Counting
(Mathematics & Logic A)

RWK/MRQ

December 6, 2002

Counting

"Counting" is also known as *Combinatorics.*

Inclusion-Exclusion Rule *Let A and B be finite sets. Then*

$$|A \cup B| = |A| + |B| - |A \cap B|\,.$$

If $A \cap B = \emptyset$, then $|A \cup B| = |A| + |B|$. This gives us a "sum rule" and hence a general principle for some counting problems, as follows:

- Given a difficult counting problem:
    i. Break it up in a natural way into disjoint subsets,
    ii. count each subset separately,
    iii. add the answers.

(If we can't get disjoint subsets, we should use the Inclusion-Exclusion Rule to deal with the overlap.)

Product Rule

Product Rule: If each object being counted can be uniquely labelled by an ordered pair (a, b) where

- there are m possible first coordinates, and
- for each first coordinate "a" there are n possible second coordinates "b",

then the total number of object to be counted is $m \times n$.

Product Rule (Triple Version): If each object being counted can be uniquely labelled by an ordered triple (a, b, c) where

- there are ℓ possible first coordinates, and
- for each first coordinate, there are m possible second coordinates, and
- for each first and second coordinate, there are n possible third coordinates,

then the total number of objects is $\ell \times m \times n$.

Factorials and Binomial Coefficients

Definition 1. If n is a positive integer, we define $n!$ (pronounced '*n factorial*') to be

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1.$$

By convention, we define $0! = 1$.

(The definition that $0! = 1$ is just a convention to make sure various formulae work properly. We shall meet some of these in this and the next lecture.)

Definition 2. The *binomial coefficient* $\binom{n}{r}$ is given by

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

for $0 \leq r \leq n$ (with r and n both integers).

(This definition is one of the reasons why we define $0!$ to be 1. The definition makes sure that we can divide by $0!$, so that $\binom{n}{n}$ makes sense and produces the correct result in the following theorem.)

$\binom{n}{r}$ is pronounced "*n choose r*". It is probably denoted nCr or ${}^n\mathrm{C}_r$ on your calculator.

Theorem 3. The number of ways of choosing a set of r objects from a collection of n (distinguishable) objects equals

$$\binom{n}{r}.$$

Pascal's Triangle

Consider the following triangle whose entries are binomial coefficients:

$$\begin{array}{ccccccccc}
 & & & & \binom{0}{0} & & & & \\
 & & & \binom{1}{0} & & \binom{1}{1} & & & \\
 & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & \\
 & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & \\
\binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} \\
\vdots & & & & \vdots & & & & \vdots
\end{array}$$

This is *Pascal's Triangle*. Replacing these coefficients with their actual values, we have the following triangle:

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
\vdots & & & & & \vdots & & & & & \vdots
\end{array}$$

Note that each entry is equal to the sum of the two entries directly above it. This is embodied in the following result:

Lemma 4.
$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$$

The exercise sheets contain an exercise asking you to give both an algebraic proof and a combinatorial proof of this result.

The Binomial Theorem

The binomial theorem tells you how to expand a power of the form $(x+1)^n$ or $(x+y)^n$.

Theorem 5 (Binomial Theorem).

$$(a+b)^n = \binom{n}{0} a^n + \binom{n}{1} a^{n-1}b + \binom{n}{2} a^{n-2}b^2 + \cdots + \binom{n}{r} a^{n-r}b^r + \cdots + \binom{n}{n} b^n$$

Once again there are combinatorial proofs and algebraic proofs. Combinatorially, we can ask how may different ways are there of obtaining a term of the form $a^r b^s$ in $(a+b)^n$? Answer: $\binom{n}{r}$ of them, provided $r+s=n$. Algebraically, we can perform an induction on n. For the induction step use $(a+b)^{n+1} = (a+b)^n(a+b) =$

$$\left(\binom{n}{0} a^n + \binom{n}{1} a^{n-1}b + \binom{n}{2} a^{n-2}b^2 + \cdots + \binom{n}{r} a^{n-r}b^r + \cdots + \binom{n}{n} b^n\right)(a+b)$$

and now expand this out. This gives $(a+b)^{n+1} =$

$$\binom{n}{0} a^{n+1} + \left(\binom{n}{0} + \binom{n}{1}\right) a^n b^1 + \left(\binom{n}{1} + \binom{n}{2}\right) a^{n-1}b^2 + \cdots +$$
$$\left(\binom{n}{r} + \binom{n}{r+1}\right) a^{n-r+1}b^r + \cdots + \left(\binom{n}{n-1} + \binom{n}{n}\right) a^1 b^n + \binom{n}{n} b^{n+1}.$$

which, using the identity $\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}$, and also $\binom{n}{0} = 1 = \binom{n+1}{0}$ and $\binom{n+1}{0} = 1 = \binom{n+1}{n+1}$, will give us the next step in the induction: $(a+b)^{n+1} =$

$$\binom{n+1}{0} a^{n+1} + \binom{n+1}{1} a^n b + \binom{n+1}{2} a^{n-1}b^2 + \cdots + \binom{n+1}{r} a^{n+1-r}b^r + \cdots + \binom{n+1}{n+1} b^{n+1}$$

Exercise: Do the base case of this induction, and hence write this proof down properly.