Graphs (Mathematics & Logic A)

RWK/MRQ

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1 Graphs and Trees

Definition 1. A *graph* G consists of a non-empty set V of *vertices* and a list E of unordered pairs of vertices called *edges*.

We usually represent a graph by a diagram where vertices are represented by filled in circles and an edge vw is represented by a line from the circle representing vertex v to the circle representing vertex w.

Example 2.

i. The *complete graph* K_n on n vertices is the graph with n vertices and a single edge between each pair of vertices.

ii. If $n \geq 3$, C_n, the *cycle of length* n, is the graph with n vertices

$$V = \{v_1, v_2, \ldots, v_n\}$$

and the following n edges

$$E = \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_nv_1\}.$$

Definition 3. If G is a graph and v is a vertex of G, the *vertex degree* at v is the number of times an edge meets v.

Definition 4. If G is a graph the *degree sequence* of G is the list of all vertex degrees of vertices of G (including repetitions) in increasing or nondecreasing order.

For example, the degree sequence of C_4 is $(2, 2, 2, 2)$.

Theorem 5 (Handshaking Lemma). In any finite graph, the sum of all the vertex degrees is equal to twice the number of edges.

Corollary 6. In a finite graph, the number of vertices with odd degree is always even.

Definition 7. A *regular* graph is a graph G in which all vertices have the same degree.

For example C_4 is regular as all vertices in C_4 have degree 2.

Corollary 8. In a finite regular graph with n vertices and e edges, in which all vertices have degree d, we have $2e = nd$.

Definition 9. A *simple graph* is a graph whose collection of edges has no loops and no multiple edges.

Example 10. (i) How many copies of C_3 are there in K_5?

(ii) How many copies of C_4 are there in K_5?

Definition 11. The *path of length* n is the graph P_n with vertex set

$$V = \{v_0, v_1, \dots, v_n\}$$

and edges

$$E = \{v_0v_1, v_1v_2, \dots, v_{n-1}v_n\}.$$

So the n in P_n counts the number of edges in the path (and the number of vertices is therefore $n + 1$).

Definition 12. Let $G = (V, E)$ be a graph (so V is the edge set and E is the list of edges). Define a relation R on V by

xRy means: there is a path in G from x to y.

Theorem 13. R is an equivalence relation on V.

Proof. We must show that R is reflexive, symmetric and transitive.

- If x is a vertex in V, we can use the path of length 0 from x to itself (i.e., start at x and don't move anywhere) to show that xRx holds. Hence R is symmetric.
- Suppose that xRy holds. This means that there is a path in G which starts at x and ends at y. Follow this path backwards: we obtain a path from y to x, so that yRx holds. Hence R is symmetric.
- Suppose that xRy and yRz both hold. This means that there is a path in G from x to y and a path in G from y to z. Join these two paths together: we obtain a path from x to z (which passes through y) and we deduce xRz holds. Hence R is transitive.

Thus R is an equivalence relation. $\square$

Since R is an equivalence relation, the equivalence classes of R form a partition of V.

Definition 14. If G is a graph, the *connected components* of G are the equivalence classes of the above equivalence relation R.

We say that G is *connected* if there is just one connected component. (This means that the equivalence class of any vertex x of G under the equivalence relation R equals the set V of all vertices.)

Definition 15. A *tree* is a connected simple graph containing no cycles.