Order-types of models of Peano arithmetic: a short survey.

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1 Introduction

This paper is a short and slightly selective survey of results on order-types of models of Peano arithmetic. We include few proofs, and concentrate instead on the key problems as we see them and possible ways of responding to the very considerable mathematical difficulties raised.

Our starting point is the following problem.

Main Question. What are the possible order-types of a model of PA?

Here, PA is the first-order theory in the language with $0, 1, +, \cdot, <$ containing finitely many basic axioms true in $\mathbb{N}$ together with the first-order induction axiom scheme. Any model of PA has an initial segment isomorphic to $\mathbb{N}$, and we will always identify $\mathbb{N}$ with this initial segment.

The theory PA is well-known not to be complete, and has $2^{\aleph_0}$ complete extensions. 'True arithmetic'—the theory $\mathrm{Th}(\mathbb{N})$ of the standard model $\mathbb{N}$—is one of these extensions, and sometimes needs to be treated differently from the others. For example, nonstandard models of $\mathrm{Th}(\mathbb{N})$ do not have any nonstandard definable elements, whereas any model of 'false arithmetic' (a model of PA not satisfying $\mathrm{Th}(\mathbb{N})$) always has nonstandard definable elements.

Kaye's book [5] provides a good background to the model theory of PA, resplendency and recursive saturation, and should be consulted for definitions and results not contained here. We shall also assume knowledge of a certain amount of standard model theory throughout this paper, as can be found in any of the standard texts.

The main question above is easily solved for countable models (see Section 3 below), so the issue is what the possible order-types of *uncountable* models of

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PA are. Remarkably little seems to be known in this area, despite the question being a natural one that has been around for some thirty years.

It is not even clear whether the question has different answers for particular complete extensions of PA, or if all completions of PA behave similarly with respect to their uncountable order-types. Indeed, this is the content of Harvey Friedman's question in this area:

Question 1 (Friedman [3]). Is the class of uncountable order-types of models of T the same for all complete extensions T of PA?

We set

$$\mathcal{C}_T = \{(M,<) \mid M \vDash T,\ |M| > \aleph_0\},$$

the class of order-types of models of T, where $T \supseteq$ PA is complete. It is tempting to guess that $\mathcal{C}_T$ is independent of the choice of T for all $T \neq \mathrm{Th}(\mathbb{N})$, but $\mathcal{C}_{\mathrm{Th}(\mathbb{N})}$ may be different. (Indeed, the corresponding thing really is the case for countable models, for rather trivial reasons!) However, there really has been no evidence for this found to date, and Friedman's question remains wide open and seems particularly difficult.

Section 3 outlines elementary results on order-types of models of PA. Despite the simplicity of these results, they seem to be important and summarise most of the 'necessary' properties of the order-type of a model of PA. Section 4 surveys some more advanced results that have a bearing on our problem.

As none of the properties obtained seem amenable to proving a converse or constructing models of arithmetic with the required order-type, we are left with the following possible approaches:

1. Give examples of 'concrete' order-types of models of PA, hopefully ones that are 'natural' and 'interesting'. For each of the examples, construct models of the given type. How many models are there of this order-type? This approach will be addressed in Section 5.

2. We may restrict our attention to models of arithmetic obtained by one or more well-known constructions, such as taking an ultrapower of a model, construction by means of the arithmetized completeness theorem (ACT), generating a model by indiscernibles, etc. The particular cases of ACT-models and 'inner models' in general are discussed in Section 7.

3. Finally, we may study classes of order-types of 'nice' models.

As an example of item 3 here, we may restrict attention to special classes of models, such as saturated models, resplendent models, etc. For example, one could ask

Question 2. Given any uncountable resplendent model $M \vDash$ PA and any completion T of PA, is there an expansion of $(M,<)$ to a model of T?

We discuss this particular question in Section 9 below.

Other related questions include: which order-types are interpreted in models of PA; and whether the class of uncountable order-types of *fragments* of PA are the same as that for the full theory? We note here that, for the second of these, the theory IOpen of open induction definitely has different order-types to PA, but know little about other fragments. The question of interpretability will be (partially) discussed further in Section 7.

The enterprise of studying order-types of models of PA is very rewarding mathematically, and increases our understanding of models of arithmetic. We shall see that the order of a model of PA is deeply interconnected with the rest of model's structure. Good illustrations of these connections may be found in Pabion's Theorem (Theorem 4.1) and also in Theorem 9.4.

2 Background definitions

Let us start off with the necessary notation and definitions.

Definition 2.1. If $(A, <)$ and $(B, <)$ are linearly ordered sets then A^* is A with the order reversed, AB is $A \times B$ with lexicographic (not antilexicographic) order. Q_λ is the saturated dense linear order of cardinality λ (which exists if and only if $\lambda^{<\lambda} = \lambda$). It is straightforward to see from this that the saturated discrete linear order with first and no last element of cardinality λ is $\mathbb{N} + Q_\lambda \mathbb{Z}$.

Definition 2.2. Let T be a consistent theory in a finite language $\mathcal{L}$. Let $A \vDash T$, $B \vDash \mathrm{PA}$. Then A is **strongly interpreted** in B if there are two formulas $\mathrm{dom}(x)$ and $\mathrm{Sat}(x, y)$ (possibly containing parameters from B) such that there is a bijection $g\colon A \to \{x \in B \mid \mathrm{dom}(x)\}$ such that for every $\varphi(\overline{x}) \in \mathcal{L}$, $\overline{a} \in A$,

$$A \vDash \varphi(\overline{a}) \;\Leftrightarrow\; B \vDash \mathrm{Sat}(\ulcorner \varphi \urcorner, \langle g(\overline{a}) \rangle).$$

If $A \vDash \mathrm{PA}$ is strongly interpreted in $B \vDash \mathrm{PA}$ we shall sometimes say that A is an **inner model** in B.

Definition 2.3. Let $M \vDash \mathrm{PA}$, T be a set of M-Gödel numbers of formulas in a language $\mathcal{L}$ definable in M by a formula $\varphi(x, t)$, where $t \in M$ is a parameter. Define $\mathrm{Con}(T)$ be the following $\mathcal{L}_{\mathrm{PA}} \cup \{t\}$-statement:

$$\begin{aligned}&\forall x \forall y\ (\ \forall i < \mathrm{len}(y)\ \varphi((y)_i, t)\ \to \\ &\qquad \to \neg \operatorname{Proof}(x, \ulcorner ((y)_0 \wedge \ldots \wedge (y)_{\mathrm{len}(y)-1}\ \to\ \exists z\ z \neq z \urcorner))).\end{aligned}$$

Notice that for every nonstandard $M \vDash \mathrm{PA}$, there is $s > \mathbb{N}$ such that

$$M \vDash \mathrm{Con}(\{x < s \mid M \vDash \text{“}x \text{ is an axiom of PA”}\}).$$

This follows from $\mathrm{PA} \vdash \mathrm{Con}(\mathrm{I}\Sigma_n)$ by an overspill argument. Of course, we also have $\mathbb{N} \vDash \mathrm{Con}(\mathrm{PA})$.

Fact 2.4 (Arithmetized Completeness Theorem). Let $M \vDash \mathrm{PA}$, and let $T \supseteq \mathrm{PA}$ be M-definable (considered as a set of possibly nonstandard Gödel-numbers). Then $M \vDash \mathrm{PA} + \mathrm{Con}(T)$ if and only if there is $N \vDash T$ which is strongly interpreted in M.

Fact 2.5. If $M, N \vDash \mathrm{PA}$ and N is strongly interpreted in M then the function $f\colon M \to N$ defined by $f(0) = 0_N$, $f(x+1) = f(x) +_N 1_N$ is a definable isomorphism between M and an initial segment of N.

Fact 2.6 (Kirby and Paris [6]). Let $M \vDash \mathrm{PA}$ be recursively saturated. Then the following are equivalent.

1. For any $f \in M$ coding a function $f\colon \mathbb{N} \to M$, there is $c \in M \setminus \mathbb{N}$ such that for all $n \in \mathbb{N}$, $f(n) > \mathbb{N} \;\Leftrightarrow\; f(n) > c$.

2. SSy(M) is closed under jump.

A model satisfying the conditions in Fact 2.6 is called **arithmetically saturated**.

3 Results on order-types obtained from elementary considerations

This section is devoted to some elementary observations on order-types of nonstandard models of PA. We start with the *very* elementary observation that a model M of PA always contains a copy of $\mathbb{N}$ as an initial segment. The nonstandard elements factorize into $\mathbb{Z}$-*blocks*, equivalence classes under

$$x \sim y \Leftrightarrow \exists n \in \mathbb{N}\ x + n > y \wedge y + n \geqslant x$$

and the set of equivalence classes is an ordered set A. Thus $(M, <) \cong \mathbb{N} + A\mathbb{Z}$, and the problem of determining the order-type of M reduces to determining the ordered set A.

Theorem 3.1. A nonstandard model of PA has order-type $\mathbb{N} + A\mathbb{Z}$, where A is a dense linear order without endpoints.

Thus, up to isomorphism, the only possibility for the order-type A if it is countable is $(\mathbb{Q}, <)$, by Cantor's theorem, and so the problem of order-types is solved in the countable case. On the other hand, it follows from Shelah's results on pseudoelementary classes [9, 11] that there are 2^λ possible order-types for every $\lambda > \aleph_0$, so there is a lot more to do in the uncountable case.

Not all order-types can occur. For example the following is common knowledge, and appears as an exercise in a paper by Smorynski [12].

Theorem 3.2. If M is a model of PA and $(M, <) \cong \mathbb{N} + A\mathbb{Z}$, then A cannot have the order-type of the reals $\mathbb{R}$.

We know of two quite different proofs of this fact. One relies on the completeness of the reals: this one takes a coded, bounded, ascending sequence of elements of M $(a_n)_{n\in\mathbb{N}}$ such that $a_i \not\sim a_j$ for all $i \neq j$ and shows using overspill that there cannot be any element $b \in A$ with $b = \lim_{i\in\mathbb{N}}(a_i/\sim)$. This proof can be sharpened with little extra effort to give the following.

Theorem 3.3. Let M be a nonstandard model of PA, and $(a_n)_{n\in\mathbb{N}}$ an ω-sequence of nonstandard elements of M coded in M. Then

1. Suppose that $X = \{a_n/{\sim} : n \in \mathbb{N}\}$ is bounded above and has no maximum element. Then there is no $a \in M$ such that $a/{\sim} = \sup X$.

2. The same, but with 'bounded below', 'minimum' and 'inf' in place of 'bounded above', 'maximum' and 'sup', respectively.

Strictly, this last result does not concern the order-type of the model but its interaction with its arithmetic structure. But it is easy to draw conclusions for the order type, as follows.

Theorem 3.4. Let M be a nonstandard model of PA and take $a < b$ from M in different $\mathbb{Z}$-blocks. Then there is a coded increasing sequence $(a_n)_{n\in\mathbb{N}}$ in M with $a < a_n < a_{n+1} < b$ for all n and $a_n/\sim$ having no maximum element. Similarly there is a coded decreasing sequence $(b_n)_{n\in\mathbb{N}}$ in M with $a < b_{n+1} < b_n < b$ for all n and $b_n/\sim$ having no minimum element.

Thus the 'places of incompleteness' in A where $(M,<) \cong \mathbb{N} + A\mathbb{Z}$ occur densely in A.

The other proof of Theorem 3.2 goes by taking an M-definable sequence $(a_j)_{j\in M}$ with $a_i \not\sim a_j$ for all $i \neq j$, such as $a_j = j\alpha$ where α is fixed and non-standard. Then (a_i, a_{i+1}) provide $|A| = |M|$-many nonempty disjoint intervals of A, something that is clearly not possible if $A = \mathbb{R}$. We thus have:

Theorem 3.5. Let M be a nonstandard model of PA. Then there is a cofinal definable subset $X \subseteq M$ (and therefore with $|X| = |M|$) such that $a/\sim < b/\sim$ for all $a < b$ in X.

Once again, slightly extra information on the order-type can be obtained from the proof. In this case, by taking $a > \mathbb{N}$ in a model M, $d = [\sqrt{a}]$ and $X = \{id : i < d\}$, and noting that the cardinalities of the sets of predecessors of a and d are the same, we can obtain

Theorem 3.6. Let M be a model of PA and let $a \in M$ be nonstandard. Then there is a cofinal subset $X \subseteq \{x \in M : x < a\}$ such that $b_1/\sim < b_2/\sim$ for all $b_1 < b_2$ in X and $|X| = |\{x \in M : x < a\}|$.

Theorems 3.4 and 3.5 give nontrivial necessary conditions on the order type of a nonstandard model of arithmetic. We can also obtain other local homogeneity conditions using addition, as follows:

Theorem 3.7. Let M be a nonstandard model of PA, and $(M,<) \cong \mathbb{N} + A\cdot\mathbb{Z}$. Then for each $a, b \in A$ there are open intervals of A: (x,y) containing a and (u,v) containing b such that $(x,y) \cong (u,v)$ via a map taking $a \mapsto b$.

Again, the proof is simple, just using the arithmetic properties of the model to construct the isomorphism.

4 More advanced results

J.-F. Pabion proved a remarkable theorem showing the saturation or otherwise of the order type of the model controls the saturation of the whole model.

Theorem 4.1 (Pabion [7]). Let $M \vDash$ PA, $\varkappa$ be a cardinal. If $(M,<)$ is $\varkappa$-saturated then M is $\varkappa$-saturated.

Theorem 4.2 (Shelah [10]). Every model M of PA has a cut I such that $\mathrm{cf}(I) = \varkappa$ and $\mathrm{cf}(M \smallsetminus I) = \varkappa$ for some cardinal $\varkappa$.

The following application of an Erdös-Rado theorem is due to Hodges.

Theorem 4.3 (Hodges [4]). Let T be a completion of PA, $M \vDash T$, $\varkappa$ be a singular strong limit cardinal, $\mathrm{card}\, M = \varkappa$. Then M contains an increasing sequence of order-type $\varkappa$.

The condition that $\varkappa$ is a singular strong limit cardinal is important. In Section 6 we shall see that there exists a family of models of cardinality 2^ω not even having increasing ω_1-sequences.

5 Examples

The first place one will look for examples is by applying the MacDowell–Specker result that says any model of Peano arithmetic has a proper elementary end-extension [5, Section 8.2]. By constructing a suitable elementary chain we obtain: for all cardinals $\lambda < \kappa$ and all models $M \models \mathrm{PA}$ of cardinality λ there is a κ-like elementary end-extension of M.

In this short section we shall look at the possibilities for more delicate constructions of this type, obtaining some specific examples of order-types of uncountable models of PA. Throughout the section we assume GCH in order not to bother about existence of saturated orders.

As we have already seen, the order-type of every countable nonstandard model is $\mathbb{N}+\mathbb{QZ}$. Also easily proved is that the order-type of any ω_1-like model is $\mathbb{N}+\omega_1\mathbb{QZ}$. The order-type of a saturated model of cardinality λ is $\mathbb{N}+Q_\lambda\mathbb{Z}$. We can easily combine the simple linear orders we know (saturated orders Q_λ and ordinals) in order to obtain other order-types.

Proposition 5.1 ([1]). For any regular λ and any $\alpha \leq \lambda,\ \beta \leq \lambda^+$, there is a model of PA of order-type

$$\mathbb{N}+(\alpha^*+\beta)Q_\lambda\mathbb{Z}.$$

Proposition 5.2 ([1]). If $T \supseteq \mathrm{PA}$ is complete and $\lambda_1, \lambda_2, \ldots, \lambda_n$ are successor cardinals such that $2^\omega \leq \lambda_1 < \lambda_2 < \cdots < \lambda_n$ then there is $M \models T$ of order-type

$$\mathbb{N}+Q_{\lambda_1}\mathbb{Z}+Q_{\lambda_2}Q_{\lambda_1}\mathbb{Z}+\cdots+Q_{\lambda_n}\ldots Q_{\lambda_2}Q_{\lambda_1}\mathbb{Z}.$$

We call these nice order-types of models of PA **canonical**. A more detailed discussion of canonical orders can be found in Bovykin's thesis [1] or in forthcoming work.

Pabion's Theorem may lead us to the suggestion that the model of order-type $\mathbb{N}+\mu Q_\lambda\mathbb{Z}$ is unique for all $\mu \leq \lambda^+$. By considering convex closures of Skolem hulls of different elements in a saturated model the following proposition can be proved

Proposition 5.3 ([1]). Let λ be regular. If $T \neq \mathrm{Th}\,\mathbb{N}$ then there are at least four pairwise non-isomorphic models of T of order-type $\mathbb{N}+\omega\mathbb{Q}_\lambda\mathbb{Z}$. There are at least three pairwise non-isomorphic models of $\mathrm{Th}\,\mathbb{N}$ of order-type $\mathbb{N}+\omega\mathbb{Q}_\lambda\mathbb{Z}$.

Our suggestion is that there are actually 2^λ pairwise non-isomorphic models of this order-type for any $T \supseteq \mathrm{PA}$. Other interesting questions about canonical orders include

Question 3. Are there models of PA of order-types

$$\mathbb{N}+\mathbb{QZ}+Q_{\omega_1}\mathbb{QZ} \quad \text{and} \quad \mathbb{N}+\omega_1\mathbb{QZ}+Q_{\omega_1}(\omega_1^*+\omega_1)\mathbb{QZ}?$$

6 Models generated by indiscernibles

Definition 6.1. If $(I,<), (A,<)$ are linear orders, $0 \in A$, then

$$(A,0)^{<I} = \{f\colon I \to A \mid \mathrm{supp}(f) \text{ is finite},\ \mathrm{supp}(f) \neq \varnothing\}$$

with order defined lexicographically: $f < g$ if for $a = \min\{i \mid f(i) \neq g(i)\}$, $f(a) < g(a)$.

Let $\mathrm{EM}(C)$ stand for the model generated by a set of indiscernibles ordered as C.

Theorem 6.2. If $(C, <)$ is a dense linear order without end-points and there is an order preserving $f\colon (C, <) \longrightarrow (C^*, <)$ and a point $0 \in C$ such that $f(0) = 0$, then $\mathrm{EM}(C)$ is order-embeddable into $(C, 0)^{<\mathbb{Q}}$.

A sketch of the proof of Theorem 6.2 was given by Charreton and Pouzet [2]. A detailed proof can be found in Bovykin's thesis [1].

Corollary 6.3. $\mathrm{EM}(\mathbb{R})$ is a model of cardinality 2^ω containing no monotonous ω_1- or ω_1^*-sequences.

7 Inner models

Here, we survey some results on order-types obtained by internalizing classical constructions. Proposition 7.2 is the internal version of the fact *"every linear order of finite cardinality is isomorphic to* $\{1, 2, \dots, n\}$ *with its natural order for some* n*."* Theorem 7.4 is the internalization of the fact *"every countable dense linear order without end-points is isomorphic to* $\mathbb{Q}$*".* Then we look at 'inner models' and express their order-type in terms of the $(<, \cdot)$-type of the original model (by internalizing the proof of *"every countable nonstandard model has order-type* $\mathbb{N} + \mathbb{QZ}$*"*). This leads to a very promising and interesting notion of order self-similarity.

Definition 7.1. A linear order A is called M**-finite** if there is $x \in M$ and $x = \langle x_1, x_2, x_3 \rangle$, where x_1 codes a bounded subset of M, x_2 codes equality, x_3 codes order so that A is order-isomorphic to the order defined by x.

Easily proved is

Proposition 7.2. If A is M-finite and order-isomorphic to one defined by a, then $A \cong [0, b]$ for some b definable from a.

Definition 7.3. Let $(A, +, \cdot, <)$ be an ordered ring, $A^+ = \{x \in A \mid x \geq 0\}$. Then $(Q(A^+), <)$ is the set

$$\{(x, y) \mid x, y \in A^+, x, y \neq 0\}$$

factored out by the following equivalence relation:

$$(x, y) \sim (z, w) \Leftrightarrow xw = yz$$

with the linear order on it defined as

$$(x, y) < (z, w) \;\Leftrightarrow\; xw < zy.$$

Theorem 7.4. If $(A, <)$ is a dense linear order without end-points interpreted in M then

$$(A, <) \;\cong\; (Q(M), <).$$

The proof of Theorem 7.4 is an internalization of Cantor's back-and-forth argument.

Corollary 7.5. Neither $\mathbb{R}$ nor $\mathbb{Q}$ is interpreted in uncountable models of PA.

Proof. Let $M \models$ PA be uncountable. $\mathbb{Q}$ cannot be interpreted in M because $Q(M)$ is uncountable, hence $\mathbb{Q} \ncong Q(M)$. $\mathbb{R}$ cannot be interpreted in M because M is order-embeddable into $Q(M)$ but not into $\mathbb{R}$, and hence $\mathbb{R} \ncong Q(M)$. $\square$

Versions of Theorem 7.4 exist for other ω-categorical theories. In particular, there is a unique random graph and a unique atomless boolean algebra interpreted in a given model of PA.

Theorem 7.6. Let $M \models$ PA and $N \models$ PA be an inner model in M. Then

$$(N, <) \cong M + Q(M)(M^* + M).$$

Proof. Since N is strongly interpreted in M, by Fact 2.5 there is $f\colon M \to N$ definable in M which determines an isomorphism between M and an initial segment of N. Hence, $(N, <) \cong M + A(M^* + M)$ for some linear order A.

For $a, b \in N$, we define $a \sim b \;\Leftrightarrow\; M \models \exists x(a -_N b = f(x))$ if $a > b$ and $a \sim b \;\Leftrightarrow\; M \models \exists x(b -_N a = f(x))$ if $b < a$. We interpret A in M by means of the following formulas:

$$\mathrm{dom}_A(x) \longleftrightarrow x \in N \;\&\; \forall y < x\, \neg(y \sim x)$$

$$o(x, y) \longleftrightarrow x <_N y.$$

Let us prove that A is dense. Take $a, b \in N, a < b, a \not\sim b$. If $\left[\frac{b-a}{2}\right]$ belonged to M (i.e. to the image of f) then so would $b - a$ because $f(M)$ is closed under addition. Hence, $a \not\sim a + \left[\frac{b-a}{2}\right] \not\sim b$.

As A is a dense order interpreted in M, by Theorem 7.4, $A \cong Q(M)$. $\square$

We do not know if the order-type of an inner model is determined by the order-type of the outer model:

Question 4. Does there exist a pair of models $A, B \models$ PA such that $A \equiv B$ and $(A, <) \cong (B, <)$, but $(A + Q(A)(A^* + A), <) \ncong (B + Q(B)(B^* + B), <)$?

Theorem 7.6 shows that from the point of view of a model of PA, there is only ONE order-type of models of PA, namely $M + Q(M)(M^* + M)$. So, from its point of view, Friedman's problem has a trivial solution: all order-types of models of PA are isomorphic.

8 Order self-similar models

What happens if some model K is strongly interpreted in N and N is an inner model in M? Is $(K, <)$ (which is equal to $N + Q(N)(N^* + N)$) a new order-type? The answer is no! K is also strongly interpreted in M, hence, by Theorem 7.6,

$$(K, <) \cong M + Q(M)(M^* + M) \cong (N, <).$$

So, N has a nice property: its inner models have order-type $(N, <)$.

Definition 8.1. A model $M \models$ PA is called order self-similar if

$$(M, <) \cong M + Q(M)(M^* + M).$$

Proposition 8.2. Concerning order self-similar models, we have:

1. Every inner model is order self-similar.

2. Every countable nonstandard model is order self-similar, because $Q(\mathbb{N} + \mathbb{QZ}, +, \cdot, <) = \mathbb{Q}$ (because if $\frac{a}{b} < \frac{c}{d}$, then $ad < bc$, then $2ad < 2ad+1 < 2bc$ then
$$\frac{a}{b} = \frac{2ad}{2bd} < \frac{2ad+1}{2bd} < \frac{2bc}{2bd} = \frac{c}{d}$$
i.e. $Q(\mathbb{N} + \mathbb{QZ})$ is dense) hence
$\mathbb{N} + \mathbb{QZ} + Q(\mathbb{N} + \mathbb{QZ})(\mathbb{QZ}) = \mathbb{N} + \mathbb{QZ} + \mathbb{QQZ} = \mathbb{N} + \mathbb{QZ}$.

3. No λ-like model is order self-similar, because $M + Q(M)(M^* + M)$ always contains an initial segment of cardinality $|M|$.

4. Every saturated model is order self-similar, because $Q(\mathbb{N} + Q_\lambda\mathbb{Z}) = Q_\lambda$. (If $E < F$ are two subsets of $Q(M)$ of cardinalities $< \lambda$ then consider
$$p(x, y) = \{ay < bx \mid \frac{a}{b} \in E\} \cup \{dx < cy \mid \frac{c}{d} \in F\}.$$
$p(x, y)$ is a type because, given $\frac{a_1}{b_1}, \ldots, \frac{a_n}{b_n} < \frac{c_1}{d_1}, \ldots, \frac{c_m}{d_m}$, by the argument from example 2 above, there are $x, y \in M$ such that
$$\max\{\frac{a_1}{b_1}, \ldots, \frac{a_n}{b_n}\} < \frac{x}{y} < \min\{\frac{c_1}{d_1}, \ldots, \frac{c_m}{d_m}\}.$$
The pair (x, y) realizing p separates E and F.) Hence
$$\mathbb{N} + Q_\lambda\mathbb{Z} + Q(\mathbb{N} + Q_\lambda\mathbb{Z})(Q_\lambda\mathbb{Z}) = \mathbb{N} + Q_\lambda\mathbb{Z} + (Q_\lambda)Q_\lambda\mathbb{Z} = \mathbb{N} + Q_\lambda\mathbb{Z}.$$
In particular, every inner model in a saturated model is again saturated by Pabion's Theorem.

Theorem 8.3. If M is order self-similar and $f\colon M \to M + Q(M)(M^* + M)$ is an order-isomorphism then there is a proper order self-similar elementary extension $N \succ M$ such that $|N| = |M|$ and the diagram

$$\begin{array}{ccc} M & \xrightarrow{f} & M + Q(M)(M^* + M) \\ \big\downarrow{\scriptstyle in} & & \big\downarrow{\scriptstyle \widetilde{in}} \\ N & \xrightarrow{\tilde{f}} & N + Q(N)(N^* + N) \end{array}$$

commutes.

Proof. Let $L = L_{\mathrm{PA}} \cup \{f\}$. If M is self-similar, let us expand it to (M, f), where f is interpreted by an isomorphism between M and $M + Q(M)(M^* + M)$. Any proper elementary extension $(N, \tilde{f}) \succ (M, f)$ of cardinality $|M|$ is as required. $\square$

It follows that any countable model has elementary order self-similar extensions of all cardinalities.

Theorem 8.4. If M is resplendent then $(M, <) \cong M + Q(M)(M^* + M)$.

Proof. Every resplendent model is order self-similar because the Σ^1_1-statement

$$\exists\, g\ (\ g \textit{ is an isomorphism between } M \textit{ and } M + Q(M)(M^* + M)\ )$$

is realised in all countable nonstandard models. $\square$

So far we know that every inner model is recursively saturated and order self-similar and every resplendent model is recursively saturated and order self-similar too. Also, if M is not λ-dense then every model of PA strongly interpreted in M will have a nonstandard initial segment of cardinality less than λ, hence is not resplendent. (This actually produces a family of examples of recursively saturated non-resplendent models.)

Question 5. Is there an uncountable order self-similar model which is not an inner model?

Question 6. Are there examples of models $M \models \mathrm{PA}$ having two elementarily equivalent non-isomorphic inner models?

The relationships between resplendency and order self-similarity are the subject of a forthcoming paper by Bovykin and Kaye.

9 Resplendency and coding

We already know from the previous section that if $M \models \mathrm{PA}$ is order self-similar, $T \supseteq \mathrm{PA}$ is coded in M and $M \models \mathrm{Con}(T)$ then there is a model $N \models T$ obtained by means of the Arithmetized Completeness Theorem such that $(N, <) \cong (M, <)$. In the resplendent case we do not need to assume $M \models \mathrm{Con}(T)$ as the following theorem shows.

Theorem 9.1. If $M \models \mathrm{PA}$ is resplendent and $c \in M$ codes a consistent theory $T \supseteq \mathrm{PA}$ then $(M, <)$ can be expanded to a model of T.

The theorem is proved by writing down the Σ^1_1-statement stating the existence of $N \models T$, $(N, <) \cong (M, <)$ and noticing that it is realized in every countable submodel of M containing c.

Corollary 9.2. If M is resplendent and $\mathrm{SSy}(M) = \mathcal{P}(\mathbb{N})$ then for any consistent $T \supseteq \mathrm{PA}$, $(M, <)$ can be expanded to a model of T.

Actually, what this Corollary requires from $\mathrm{SSy}(M)$ is that it contains all completions of PA, but this amounts to the same thing as for every $X \in \mathcal{P}(\mathbb{N})$ there is a complete $T^* \supset \mathrm{PA}$ such that X is recursive in T^*.

Corollary 9.3. If M is resplendent and ω_1-saturated then $(M, <)$ can be expanded to a model of any consistent extension of PA.

We also know that, by Pabion's Theorem, this expansion of M will have to be ω_1-saturated because it has an ω_1-saturated order-type. Can we also make it resplendent? In the next result, we use $\mathrm{lcf}\, A$ for the 'lower cofinality' of A, i.e., the cofinality of the order-reverse of the set A.

Theorem 9.4. If M is resplendent and $\mathrm{lcf}(M \smallsetminus \mathbb{N}) > \omega$ then for all $n \in \omega$, $\Pi_n \,\mathrm{Th}\,\mathbb{N}$ is coded in M.

Hence, as PA is recursive, by Theorem 9.1, for every $n \in \mathbb{N}$, $(M, <)$ is expandable to a model of $\mathrm{PA} + \Pi_n \operatorname{Th} \mathbb{N}$, i.e., $\operatorname{Th} \mathbb{N}$ can be "approximated" as closely as you want.

Theorem 9.4 is proved by induction on n [1]. The inductive step goes as follows. Suppose $\Pi_n \operatorname{Th} \mathbb{N} \in \mathrm{SSy}(M)$. Then we formulate a Σ^1_1-statement which says 'there is a model $N \models \mathrm{PA} + \Pi_n \operatorname{Th} \mathbb{N}$, $\mathrm{SSy}(N) = \mathrm{SSy}(M)$, $(N, <) \cong (M, <)$' that is consistent by properties of countable Scott sets. Then we notice that as $\operatorname{lcf}(N \setminus \mathbb{N}) > \omega$, the set of nonstandard Σ_n-definable points of N is bounded below by, say, $a \in N \setminus \mathbb{N}$ thus making the definable set $\{\ulcorner \forall x \varphi(x) \urcorner \mid \varphi \in \Sigma_n, N \models \forall x < a\ \varphi(x)\}$ equal $\Pi_{n+1} \operatorname{Th} \mathbb{N}$. Hence $\Pi_{n+1} \operatorname{Th} \mathbb{N} \in \mathrm{SSy}(M)$.

However, there is another, easier way to prove the above theorem if we know the notion of arithmetic saturation. Resplendency and uncountable lower cofinality imply arithmetic saturation, and any arithmetically saturated model codes all $\Pi_n \operatorname{Th} \mathbb{N}$. Indeed, resplendency implies recursive saturation and for any $f\colon \mathbb{N} \to M$ there is $a \in M$ such that $\forall n \in \mathbb{N}\ (f(n) > \mathbb{N} \Rightarrow f(n) > a)$ because $\operatorname{lcf}(M \setminus \mathbb{N}) > \omega$.

A consistent theory T is called arithmetic if it has an axiomatization S such that $S = \{n \in \mathbb{N} \mid \mathbb{N} \models \theta(n)\}$ for some formula $\theta(x) \in L_{\mathrm{PA}}$. Recursive extensions of PA are examples of arithmetic theories. Also, there are complete arithmetic theories, but these are necessarily incomplete. However

Corollary 9.5.
For any arithmetic theory $T \supseteq \mathrm{PA}$, if $M \models \mathrm{PA}$ is resplendent and $\mathrm{SSy}(M)$ is closed under jump then there is $N \models T$ such that $(N, <) \cong (M, <)$.

Proof. Let $T = \{n \in \mathbb{N} \mid \mathbb{N} \models \theta(n)\}$. As $\mathrm{SSy}(M)$ is closed under jump, T is coded in M. Hence, as M is resplendent, $(M, <)$ is expandable to a model of T, by Theorem 9.1. $\square$

Finally, we have the following theorem, which appeared in [1] and will also appear elsewhere in due course.

Theorem 9.6. *Let T be a completion of PA. Suppose M is resplendent, $\mathrm{cf}(M) = \omega$ and M codes $\Sigma_n T$ for all n. Then $(M, <)$ is expandable to a model of T.*

Question 7. If M is resplendent and codes $\Sigma_n T$ for all n, is $(M, <)$ expandable to a model of T?

The obvious attempt to generalise Theorem 9.6 to higher cofinalities by proving that if $I \models T$ is an initial segment of $M \not\models T$ then there is an initial segment $J \succ I$ fails, because given $I_0 = I$ we define $I_j = \bigcup_{i<j} I_i$ if j is a limit ordinal, $I_{i+1} =$ an initial segment $J \succ I_i$ if $I_i \neq M$, which gives us an elementary chain of some length γ. Now, $M = \bigcup_{i<\gamma} I_i \models T$, contradiction. (The initial segments satisfying T exist as far up as one wants (by the variation of Friedman's Theorem) but they are not elementary substructures of each other.)

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