2003 MSM1F3 Exam (Maths & Logic A)

3 hours — No calculator — Do all the questions

Section A (1.5 hours)

Question 1

(a) (i) Give the exact decimal for each of the following fractions:

$$\frac{2}{3}, \quad \frac{3}{4}, \quad \frac{2}{15}, \quad \frac{3}{40}.$$

(ii) Find (in its simplest form) the fraction $\frac{p}{q}$ which has decimal $0 \cdot \dot{3}\dot{6}$.

(iii) Simplify $(\sqrt{3} - \sqrt{2})(3 + \sqrt{6})$. [5]

(b) How many elements are there in each of the following sets?

(i) $\{\mathbb{Q}\}$

(ii) $\mathbb{Z}$

(iii) $\{x \in \mathbb{Z} : x^2 \leq 2x + 3\}$ [3]

(c) Decide for each of the following statements whether it is true or false. If the statement is true, prove it; if the statement is false, give a couterexample.

(i) If $A \in B$ and $B \in C$, then $A \in C$.

(ii) If $A \subseteq B$, then $B^c \subseteq A^c$.

(iii) If $A, B \subseteq \mathcal{U}$, then $(A \cap B)^c = A^c \cup B^c$. [9]

Question 2

(a) Define two functions $f : \mathbb{N} \to \mathbb{N}$ and $g : \mathbb{R} \to \mathbb{R}$ as follows:

$$f(x) = 2x - 1; \qquad g(x) = x^3.$$

(i) Specify the range of f and the range of g.

(ii) Which of f, g is/are one-one?

(iii) Which of f, g is/are onto? [4]

(b) Define two relations R_1, R_2 on $\mathbb{Z}$ as follows:

xR_1y means "x divides y^2";

xR_2y means "$x - y$ is a multiple of 2".

(i) Decide which of R_1, R_2 is an equivalence relation, and prove your assertion.

(ii) Decide, giving reasons, whether the other is an order relation.

(iii) For the equivalence relation, give a simple description of the equivalence classes [0], [1], [4]. [8]

(c) Prove by induction that

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2$$

for each $n \in \mathbb{N}$. [5]

Question 3

(a) (i) How many 'anagrams' are there of the word EXAM?

(ii) How many 'anagrams' are there of the word EXAMINATION?

(iii) Give the exact values of the binomial coefficients $\binom{5}{2}$, $\binom{6}{3}$.

(iv) Use the binomial theorem to write down the expansion of $(a + b)^5$, evaluating all the coefficients. Expand $(x - 3y)^4$ in the same way. [7]

(b) (i) How many integers between 100 and 1000 do not involve the digit "0"?

(ii) How many of the integers between 100 and 1000 which do not involve the digit "0" have no repeated digit? [5]

(c) (i) Draw and label: a complete graph K_5 on 5 vertices; a circuit/cycle C_5 on five vertices.

(ii) How many copies of C_4 are there in K_5? How many copies of C_4 are there in K_n for $n \geq 5$?

(iii) Draw all trees on 5 vertices. [5]