On the Arithmetic of Lexicographic Exponentiation

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In [H] Hausdorff developed several arithmetic operations on totally ordered sets, generalizing Cantor's ordinal arithmetic. Many open questions arise naturally, that we have been studying in the last few years. This talk will give an overview of our main results: In [K], we studied lexicographic powers of the form $\mathbb{R}^\Gamma$, and asked whether the exponent is an **isomorphism invariant**:

Theorem 1 *Let α be an ordinal, and J a chain in which the chain $\mathbb{R}$ does not embed. Assume that φ is an embedding of $\mathbb{R}^\alpha$ in $\mathbb{R}^J$. Then α embeds in J. In particular, if α and β are distinct ordinals, then the chains $\mathbb{R}^\alpha$ and $\mathbb{R}^\beta$ are nonisomorphic.*

This theorem is used in [W] to classify the **convex congruences** of such powers. On the other hand, after establishing further **arithmetic rules**, we study in [HKM] nonisomorphic chains for which the corresponding lexicographic powers are isomorphic: for a countable infinite ordinal α, we show that $\mathbb{R}^{\alpha^*+\alpha}$ and $\mathbb{R}^\alpha$ are isomorphic. We show that $\mathbb{R}^\mathbb{R}$ and $\mathbb{R}^\mathbb{Q}$ are nonisomorphic. We show that $\Delta^\mathbb{R}$ is **2-homogeneous**, where Δ is a countable ordinal ≥ 2. Further related open questions arise while studying the question of defining an exponential function on a **power series field**: in [KKS2] we study **convex embeddings** of a chain Γ in a lexicographic power Δ^Γ and prove

Theorem 2 *Let Γ and Δ_γ, $\gamma \in \Gamma$, be nonempty totally ordered sets. For every $\gamma \in \Gamma$, fix an element 0_γ which is not the last element in Δ_γ. Suppose that Γ has no last element and that Γ' is a cofinal subset of Γ. Then there is no convex embedding $\iota : \Gamma' \to \mathbf{H}_{\gamma\in\Gamma}\Delta_\gamma$.*

In [KKS1] this result applies to prove that (full) power series fields never admit an **exponential function**. However, in [KS] we consider proper subfields consiting of series of length bounded by an uncountable regular κ. We show that these subfields can be naturally endowed with 2^κ pairwise distinct exponentials making them into models of real exponentiation. This is done by constructing lexicographic chains with many increasing automorphisms of distinct σ-ranks.

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