Algebra of Sets (Mathematics & Logic A)

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Note. These notes are adapted (with thanks) from notes given last year by my colleague Dr Martyn Quick. Please feel free to ask *me* (not Dr Quick) if there is something in these notes that you do not understand.

These notes are provided as additional examples of proofs. For this reason there are no Venn diagrams. *A diagram does not constitute a proof.* You may find it useful to draw diagrams as you read the text to understand the overall shape of the proof, but an adequate proof should be completely watertight without a diagram. When you have read the proofs here you might, for extra practice, like to put these notes aside and see if you can reproduce a proof of the statements here on your own.

Sets. Remember the definitions of subsets and of when two sets are equal:

- $A = B$ means: if $x \in A$ then $x \in B$, *and* if $x \in B$ then $x \in A$.
- $A \subseteq B$ means: if $x \in A$ then $x \in B$.

Accordingly when we prove that two sets A and B are equal, the proof often breaks into two parts. The first step is to show that A is a subset of B (we usually pick an arbitrary element of A and prove that it must belong to B), while the second step is to show that B is a subset of A (by a similar method).

Lemma 1. Let A, B and C be sets.

i. $A \cup B = B \cup A$

ii. $A \cap B = B \cap A$

iii. $A \cup (B \cup C) = (A \cup B) \cup C$.

iv. $A \cap (B \cap C) = (A \cap B) \cap C$.

Proof. (i) Recall that

$$A \cup B = \{\, x \colon x \in A \text{ or } x \in B \,\}$$
$$B \cup A = \{\, x \colon x \in B \text{ or } x \in A \,\}.$$

This means that an element lies in the union $A \cup B$ precisely when it lies in one of the two sets A and B. Equally an element lies in the union $B \cup A$ precisely when it lies in one of the two sets A and B. Hence

$$A \cup B = B \cup A.$$

(ii) Recall that

$$A \cap B = \{ x \colon x \in A \text{ and } x \in B \}$$
$$B \cap A = \{ x \colon x \in B \text{ and } x \in A \}.$$

This means that an element lies in the intersection $A \cap B$ precisely when it lies in both of the two sets A and B. Equally an element lies in the intersection $B \cap A$ precisely when it lies in both of the two sets A and B. Hence

$$A \cap B = B \cap A.$$

(iii) **Step 1:** If $x \in A \cup (B \cup C)$, then either $x \in A$ or $x \in B \cup C$. We analyze these two possibilities separately.

If $x \in A$, then $x \in A \cup B$ (since by definition of the union, $A \cup B$ includes all elements which belong to A). From this fact we deduce that $x \in (A \cup B) \cup C$ (since the union $(A \cup B) \cup C$ contains all elements belonging to $A \cup B$).

If $x \in B \cup C$, then either $x \in B$ or $x \in C$. If $x \in B$, then $x \in A \cup B$ (as this union contains all elements in B), and it follows that $x \in (A \cup B) \cup C$. If $x \in C$, then $x \in (A \cup B) \cup C$ (since the union $(A \cup B) \cup C$ contains all elements belonging to C).

Hence we have shown that if $x \in A \cup (B \cup C)$, then $x \in (A \cup B) \cup C$ (irrespective of which of the above cases holds). So

$$A \cup (B \cup C) \subseteq (A \cup B) \cup C.$$

Step 2: If $x \in (A \cup B) \cup C$, then either $x \in A \cup B$ or $x \in C$. We analyze these two possibilities separately.

If $x \in A \cup B$, then either $x \in A$ or $x \in B$. If $x \in A$, then $x \in A \cup (B \cup C)$ (since the union $A \cup (B \cup C)$ certainly contains all elements from A). On the other hand, if $x \in B$, then $x \in B \cup C$, so we deduce $x \in A \cup (B \cup C)$.

If $x \in C$, then $x \in B \cup C$ and we then deduce $x \in A \cup (B \cup C)$.

Hence we have shown that if $x \in (A \cup B) \cup C$ then (irrespective of which of the above cases holds) $x \in A \cup (B \cup C)$. So

$$(A \cup B) \cup C \subseteq A \cup (B \cup C).$$

We now put the conclusions of Step 1 and Step 2 together. We have shown that

$$A \cup (B \cup C) = (A \cup B) \cup C.$$

(iv) **Step 1:** If $x \in A \cap (B \cap C)$, then $x \in A$ and $x \in B \cap C$ (since an element lies in an intersection of two sets precisely when it lies in both the sets concerned). Since $x \in B \cap C$, this means $x \in B$ and $x \in C$. We now put this information back together.

Since $x \in A$ and $x \in B$, we have $x \in A \cap B$. Now we have $x \in A \cap B$ and $x \in C$, so we deduce $x \in (A \cap B) \cap C$.

Hence we have shown that if $x \in A \cap (B \cap C)$, then $x \in (A \cap B) \cap C$. So

$$A \cap (B \cap C) \subseteq (A \cap B) \cap C.$$

Step 2: If $x \in (A \cap B) \cap C$, then $x \in A \cap B$ and $x \in C$. Since $x \in A \cap B$, this means $x \in A$ and $x \in B$. Now from $x \in B$ and $x \in C$, we deduce $x \in B \cap C$. Now since $x \in A$ and $x \in B \cap C$, we deduce that $x \in A \cap (B \cap C)$.

Hence we have shown that if $x \in (A \cap B) \cap C$, then $x \in A \cap (B \cap C)$. So

$$(A \cap B) \cap C \subseteq A \cap (B \cap C).$$

Putting the conclusions of Step 1 and Step 2 together, we deduce that

$$A \cap (B \cap C) = (A \cap B) \cap C.$$

□

Comments about the proof of Lemma 1. I have tried to spell out *all* the details here. Many people would be tempted to say that (i) and (ii) particularly were 'obvious', but this always hides the real reason why they are true!

Theorem 2. Let A, B and C be sets.

i. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

ii. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Proof. (i) See the solution to Question 6(a) on Problem Sheet 3.

(ii) **Step 1:** Let $x \in A \cap (B \cup C)$. Since an element in an intersection is in both sets involved, we must have $x \in A$ and $x \in B \cup C$. Now the fact that x lies in the union $B \cup C$, means that either $x \in B$ or $x \in C$. We analyze these two possibilities separately (using the fact that $x \in A$ holds in whichever case we consider).

If $x \in B$, then we have $x \in A$ and $x \in B$. Hence $x \in A \cap B$.

If $x \in C$, then we have $x \in A$ and $x \in C$. Hence $x \in A \cap C$.

Therefore we see that either $x \in A \cap B$ or $x \in A \cap C$. This means $x \in (A \cap B) \cup (A \cap C)$.

So in conclusion (for Step 1) we have shown that if $x \in A \cap (B \cup C)$, then $x \in (A \cap B) \cup (A \cap C)$. Hence

$$A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C).$$

Step 2: This could be conducted in the same manner as Step 1. What I present below is, however, an alternative way to produce the required inclusion of sets. (I do present after the proof the more familar way of completing this step.)

The definition of intersection tells us that $A \cap B \subseteq A$ and $A \cap C \subseteq A$. This means that every element in $A \cap B$ and every element in $A \cap C$ belongs to A, so if we form their union we are still going to end up with elements that belong to A. Hence

$$(A \cap B) \cup (A \cap C) \subseteq A. \tag{1}$$

Also the definition of intersection tells us that $A \cap B \subseteq B$, while the definition of union tells us that $B \subseteq B \cup C$. Hence

$$A \cap B \subseteq B \cup C.$$

Similarly the definitions of intersection and of union tells us that $A \cap C \subseteq C \subseteq B \cup C$. Hence

$$A \cap C \subseteq B \cup C.$$

These two tell us that every element in $A \cap B$ and every element in $A \cap C$ belong to $B \cup C$. Hence if we form the union of $A \cap B$ and $A \cap C$, we can only end up with elements belonging to $B \cup C$. Therefore

$$(A \cap B) \cup (A \cap C) \subseteq B \cup C. \tag{2}$$

Equations (1) and (2) tell us that the set $(A \cap B) \cup (A \cap C)$ is a subset of both A and in $B \cup C$ simultaneously. Consequently, it must be contained in the union of these two sets:

$$(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C).$$

We now put the conclusions of Step 1 and Step 2 together and deduce

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

□

[**Alternative proof of Step 2:** Let $x \in (A \cap B) \cup (A \cap C)$. This means that either $x \in A \cap B$ or $x \in A \cap C$. We consider each of these possibilities separately.

If $x \in A \cap B$, then $x \in A$ and $x \in B$. The fact that $x \in B$ implies that $x \in B \cup C$. Hence we have $x \in A$ and $x \in B \cup C$, so that $x \in A \cap (B \cup C)$.

If $x \in A \cap C$, then $x \in A$ and $x \in C$. The fact that $x \in C$ implies that $x \in B \cup C$. Hence we have $x \in A$ and $x \in B \cup C$, so that $x \in A \cap (B \cup C)$.

In conclusion, this shows that if $x \in (A \cap B) \cup (A \cap C)$, then $x \in A \cap (B \cup C)$ (no matter which case we are in). Therefore

$$(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C).$$

The proof of Theorem 2.20(ii) can then be completed in the same manner as before.]

Lemma 3. Let $\mathcal{U}$ be a set and A and B be subsets of $\mathcal{U}$. If $A \subseteq B$, then $B^c \subseteq A^c$.

[Remember that the complement of A is defined to be

$$A^c = \{\, x \in \mathcal{U} \colon x \notin A \,\}.]$$

Proof. Suppose $A \subseteq B$. Let $x \in B^c$. This means that $x \notin B$. Now if it *were* the case that $x \in A$, we would have a contradiction since $A \subseteq B$. Thus $x \notin A$; that is, $x \in A^c$.

Hence we have shown that if $x \in B^c$, then $x \in A^c$. Therefore

$$B^c \subseteq A^c$$

holds whenever $A \subseteq B$. □

Theorem 4 (de Morgan's Laws). Let $\mathcal{U}$ be a set and A and B be subsets of $\mathcal{U}$.

i. $(A \cap B)^c = A^c \cup B^c$

ii. $(A \cup B)^c = A^c \cap B^c$

Proof. (i) **Step 1:** Let $x \in (A \cap B)^c$. Then $x \notin A \cap B$. Now if x *were* to belong to both A and B, then it would belong to the intersection. Hence either $x \notin A$ or $x \notin B$. If $x \notin A$ then $x \in A^c$, while if $x \notin B$ then $x \in B^c$. Hence $x \in A^c$ or $x \in B^c$. This means that $x \in A^c \cup B^c$.

So we have shown that

$$(A \cap B)^c \subseteq A^c \cup B^c.$$

Step 2: Let $x \in A^c \cup B^c$. Then either $x \in A^c$ or $x \in B^c$. If $x \in A^c$, this means that $x \notin A$ and by definition of the intersection $x \notin A \cap B$. On the other hand, if $x \in B^c$, this means that $x \notin B$ and by definition of the intersection $x \notin A \cap B$. Hence we have $x \notin A \cap B$ (irrespective of which of $x \in A^c$ or $x \in B^c$ is true). Therefore $x \in (A \cap B)^c$.

So we have shown that

$$A^c \cup B^c \subseteq (A \cap B)^c.$$

Putting together the conclusions of Step 1 and Step 2, we deduce

$$(A \cap B)^c = A^c \cup B^c.$$

(ii) **Step 1:** Let $x \in (A \cup B)^c$. Then $x \notin A \cup B$. Now if x *were* to belong to either of A or B, then it would belong to the union. Therefore $x \notin A$ *and* $x \notin B$. This means $x \in A^c$ and $x \in B^c$, so $x \in A^c \cap B^c$.

So we have shown that

$$(A \cup B)^c \subseteq A^c \cap B^c.$$

Step 2: Let $x \in A^c \cap B^c$. Then $x \in A^c$ and $x \in B^c$. The fact that $x \in A^c$ tells us that $x \notin A$, while the fact that $x \in B^c$ tells us that $x \notin B$. So we see that x neither belongs to A nor to B, so that $x \notin A \cup B$. Therefore $x \in (A \cup B)^c$.

So we have shown that

$$A^c \cap B^c \subseteq (A \cup B)^c.$$

Putting together the conclusions of Step 1 and Step 2, we deduce

$$(A \cap B)^c \subseteq A^c \cup B^c.$$

$\square$