paper of which [20] is the first part. Due to ill health, Gödel has never written this second half. However, Kleene gives an exact statement RWK: I have lost the reference I had for this! Presumably it is from a later version of the dialectica interpretation.

14

of a set of assumptions sufficient for the proof of Gödel's First Theorem. Unfortunately they are phrased in terms of general recursive functions, and are illuminating only to someone who is thoroughly familiar with the theory of general recursive functions.

But according Gödel's own footnote 48a in his 1931 paper, this was not all. The footnote reads:

As will be shown in Part II of this paper, the true reason for the incompleteness inherent in all formal systems of mathematics is that the formation of every higher type levels can be continued into the transfinite (Hilbert [35]) while in any formal system as most denumerably many of them are available. For it can be shown that the undecidable propositions constructed here become decidable whenever appropriate higher types are added (for example, the type ω to the system P). An analogous situation prevails for the axiom system of set theory.

In the end, Gödel never published any account of these ideas, though as mentioned earlier, there is much in Gödel's later work to show that did spend much time thinking about modifications to Hilbert's programme. The intriguing thing about this footnote for me is that it hints at more than this: it hints at some sort of completeness result for the totality of all formal systems at higher types. But though such systems are discussed at length in later papers, and in particular the idea of using higher types to decide previously undecidable propositions, I could not find any later reference implying any such completeness result. Turing [59] did, however, find a result of this type, which will be discussed shortly.

1.2 The development of first-order arithmetic

Gödel's 1931 paper can be seen as the beginnings of the modern study of first-order arithmetic, in that it was here that the two main techniques of *arithmetization* and *diagonalization* are first introduced and used to effect.

The surprise concerning arithmetization seemed to be Gödel's argument using that Chinese remainder theorem that ordinary first-order arithmetic of + and \cdot for $\mathbb N$ is so expressive. This result not only forced a complete re-think in the Logisicts' and Formalists' programme, put it also put severe bounds on Tarski's programme of applying the method of quantifier elimination to natural mathematical structures to obtain undecidability results.

Of course, Gödel had used the system P with its higher types to simplify his work in the 1931 paper, but shortly afterwards he seems to have realized that P can be replaced by what is now known as first-order Peano-arithmetic, PA, (which Gödel called Z) and he published a short extract of these results the next year, in 1932 [21]. This 1932 paper might be seen as the true beginnings of (first-order) PA. Thus the importance of Gödel's method of arithmetization is not only to show that certain first-order arithmetic statements are formally undecidable, but that the system of first-order arithmetic, PA, is strong and worthy of study in its own right.

Gödel was a visiting member of the Institute of Advanced Study at Princeton from October 1933 to May 1934, and during this period re-worked and extended his work of 1931; between February and May he gave lectures on the incompleteness theorems. These lectures were, at the suggestion of Veblen, written up by Rosser and Kleene and approved by Gödel, who added two pages of 'notes and errata'.

One feature of Gödel's lectures was explicit mention to what is now known as the diagonalization lemma,

Lemma 1.1 (Diagonalization) For all $\theta(x)$ there is D such that

$$PA \vdash D \leftrightarrow \theta(\lceil D \rceil).$$

and its immediate consequence,

Theorem 1.2 (The undefinability of truth) For all $\theta(x)$ there is D such that

$$PA \vdash D \leftrightarrow \neg \theta(\lceil D \rceil).$$

Specifically, in a model of a suitably rich theory, T, there is no formula θ such that $M \models \sigma \Leftrightarrow \theta(\lceil \sigma \rceil)$ for all sentences σ of the language of T.

Interestingly, in footnotes added later, Gödel attributes the diagonalization lemma to Carnap [4] and the undefinability of truth to Tarski [53] and Carnap [3], but it is clear from his own remarks (for example the remarks made to Wang presented above on page 5) that Gödel had already noted these results for himself by 1930, but had not seen fit to publish them.

Ed: Please note page reference here

In section 8 of his Princeton lecture notes, Gödel introduces his β function notation,

 $\beta(c,d,i)$ = the least non-negative residue of c modulo 1+(i+1)d.

Gödel also takes the arithmetization a stage further showing that the undecid-

Preliminary draft 17

able stament may be taken to be arithmetic with a diophantine matrix.

$$Q_1x_1 Q_1x_1 \dots Q_1x_n p(x_1, x_2 \dots, x_n) = q(x_1, x_2 \dots, x_n)$$

for polynomials p, q with natural number ocefficients and each Q_i being either \forall or \exists . Finding better polynomial representations of these statements (or, equivalently, better polynomial representations of the r.e. sets) would eventually become a major theme in first-order arithmetic, with more unpublished work done later by Gödel himself [30] and also important published work by Julia Robinson, Martin Davis, Hilary Putnam [8, 9], ultimately concluded by Matiyasevich [41] in his solution to Hilbert's tenth problem.

Hilbert and Bernays [34] (pages 283–340) carried out all the details of the proof of the second incompleteness theorem for the systems Z_{μ} and Z of number theory. Bernays would go on to further develop mathematics inside formal number theory with his arithmetized completeness theorem, which would later have important consequences for the model theory of PA.

Another obvious question left over from Gödel's work was whether in the first incompleteness theorem it is possible to reduce the slightly mysterious condition of ω -consistency to ordinary consistency. In other words, is it possible to have a primitive recursive extension of PA which is consistent and complete for first-order arithmetic. (Note that by Gödel's work, such a system cannot be sound, or even ω -consistent, so it is not at all obvious if such systems, and their existence or non-existence, would have any significant foundational importance, except to clarify the idea of ω -consistency.) This question was settled by Rosser [44] negatively. As a consequence, it follows that there are 2^{\aleph_0} complete extensions of a theory such as PA.

Rosser's trick may be seen as typical of some techniques used in recursion theory later on, and indeed from a modern recursion theoretic viewpoint, Rosser's theorem is essentially equivalent to the statement that there are disjoint, r.e., recursively inseparable subsets $A, B \subseteq \mathbb{N}$. Much later on, this key result of recursion theory would be particularly important in the model theory of arithmetic, including Tennenbaum's theorem that nonstandard models are nonrecursive [56] and the failure of the joint embeddability property for models of PA.

From the modern persepctive, the most obvious ingredient missing in Gödel's original paper was the link with computability theory. The story of how the notion of computability and its link with Gödel's incompleteness theorems were found is told at much greater length elsewhere in this volume, but to summarize briefly, the notion of a computable function arose in four quite different contexts: from Church's λ -calculus, from Gödel's general recursive functions, from Post's

RWK: Look up refs for the arithmetized completeness theorem.

Ed: please add cross reference

analysis of formal systems, and from Turing's analysis of what are now called Turing machines. Church [5] proposed that his class of λ -definable functions should be identified with the idea of intuitively computable functions and—using this identification—solved Hilbert's last outstanding foundational problem, the Entsheidungsproblem, negatively. Despite the success when Kleene [39] verified that the λ -definable functions and Gödel's general recursive functions agreed, Gödel had remained sceptical, until he had seen Turing's argument [58, 57] using his Turing machines. Sieg [47] presents a nice account of this period.

This work was to spawn the new subject of computability theory. Work by Post and Kleene in particular [38, 42] gave impetus to the new theory, which at least in its early stages was very much influenced by Gödel's results, through Post's analysis of formal systems and Kleene's recursion theorem which can be seen as an elegant and useful an application of Gödel's diagonal method in the realm of computable functions.

A major topic to grow out of Gödel's incompleteness theorems was the study of logics of higher types and the revision of Hilbert's programme. Much of this falls under the field of proof theory and its history is discussed elsewhere in this volume, but I want to return to the topic metioned earier: Turing's ordinal logics and in particular his completeness theorem.

Ed: provide cross ref

The history of how Turing wrote his paper on ordinal logics [59] is written up by Feferman [12] and Hodges [36]. Turing worked on his Ph.D. thesis in Princton under Church from September 1936, and was granted the degree in June 1938. Turing never met Gödel, nor, apparently did he meet Kleene, Bernays or Rosser, and seems to have taken up the challenge of working on ordinal logics at Church's suggestion. In broad terms, the idea is as follows. Suppose T_{α} is a formal system for arithmetic that is known to be sound (i.e., only proves true statements), and let $\operatorname{Prov}_{T_{\alpha}}(x)$ denote 'the fomula with Gödel number x is provable in T_{α} ' and $\lceil \phi \rceil$ denote the Gödel number of ϕ . Then consider the scheme of local reflection for T_{α} :

$$\operatorname{Prov}_{T_{\alpha}}(\lceil \phi \rceil) \to \phi$$

for all arithmetic statements ϕ . This scheme is also sound when added to T_{α} , since If $\operatorname{Prov}_{T_{\alpha}}(\lceil \phi \rceil)$ then ϕ is provable in T_{α} so ϕ is true, as T_{α} is sound. On the other hand, the special case of scheme of local reflection for T_{α} with ϕ set to 0=1 implies the consistency of T_{α} , so local reflection is not provable in T_{α} . Thus we obtain a new system $T_{\alpha+1}$ which is known to be sound. At limit stages we can take unions $T_{\lambda} = \bigcup_{\alpha < \lambda} T_{\alpha}$ and obtain yet another sound theory. This can be carried on, getting stronger and stronger theories that appear to decide

more and more of the Gödel statements.

There are some important technical issues though: to define $\operatorname{Prov}_{T_{\alpha}}(x)$ it is necessary have a recursive (i.e., computable) axiomatization of T_{α} , so it is important that, at limit stages, the union $\bigcup_{\alpha<\lambda}T_{\alpha}$ is recursive. What this amounts to saying is that the inductive construction must be carried out not so much on ordinals α but on recursive notations for ordinals.

These are very natural ideas that might have occurred to many people at the time, but are particularly reminiscent of the 48b footnote in Gödel's 1931 paper. In the particular case here, it is quite natural to wonder if they arose out of discussions between Church and Gödel when Gödel had been visiting Princeton prior to Turing's visit. Unfortunately it seems impossible to answer this question at present, though some information on it may come to light in the future. In any case Turing set about studying ordinal logics, and it became the subject of his Ph.D. thesis and his 1939 paper.

Turing's work concerns ordinal logics in a very general setting, but I will simplify the discussion by simply picking out one or two highlights of it concerning the logics built on the local reflection principles above. Firstly, a notation for an ordinal is some a which codes both an algorithm to decide whether an $x \in \mathbb{N}$ is in the domain of some well-ordering $<_a$ on a subset of \mathbb{N} and an algorithm that decides if $x <_a y$ for x, y in this domain. (It is not an easy matter to decide whether a give a has this property. Turing also considers the better-behaved notations for recursive ordinal given by Church and Kleene [6].) Turing considers several specific examples of ordinal logics, but in his presentation of the reflection scheme, he only considers it for formulas ϕ that are what is now called Π_2 (or Π_2^0) statements, i.e., of the form $\forall x \exists y \ R(x,y)$ for R primitive recursive. (He gives a very pretty argument showing that an arbitrary Π_2 statement can be put in the form

$$\forall x \; \exists y > x \; R(y)$$

with R(y) primitive recursive, stating that the set of y satisfying R(y) is infinite.) With these preliminaries, he can define the ordinal logic called Λ_P for local reflection by the above scheme, taking for T_1 the system P used by Gödel in his 1931 paper, and extending to T_a for all recursive notations a for ordinals.

The most interesting result Turing proves is a completeness result, that if σ is a true Π_1 statement then for some ordinal notation representing an ordinal of value at most $\omega + 1$ we have $T_a \vdash \sigma$. (The question Turing poses of whether this is also true of all true Π_2 statement σ was later answered negatively by Feferman [11].) The proof is a very clever and delicate argument based on a

diagonalization using the recursion theorem of Kleene and an application of the Gödel second incompleteness theorem, and seems at first sight to provide just what Gödel was looking for as a way out of the incompleteness phenomena into a complete system based on some finitistically justifiable logic (but not formalizably so).

But, as nice as Turing's proof is, the result unfortunately does not live up to any of these expectations. Part of the reason is in the necessity to work with ordinal notations rather than ordinals: in fact this is essential, since not only (as Turing shows) does Λ_P fail to be invariant—in the sense that there are notations a and b for the same ordinal α such that T_a and T_b are inequivalent—but this property of invariance must fail for all complete ordinal logics. What's more, Turing's completeness proof comes by means of a trick that does not tell you how to proceed in any real case of determining whether a Π_1 statement σ is true or false, and the foundational issue of deciding which systems T_a one might believe sound boils down to the issue of determining which a are valid notations for ordinals (in particular determine a linear order that is wellfounded), and this question is just as tricky as the original one of deciding truth or falsity of Π_1 statements of arithmetic. Turing also presents an ordinal logic based on transfinite type theory which a invariant, but of course because of this invariance it is necessarily incomplete.

Turing's paper contains a large number of other important ideas (including the idea of an oracle machine and the essence of a proof of (in modern terminology) the statement that the arithmetical hierarchy of formulas is proper, as well as much more on logics, and I cannot possibly do justice to it in these few paragraphs. But as far as the topic ordinal logics is concerned, Turing seems to have given up the situation as 'hopeless'—the Gödel incompleteness phenomena are too pervasive—and did not return to the subject again.

I don't believe that Gödel ever referred to Turing's work. As far as Gödel was concerned, his incompleteness had 'explained the nature of arithmetic'; the next foundational questions were to understand set theory and how logic higher types can settle statements concerning objects of lower type, such as consistency statements. Gödel would go on to look at consistency and independence results in set theory. On the other hand, the work had spawned exciting new directions in proof theory, consistency proofs and the *partial* realizability of Hilbert's programme.

One issue that was left in the air for some time after Gödel's incompleteness results was exactly how complicated independent statements might be, and whether one could find independent statements in arithmetic of genuine mathRWK: what is the reference here?

21

ematical (rather than metamathematical) interest. In some important cases, it turned out that mathematically interesting statements equivalent to certain Gödel statements could be found, and their analysis was initially done using a new theory in mathematical logic that was only just starting up in the 1930s: the theory of models of arithmetic. This story will be taken up in the next section.

2 Models of arithmetic

We have seen how foundational questions in mathematics—Hilbert's formalist programme and Russell's logisist programme in particular—lead to the study of provability, computability and interpretations of first-order arithmetic, via the key results and methods due to Gödel.

These three strands of arithmetic have proved very profitable areas of mathematical research. The proof-theoretic aspects are explored in more detail by Sieg's article in the current volume, and the computational aspects are covered by Davis and others here. This section is devoted to interpretations of first-order arithmetic, and the development of the theory of models of arithmetic. As we will see, a clearly defined subject arises, one whose role is certainly related to the other metamathematical aspects of arithmetic and model theory. We will examine that role, and areas where models of arithmetic have been particularly influential.

Despite its obvious relevance, the area of models of arithmetic and nonstandard models has sometimes found itself peripheral to model theory itself, and is not always the tool of first choice for proving independence results in arithmetic. Nevertheless, the major advance in the subject of first-order arithmetic, the Paris–Harrington theorem of 1977 giving the first mathematically interesting statement independent of PA, was discovered via the study of models of arithmetic and its history is intimately connected with several important features of the theory of nonstandard models. My aim in this section is to outline this history of ideas concerning models of arithmetic, focusing particularly on the ones that lead up to the Paris–Harrington theorem, and explain how the Paris–Harrington theorem appears rather natural—indeed, perhaps inevitable—from the point of view of model theory.

2.1 Constructions of models of arithmetic

If we trace back the idea of a *nonstandard model of arithmetic*, we find three distinct ideas and constructions that were available to mathematicians in the

1930s.

In all cases, the construction of a model of arithmetic is somewhat non-effective, and as such might have seemed counterintuitive or unappealing at the time. Nowadays, we know now that nonstandard models of arithmetic are necessarily nonrecursive and highly complicated mathematical objects, and that this is very much related to the Gödel's incompleteness phenomena. But this was not at all clear in the 1920s and 30s, and Skolem (who was first to construct such a model) and many of his contemporaries must have wondered whether a simpler construction might not suffice.

The first explicit construction of a nonstandard model was by Skolem [51, 48]. The methods he used originate from the idea of what are now called Skolem functions, but with an extra twist rather suggestive of ultraproducts (though there are some essential differences between his methods and Łoś's theorem that would come later).

The motivation for constructions of models of arithmetic come from foundational questions concerning what logic is required to characterize the natural numbers. It was well known from a result going back to Dedekind in 1888, that the second-order theory of the natural numbers, \mathbb{N} , suffices to characterize it up to a canonical isomorphism, and this had been used to give a satisfactory definition of the natural numbers in set theory. As Skolem later wrote [49, p1] (referring to the 'Skolem paradox' in set theory),

This definition cannot then be conceived as having an absolute meaning, because the notion set and particularly the notion subset in the case of infinite sets can only be asserted to exist in a relative sense. It was then to be expected that if we try to characterize the number series by axioms, for example by Peano's, using the reasoning with sets given axiomatically or what amounts to the same thing given by some formal system, we would not obtain a complete characterisation. By closer study I succeeded in showing that this really is so. This fact can be expressed by saying that besides the usual number series other models exist of the number theory given by Peano's axioms or any similar axiom system.

Skolem's method was ingenious, but other methods existed at the time. The surprise was that noone seemed to have realised their relevance to these problems.

Perhaps the simplest approach, and the method of choice for modern model theory textbooks, is a simple application of Gödel's completeness and compactness theorems of 1930 [18]. Here, one need simply observe that by compactness

one may add the sentences $\{c > n \mid n \in \mathbb{N}\}$ to Th(N) (where c is a new constant symbol) to obtain a consistent theory with a (necessarily nonstandard) model. Noone seemed to notice this at the time though Mal'stev was using compactness aruments of this type around this time; Kleene [37] attributes the first application of compactness to build nonstandard models of arithmetic to Henkin [32] as late as 1947^4 . A third method for constructing nonstandard models was given—by Gödel himself—in his review of Skolem's 1934 paper [23], where he points out that by the incompleteness theorems, PA + \neg Con(PA) is consistent, so has a (necessarily nonstandard) model by the completeness theorem. But again, Gödel fails to spot the purely model theoretic argument.

RWK: look this up

To appreciate these points properly, it might help to contrast these methods. Both the Skolem method and the compactness argument easily give a model of true arithmetic, that is, a single nonstandard model which satisfies all first-order statements true in \mathbb{N} . But on the other hand, the only definable elements in such a model are the standard ones. On the other hand, Gödel's methods cannot give a model of true arithmetic (they always satisfy some statement $\neg Con(T)$ for some consistent theory T) but they do construct models with nonstandard definable elements. These distinctions were not noticed at the time, it seems at least in part because the idea of 'all true arithmetical statements' was not on the logicians' agenda. (As we have just seen, even in 1955 Skolem phrased his results by saying that 'models exist of the number theory given by Peano's axioms or any similar axiom system'. Much later on, Skolem's method would be important for the construction of elementary end extensions of models of PA by MacDowell and Specker and later by Gaifman, whereas Gödel's method would lead to results about the non-finite axiomatization of PA proved by Ryll-Nardzewski and Mostowski.

At around the same time as Skolem's constructions of models of arithmetic, work was being carried out on more constructive models and theories of arithmetic. One line here was again begun by Skolem, in his ideas of free-variable arithmetic, and in particular what is now known as Primitive Recursive Arithmetic [50]. Tarski's student Presburger looked at what is now known as Presburger arithmetic [43], essentially the theory of $(\mathbb{N},0,1,+,<)$ without the multiplication operation, and though it did not seem to be of interest at the time, it is worthwhile pointing out that Presburger's characterization of models of this theory can be used to obtain effective models. Skolem also looked at the theory of $(\mathbb{N},0,1,\cdot,<)$ without the addition operation, with similar results. Skolem presented constructions of nonstandard models.

RWK: Check!

RWK: did

he?

⁴Hodges informs me that Kemeny had some similar constructions soon afterwards

The search for simple or effective constructions for models of arithmetic has continued to this day. In one line of work, there are questions concerning the decidability—or otherwise—of expansions of Presburger (adding a predicate for primality, for example). Another line continues Skolem's idea of quantifier-free arithmetic. This was emphasised by Kreisel, who also pointed out the connections between constructions of models and effectivity of a theory, and work was continued by Shoenfield and Shepherdson. In particular, Shepherdson managed to relate the decidability of quantifier-free Peano arithmetic (with the usual language of Peano, but no quantifiers, and induction expressed as a rule

$$\frac{\Gamma, \theta(x) \vdash \theta(x+1), \Delta}{\Gamma, \theta(0) \vdash \theta(t), \Delta}$$

with x not free in the sequent on the bottom) with an ordinary first order theory, IOpen. The statements provable in quantifier free arithmetic should be interpreted in ordinary logic by their universal closure. These then correspond exactly to the universal statements provable in IOpen. Shepherdson also went on to construct recursive models of IOpen, and provide indepence results for quantifier-free PA. The key question here, of the decidability of QFPA is still open, however.

On the other hand, more recent work based on what is effectively the model-theoretic and recursion-theoretic analysis of the incompleteness theorems has led to the idea that nonstandard recursive models of strong theories such as PA do not exist, as proved by Tennabaum [56], using a pair of r.e., recursively inseparable sets—the recursion theoretic analogue of Rosser's incompleteness theorem.

erences required

ref-

RWK:

2.2 Towards a structure theory for models of arithmetic

As we have seen, early interest in models of arithmetic from the point of view of foundational studies was low. Thus, models of arithmetic were used as important examples in the study of the new subject of model theory.

Each complete extension of PA has continuum-many countable models: to see this, notice that by compactness any set $A \subseteq \mathbb{N}$ can be coded in some such model. There is also a nice link with computability and diagonalization here: any noncomputable set is not coded in *some* nonstandard model of PA, though every computable set is coded in every nonstandard model. Recursion theoretic results also enable PA to be used as a source of other model theoretic examples. For example, Post's construction of a simple set [?] enables one to show that PA fails a joint embedding property: there are models M_1 and M_2 both elementary

Ed: Link with Hodges' article RWK: Who first did this? RWK: Who first did this?

extensions of \mathbb{N} which cannot be jointly embedded in a third model M of PA.

List of topics to include in the next version of this paper At this point, I intend to mention at least briefly the following:

RWK: a section needs to be written here.

- Results of Ryll-Nardewsi and Mostowski: the first to use models of PA to give results of foundational interest.
- MacDowell–Specker theorem.
- Rabin and diophantine-incorrect models of PA.
- Abraham Robinson, and especially overspill.
- Tennenbaum's theorem and some later stuff on the Tennenbaum phenomena.
- Dana Scott's work, espically on relativizations of the completeness theorem and 'Scott sets'.

As Hodges points out in his article in this volume, the beginning a more 'modern' view in model theory concerns the different kinds of relationships that can occur between two or more models and the structural properties of the models that these relationships show. In the case of models of arithmetic, PA in particular, The MacDowell–Specker theorem was an early example of this sort of result. But the first person to study models of PA in this way as abstract objects 'for their own sake' was Gaifman. His work would have quite profound consequences for applications of models of PA too, as we shall see.

Gaifman's splitting theorem [14] concerns the kind of extensions of models of PA and shows that essentially there are only two kinds. The model theoretic extensions of interest in the study of PA are the end extensions and the cofinal extensions, and moreover cofinal extensions are always elementary. To prove these results he first observes that the then-new MRDP theorem answering Hilbert's tenth problem can be carried out in PA, and hence

Theorem 2.1 For all $\theta(\bar{x})$ in Σ_1 there is $\theta_{\exists}(\bar{x})$ in \exists_1 with the same free-variables such that $PA \vdash \forall \bar{x} (\theta(\bar{x}) \leftrightarrow \theta_{\exists}(\bar{x}))$.

From this, he deduces

Theorem 2.2 If $M_1 \subseteq M_2$ and both M_1, M_2 satisfy PA then $M_1 \prec_{\Delta_0} M_2$ and there is $\overline{M_1} \subseteq_{\operatorname{e}} M_2$ such that $M_1 \prec_{\operatorname{cf}} \overline{M_1} \subseteq_{\operatorname{e}} M_2$.

Ed: cross ref

This paper, which is entirely and unashamedly model theoretic, is particularly useful for extending results about countable models to the uncountable, but at first sight doesn't seem to say much about foundational issues. There is one important methodological point that can be drawn from this paper however: this is that since cofinal extensions are elementary and hence preserve all first-order sentences, to find a model satisfying a new theory, it suffices to look at initial segments and end-extensions.

Gaifman's second paper concerns types. In it he introduces the idea of a definable type—now a mainstay of stability theory—and a minimal type, and uses these ideas to produce a huge variety of models of arithmetic with almost any automorphism group $G = \operatorname{Aut}(S, <)$ one cares to take provided the group acts faithfully on a linearly ordered set. [15] Here was perhaps the first major application of Ramsey's theorem to models of PA: the types are constructed by an elegant and powerful extension of Skolem's original method, but working from within the formal theory PA itself not from 'the outside'. Also, rather than Skolem's straightforward combinatorial principle, Gaifman used a formalization of a version of Ramsey's theorem to construct his formulas one-by-one. We thus see Gödel's β -function used in a most essential way to identify the algebraic theory of the semiring $(\mathbb{N}, 0, 1, +, \cdot)$ with the combinatorial theory of finite sets and definable infinite sets of finite sets.

2.3 Initial segments and Mathematical independence results

At the Cambridge summer school of 1971 Harvey Friedman presented several startling and highly influential results concerning initial segments of models of PA.

Friedman had, without being aware of Scott's previous work, rediscovered the correct closure properties of the standard system $\mathrm{SSy}(M)$ of a model M. He also realized the importance of $\mathrm{SSy}(M)$ to the model theory of a nonstandard model M, and to realizing types in particular. If a type (by definition, here, a finitely satisfied set of formulas with at most finitely many parameters) $p(\bar{x})$ is coded in M, i.e., the set of Gödel numbers $\{ \lceil \theta(\bar{x}) \rceil \mid \theta(\bar{x}) \in p(\bar{x}) \}$ is in $\mathrm{SSy}(M)$ and is of bounded complexity, then an overspill argument together with a Σ_n -universal formula $\mathrm{Sat}_{\Sigma_n}()$ shows that $p(\bar{x})$ is realized in M. This allowed Friedman to prove several striking results on initial segments of countable models of PA. Here are two samples of these kinds of results.

Theorem 2.3 If M is a nonstandard model of PA, and T is a complete exten-

sion of PA which is coded in M, and $M \models \Sigma_1$ -Th(T) then for all nonstandard $b \in M$ with

27

$$T \vdash \exists \bar{x} \ \theta(\bar{x}) \Rightarrow M \vDash \exists \bar{x} < b \ \theta(\bar{x})$$

for all $\theta(\bar{x})$ in Δ_0 , the model M has a nonstandard proper initial segment I satisfying T not containing b (and such b always exist).

In particular, there are arbitrarily small nonstandard initial segments of M satisfying PA.

Theorem 2.4 If M is a nonstandard model of PA then there are arbitrarily large proper initial segments I of M isomorphic to M.

These result, it seems, came as a great surprise. Although Tennenbaum's theorem had shown that nonstandard models of PA have to be nonrecursive, it appears that no-one expected the initial-segment structure of such models to be so complicated.

Friedman's results apply to models of second-order arithmetic and to models of set-theory too. The techniques he used would have very far-reaching consequences and spawn new research in its own right. One of these new areas was that of admissible sets and structures, and admissible languages (Barwise), which lead Barwise and Schlipf to the idea of recursive saturation. The idea of recursive saturation was independently discovered by others about this time: recursive saturation is what one obtains when one drops the condition that a coded type be of bounded complexity in the Friedman type-realization lemma referred to above, and is sufficient in the case of countable models to imply the powerful notion of resplendency (Ressayre). Finally, recursive saturation arises very naturally when one investigates the extenal 'extension' of models M within an ω -nonstandard model of set-theory (Wilmers). Recursive saturation has become a useful tool in the model theory of arithmetic: countable recursively saturated models are, by and large, the 'nice' well-behaved models to work with, and are available in sufficient supply so that they can be used to study arbitrary first-order theories of arithmetic (unlike the notion of a saturated model which is more commonly used in mainstream model theory to make 'constructions come out right', but which only exist in certain cardinalities and under special set-theoretic assumptions for theories such as arithmetic).

But Friedman's results on initial segments were to have more immediate impact in the mid 1970s, in that the sort of structure theory he had been developing turned out to be the key to a new series of independence results for first-order arithmetic.

Jeff Paris, who had been present at the Cambridge summer school, became intrigued by properties of initial segments I of countable models $M \vDash \mathrm{PA}$ analogous to (large-) cardinal hypotheses in set theory.⁵ For example, the notion of a measurable cardinal in st theory corresponds to an initial segment $I \subseteq_{\mathrm{e}} M$ of a model M of PA such that there is an elementary extension $K \succ M$ such that I is still an initial segment of K but there is $b \in K$ with a < b < c for all $a \in I$, $c \in M \setminus I$.⁶

The immediate question was: what first-order theory must be satisfied by such a cut I? The answer to this question, and several similar ones, were worked out by Paris and his research student Laurie Kirby. It turned out that the first-order theory of regular cuts (as cuts I with the property above were called) is the fragment $B\Sigma_2$ of PA. This is perhaps surprisingly weak, but nevertheless it is interesting that the first-order theory of such cuts comes out to be so straightforward to state. Kirby and Paris also studied other combinatorial properties of cuts, the chief ones being semiregular cuts (corresponding to the subtheory $I\Sigma_1$ and the strong cuts (corresponding to PA itself).

The other main feature of this early work of Paris and Kirby was their detailed analysis of some other the other methods used by Friedman. Friedman had shown how, given a < b in $M \models PA$ and a recursively axiomatized theory T of arithmetic there is an initial segment $I \subseteq_e M$ with $a \in I < b$ satisfying T if and only if

$$T \vdash \forall x \; \exists y \; \theta(x,y) \Rightarrow M \vDash \exists y < b \; \theta(a,y)$$

for all Δ_0 formulas $\theta(x, y)$. Using formalization of syntax similar to Gödel, Paris and Kirby internalized this argument and came up with the idea of an *indicator*. For example,

$$Y_T(a,b) = \begin{cases} \text{the least pair } \langle p,g \rangle \text{ where } g = \ulcorner \theta(x,y) \urcorner, \ \theta \text{ is } \Delta_0, \\ p \text{ is the G\"{o}del number of a proof of } \forall x \ \exists y \ \theta(x,y) \\ \text{from } T \text{ and } M \vDash \neg \ \exists y < b \ \theta(a,y). \end{cases}$$

⁵This perhaps explains why Jeff Paris and many other workers in models of arithmetic prefer to draw diagrams of models of arithmetic going up the page, whereas a left-right diagram might be more natural if we think of nonstandard naturals as part of an extended number-line.

⁶This cannot be achieved by an initial segment with a top element x, for if x was the top element of an initial segment I then x + 1 would be the least element not in I, for all models K containing I.

Then it turns out that

$$Y_T(a,b) > \mathbb{N} \Leftrightarrow \exists I \subseteq_{\mathrm{e}} M \ I \vDash T \& a \in I < b$$

i.e., the value $Y_T(a, b)$ 'indicates' whether there is a cut $I \subseteq_e M$ with $a \in I < b$ and $I \models T$.

The combinatorial properties of a cut being semiregular, regural, or strong, also have indicators; in fact $Y_{\text{I}\Sigma_1}$ is an indicator for both regular and semiregular cuts⁷ and Y_{PA} is an indicator for strong cuts, although it turned out that mathematically simpler indicators could be found by working directly from the combinatorial properties rather that from first-order syntax. Paris and Kirby, in their construction of semiregular cuts essentially show that

$$Y_G(a, b) = (\max n) F_n(a) < b$$

is an indicator for semiregular cuts, where F_n is the fast-growing or Grzegorczyk hierarchy, $F_0(x) = x + 1$, $F_{n+1}(x) = F_n^{(x)}(x)$.

Paris and Kirby were also aware that indicators give independence results: if Y(a,b) is an indicator for T then

$$T \not\vdash \forall a \; \exists b \; Y(a,b) \geqslant a.$$

Here then is the power of the indicator method: by concentrating on combinatorial properties of cuts, indicators can be defined quite simply, but mathematically interesting independence results may then be derived from them rather quickly. The first such for PA itself was provided by Paris [?] when he provided an indicator for strong cuts. Harrington simplified the arguments and the two collaborated by providing the most famous of all the independence results that followed by this method, the Paris–Harrington theorem [?].

The main device required to find an indicator for the key case of strong cuts is a way of relating truth in the initial segment I to truth in the model M. in the Paris–Harrington approach, this is achieved using *indiscernibles*. By repeatedly finding a subset of some coded $A = \{a_0, a_1, \ldots, a_n\} \subseteq M$ of nonstandard size n, one obtains an internal increasing sequence (c_i) in M such that for

$$I \vDash \{x \in M \mid \exists i \in \mathbb{N} \ x < c_i\} \tag{1}$$

 $^{^7}$ But note that a semiregular cut need not be regular: these classes of cuts are said to be symbiotic, i.e., they have the same indicators.

and any parameter $a < c_i \in I$ we have

$$I \vDash \exists x_1 \ \forall x_2 \ \exists x_3 \dots \theta(x_1, x_2, x_3, \dots, a)$$

$$\Leftrightarrow M \vDash \exists x_1 < c_{i+1} \ \forall x_2 < c_{i+2} \ \exists x_3 < c_{i+3} \dots \theta(x_1, x_2, x_3, \dots, a)$$
(2)

With this idea in mind, it's clear that the indicator for strong cuts will be based on a variation of the finite Ramsey theorem. In fact, Paris and Harrington used

$$Y(a,b) = (\max n) \begin{cases} \text{for any (internal) partition of } \left[[a,b]\right]^n \text{ into } n \text{ pieces} \\ \text{there is an } n\text{-subset } A \text{ of } [a,b] \text{ such that } [A]^n \text{ is in} \\ \text{one part, and } A \text{ is } relatively \ large, i.e., } \operatorname{card} A \geqslant \min A. \end{cases}$$

As for Gödel's first incompleteness theorem, the Paris–Harrington theorem can be formalized in PA, and it's worthwhile looking to see what this gives. Although the cut I in (1) cannot be defined in M (since \mathbb{N} , nor any proper cut closed under successor can be so defined) the truth definition (2) provides an interpretation or formulas of unbounded quantifier complexity in I in terms of truth for formulas of bounded complexity in M. This is enough to formalize the soundness theorem: if p is a proof in a model M of PA, p is a proof that PA $\vdash \sigma$, and $a \in M$ is larger than the complexity of all formulas in p, then provided p with p is a exist, (2) gives an interpretation of the sentences in p, and an induction on the length of p using (2) and the p0 truth prediacte Satp0 shows that all statements in p1 and in particular the statement p2 is true according to (2).

Refining this argument only slightly, we have the stronger statement that, in PA,

$$\forall x \; \exists y \; Y(x,y) \geqslant x \Rightarrow \forall \sigma \in \Pi_1 \left(\operatorname{Sat}_{\Pi_1}(\sigma) \rightarrow \operatorname{Con}(\operatorname{PA} + \sigma) \right).$$

In other words, PA + PH proves a reflection principle, the 1-consistency of PA. In fact, there is a converse also:

$$PA \vdash \forall x \exists y \ Y(x,y) \geqslant x \Rightarrow \forall \sigma \in \Pi_1 \left(Sat_{\Pi_1}(\sigma) \rightarrow Con(PA + \sigma) \right).$$

The 1-consistency of PA is a Π_2 statement, as in PH itself: it asserts the totality of a very fast-growing recursive function f. Such a Π_2^0 -analysis of a theory like PA had been the realm of proof theorists, and the proof theorists were very quick to find their own arguments showing PA $\not\vdash$ PH. Many very slick and accessible arguments like this have been found, but the original arguments were entirely model theoretic. In particular, as we saw, the reason for

looking at variations of Ramsey theorems was to construct special sequences of indiscernibles. This to my mind, was the real success of the *model theoretic* Π_2 analysis of PA via cuts: it directly suggests the indicators (or independent statements) that should be considered.

Gödel's first and second incompleteness theorems provided independent Π_1 statements, in particular Con(PA). To produce such independent results by analysis of cuts is impossible since the truth of a Π_1 statement is a model $M \models PA$ is preserved to initial segments of M. The study of cuts is presisely the model theoretic version of the Π_2 analysis of arithmetic.

The main thrust of work in models of arithmetic in the 1980s was to relate theories of arithmetic such as PA, but also very weak subsystems of PA such as $I\Delta_0$ and related theories, to the computational problems of complexity and practical feasability—just as Gödel had related his independence results to theoretical computability. But the problems in computational complexity (such as P = NP?) which were the motivation of this research remain as intractable as ever. The result is that the model theory of first-order arithmetic has splitinto two quite separate areas: very weak theories, such as IOpen, with a characteristic algebraic flavour, and strong theories such as PA itself.

Preliminary draft 32

References

[1] Wilhelm Ackermann. Begründung des "tertium non datur" mittels der Hilbertschen Theorie der Widerspruchsfreiheit. *Mathematische Annalen*, 93:1–36, 1924.

- [2] Wilhelm Ackermann. Zum Hilbertschen Aufbau der reellen Zahlen. Mathematische Annalen, 99:118–133, 1928. English translation by Stefan Bauer-Mengelberg [60, 493–507].
- [3] Rudolf Carnap. Die Antinomien und die Unvollständigkeit de Mathematik. Monatshefte für Mathematik und Physik, 41:263–284, 1934.
- [4] Rudolf Carnap. Logische Syntax de Sprache. Springer, Vienna, 1934.
- [5] Alonzo Church. A note on the Entscheidungsproblem. *The journal of symbolic logic*, 1:40–41, 1936. reprinted in Davis 1965 [7, 110–115].
- [6] Alonzo Church and Stephen Cole Kleene. Formal definitions in the theory of ordinal numbers. *Fundamenta Mathematicae*, 28:11–21, 1936.
- [7] Martin Davis, editor. The undecidable. Basic papers on undecidable propositions, unsolvable problems and computable functions. Raven Press, Hewlett, N.Y., 1965.
- [8] Martin Davis and Hilary Putnam. Reductions of Hilbert's tenth problem. Journal of Symbolic Logic, 23:183–187, 1958.
- [9] Martin Davis, Hilary Putnam, and Julia Robinson. The decision problem for exponential diophantine equations. *Ann. of Math.* (2), 74:425–436, 1961.
- [10] Richard Dedekind. Was sind und was sollen die Zahlen? Brunswick, 1888.
- [11] Solomo Feferman. Transfinie recursive progressions of axiomatic theories. The journal of symbolic logic, 27:259–316, 1962.
- [12] Solomon Feferman. Turing in the land of O(z). In The universal Turing machine: a half-century survey, pages 113–147. Oxford Univ. Press, New York, 1988.
- [13] Paul Finsler. Formale beweise und die entscheidbarkeit. Mathematische Zeitschrift, 25:676–682, 1926. In van Heijenoort [60] pp438–445.

[14] Haim Gaifman. A note on models and submodels of arithmetic. In Conference in Mathematical Logic—London '70 (Proc. Conf., Bedford Coll., London, 1970), pages 128–144. Lecture Notes in Math., Vol. 255. Springer, Berlin, 1972.

- [15] Haim Gaifman. Models and types of Peano's arithmetic. Ann. Math. Logic, 9(3):223–306, 1976.
- [16] Gerhard Gentzen. Die widerspruchsfreiheit der reinen zahlentheorie. Mathematische Annalen, 112:493–565, 1936. English translation by M. E. Szabo [17], 132-213.
- [17] Gerhard Gentzen. The collected papers of Gerhard Gentzen, edited and translated into English by M. E. Szabo. North-Holland, Amsterdam, 1969.
- [18] Kurt Gödel. Die Vollständigkeit der Axiome des logischen Funktionenkalküls. Monatshefte für Mathematik und Physik, 37:349–360, 1930.
- [19] Kurt Gödel. Einige metamathematische Resultate über Entscheidungsdefinitheit und Widerspruchsfreiheit. Anzeiger der Akademie der Wissenschaften in Wien, 67:214–215, 1930.
- [20] Kurt Gödel. Über formal unentscheidbare Sätze der Principia Mathematica und verwandter System I. Monatshefte für Mathematik und Physik, 38:173– 98, 1931.
- [21] Kurt Gödel. über Vollständigkeit und Widerspruchsfreiheit. Ergebnisse eines mathematischen Kolloquiums, 3:12–13, 1932.
- [22] Kurt Gödel. On undecidable propositions of formal mathematical systems. 1934. Mimeographed lecture notes, taken by Stephen C. Kleene and J. Barkley Rosser; reprinted with revisions in 1965 [7, pp39–74].
- [23] Kurt Gödel. Review of Skolem 1934 [48]. Zentralblatt für Mathematik und ihre Grenzgebiete, 10, 1935.
- [24] Kurt Gödel. über eine bisher noch nicht benützte Erweiterung des finiten Standpunktes. *Dialectica*, 12:280–287, 1958. Revised and expanded in English [25], translated into English by Wilfrid Hodges and Bruce Watson [26].
- [25] Kurt Gödel. On an extension of finitary mathematics which has not yet been used. 1972. To have appeared in Dialectica; first published in Gödel 1990[28, pp271–280].

Preliminary draft 34

[26] Kurt Gödel. On a hitherto unexploited extension of the finitary standpoint. Journal of philosophical logic, 9:133–142, 1980.

- [27] Kurt Gödel. Collected works. Vol. I. The Clarendon Press Oxford University Press, New York, 1986. Publications 1929–1936, Edited and with a preface by Solomon Feferman.
- [28] Kurt Gödel. Collected works, volume II: Publications 1938–1974. Oxford University Press, 1990. edited by Solomon Feferman, John W. Dawson, Jr., Stephen C. Kleene, Gregory H. Moore, Robert M. Solovay, and Jean van Heijenoort.
- [29] Kurt Gödel. The present situation in the foundations of mathematics. In Soloman Feferman et al, editor, Kurt Gödel's Collected works, volume III: Unpublished essays and lectures, pages 36–53. Oxford University Press, 1995. Text of an invited lecture to the Mathematical association of America, December 1933.
- [30] Kurt Gödel. Undecidable diophantine propositions. In Soloman Feferman et al, editor, Kurt Gödel's Collected works, volume III: Unpublished essays and lectures, pages 156–175. Oxford University Press, 1995.
- [31] Kurt Gödel. Vortrag bei Zilsel. In Soloman Feferman et al, editor, Kurt Gödel's Collected works, volume III: Unpublished essays and lectures, pages 62–113. Oxford University Press, 1995. Text of a lecture given in Vienna, 29th January 1938.
- [32] Leon Henkin. The completeness of formal systems. Ph.d. thesis, Princton, 1947.
- [33] Jacques Herbrand. Sur la non-contradiction de l'arithmétique. *Journal für die reine und angewandte Mathematik*, 166:1–8, 1931. English translation by van Heijenoort [60, pp.618–628].
- [34] Hilbert and Bernays. Grundlagen der Mathematik, vol II. Springer, Berlin, 1939.
- [35] David Hilbert. über das Unendliche. Mathematische Annalen, 95:161–190, 1926. English translation by Stefan Bauer-Mengelberg [60, 367–392].
- [36] Andrew Hodges. Alan Turing: the enigma. Simon & Schuster, New York, 1983.

[37] Stephen C. Kleene. The work of Kurt Gödel. J. Symbolic Logic, 41(4):761–778, 1976.

- [38] Stephen Cole Kleene. General recursive functions of natural numbers. Mathematische Annalen, 112:727–742, 1936.
- [39] Stephen Cole Kleene. λ-definability and recursiveness. Duke Mathematical Journal, 2:340–353, 1936.
- [40] Stephen Cole Kleene. Introduction to metamathematics. D. Van Nostrand Co., Inc., New York, N. Y., 1952.
- [41] Yu. V. Matiyasevich. Enumerable sets are diophantine [in Russian]. *Doklady Akademii Nauk S.S.S.R*, 191:279–282, 1970.
- [42] Emil Leon Post. Finite combinatory processes, formulation i. *The journal of symbolic logic*, 1:103–105, 1936. reprinted in Davis 1965 [7, 288–291].
- [43] Mojzesz Presburger. über die Vollständigkeit eines gewissen Systems der Arithmetik ganzer Zahlen, in welchem die Addition als einzige Operation hervortritt. Sprawozdanie z I Kongresu matematyków kraów słowiańskich, Warszawa 1929, pages 92–101, 395, 1930.
- [44] J. B. Rosser. Extensions of some theorems of Gödel and Church. J. Symbolic Logic, 1:87–91, 1936.
- [45] J. B. Rosser. An informal exposition of proofs of Gödel's theorem and Church's theorem. J. Symbolic Logic, 4:53–60, 1939.
- [46] Bertrand Russell. Letter to Frege. In From Frege to Gödel: a source book in mathematical logic, 1879–1931, pages 124–5. Harvard University Press, Cambrige, Mass, 1902.
- [47] Wilfried Sieg. Step by recursive step: Church's analysis of effective calculability. *Bull. Symbolic Logic*, 3(2):154–180, 1997.
- [48] T. A. Skolem. über die Nichtcharakterisierbarkeit der Zalenreihe mittels endlich oder abzählbar unendlich vielen Assagen mit asschliesslich Zahlvariablen. Fund. Math., 23:150–161, 1934. English version published in 1955 [49].
- [49] Th. Skolem. Peano's axioms and models of arithmetic. In *Mathematical interpretation of formal systems*, pages 1–14. North-Holland Publishing Co., Amsterdam, 1955.

Preliminary draft 36

[50] Thoralf Skolem. Begründung der elementaren Arithmetik durch die rekurrierende Denkweise ohne Anwondung scheinbarer Veränderlichen mit unendlichem Ausdehnungsbereich. Videnskapsselskapets skrifter, I. Matematisk-naturvidenskabelig klasse, 6, 1923. English translation by Stefan Bauer-Mengelberg [60, 302–333].

- [51] Thoralf Skolem. über die Unmöglichkeit einer vollständigen Charakterisierung der Zahlenreihe mittels eines endlichen Axiomensystems. Norsk matematisk forenings skrifter, series 2, 10:73–82, 1933. reprinted in 1970 [52, 345-354].
- [52] Thoralf Skolem. Selected works in logic. Universitetsforiaget, Oslo, 1970. edited by Jens Erik Fenstad.
- [53] Alfred Tarski. Pojecie prawdy w jezykach nauk dedukcyjnych [the concept of truth in the languages of deductive sciences]. *Prace Towarzystwa Naukowego Warszawskiego, wydział III*, 34, 1933. translated into German by L. Blaustein [54]; English translation by Joseph H. Woodger [55, 152–278].
- [54] Alfred Tarski. Der Wahrheitsbegriff in den formalisierten Sprachen. *Studia philosophica (Lemberg)*, 1:261–405, 1935.
- [55] Alfred Tarski. Logic, semantics, metamathematies: papers from 1923 to 1938. Clarendon Press, 1956. translated into English and edited by Joseph H. Woodger.
- [56] Stanley Tennenbaum. Non-archimedian models for arithmetic. Notices of the American Math. Soc., 270, 1959.
- [57] Alan M. Turing. Computability and λ -definability. *Journal of Symbolic logic*, 2:153–163, 1937.
- [58] Alan M. Turing. On computable numbers, with an application to the Entsheidungsproblem. *Proceedings of the London Mathematical Society*, 2nd series, 42:230–265, 1937.
- [59] Alan M. Turing. Systems of logic based on ordinals. Proceedings of the London Mathematical Society, 2nd series, 45:161–228, 1939.
- [60] Jean van Heijenoort, editor. From Frege to Gödel: a source book in mathematical logic, 1879–1931. Harvard University Press, Cambrige, Mass, 1967.

Preliminary draft 37

[61] John von Neumann. Zur Hilbertschen Beweistheorie. Mathematische Zeitschrift, 26:1–46, 1927. reprinted in von Neumann's collected works [62, pp256–300].

- [62] John von Neumann. Collected works, volume 1. Pergamon Press, 1961.
- [63] Hao Wang. Some facts about Kurt Gödel. J. Symbolic Logic, 46(3):653–659, 1981.
- [64] Hao Wang. Reflections on Kurt Gödel. MIT Press, Cambridge, MA, 1987.
- [65] Hao Wang. Reflections on Kurt Gödel. MIT Press, Cambridge, MA, second edition, 1988.
- [66] Hao Wang. Kurt Gödel. Armand Colin, Paris, 1990. Translated from the 1987 English edition by Laura Ovion and Michel Mériaux.
- [67] Hao Wang. A logical Journey: from Gödel to Philosophy. The MIT press, Cambridge, Mass, 1996.