Relations (Mathematics & Logic A)

RWK/MRQ

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Note. The concepts in this lecture are quite abstract. It is *very important* that you attempt to learn them and ensure you can distinguish between them (even when they might appear to have similar names).

1 Relations and Functions

Definition 1. A *relation* on a set A is a collection of one-way links between certain pairs of elements of A.

Definition 2 (Formal Version). A *relation* on a set A is a subset of $A \times A$.

The correspondence between these two definitions is as follows: If we have a collection of one-way links (drawn with arrows, say $x \to y$), then the corresponding ordered pairs are given by the end-points of the links. So if we have a link from x to y, we record the ordered pair (x, y) in the subset R.

Conversely, if we have a subset R of $A \times A$, then for each ordered pair (x, y) in R we draw an arrow from x to y.

If R is a relation on a set A, we write xRy to indicate that x and y are related under R. (Thus xRy means that the ordered pair (x, y) belongs to the corresponding subset of $A \times A$; that is, there is a link from x to y in the sense of the first definition.)

Definition 3. Let R be a relation on a set A.

i. R is *symmetric* if whenever xRy holds, then also yRx holds.

ii. R is *transitive* if whenever xRy and yRz hold, then also xRz holds.

iii. R is *reflexive* if for all $x \in A$, xRx holds.

Definition 4. An *equivalence relation* on a set A is a relation R which is symmetric, transitive and reflexive.

Definition 5. Let A be a set and R be an equivalence relation on A. If $x \in A$, the *equivalence class* of x is the subset

$$[x] = \{ y \in A \colon xRy \}$$

So the equivalence class of x contains all elements y which are related to x under R.

Lemma 6. Let R be an equivalence relation on the set A and let $x \in A$. If $x' \in [x]$, then $[x'] = [x]$.

Proof. Since we wish to prove that two sets are equal, we follow the usual method of showing all the elements of one set lie in the other and vice versa.

Note first our starting hypothesis is that we have an element x' in the equivalence class of x. This means that our x' satisfies xRx'.

Step 1: Let $y \in [x']$. This means that $x'Ry$. We then have two instances of elements being related under R: both xRx' and $x'Ry$ hold. Since R is transitive, this implies that xRy. Hence $y \in [x]$.

We have shown that if $y \in [x']$, then $y \in [x]$. This means $[x'] \subseteq [x]$.

Step 2: Let $y \in [x]$. This means that xRy holds. Now we know that xRx' holds (since this is our original assumption on x'). Since R is symmetric, this implies that $x'Rx$ holds. We now have both $x'Rx$ and xRy holding, so we deduce that $x'Ry$ holds (as R is transitive). This means that $y \in [x']$.

We have shown that if $y \in [x]$, then $y \in [x']$. This means $[x] \subseteq [x']$.

Finally, if we put Steps 1 and 2 together, we obtain $[x'] = [x]$, as required.

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