Advanced Studies in Pure Mathematics 24, 1996
Progress in Algebraic Combinatorics
pp. 0–0

Generation Theorems for Finite Groups

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Dedicated to Professor Michio Suzuki

§1. Introduction

This article is a survey of the author's work on generation theorems for finite groups. The starting point is:

Theorem A (J. G. Thompson 1968)**.** *A finite group is soluble if and only if every two elements generate a soluble subgroup.*

Thompson obtained this result as a corollary of his classification of the minimal simple groups [12]. A direct proof has been obtained by the author [3]. A natural question to ask is:

what happens if we keep one of the generators fixed?

For a finite group G we define

$$\mathrm{sol}(G)$$

to be the largest normal soluble subgroup of G.

Conjecture B. *Let x be an element of the finite group G. Then*

$$x \in \mathrm{sol}(G) \quad \textit{if and only if} \quad \langle x, y\rangle \quad \textit{is soluble for all} \quad y \in G.$$

The author has not yet been able to prove this conjecture. However, much progress has been made and will be described in what follows.

In order to illustrate one of obstacles to proving Conjecture B, we present a small but crucial part of the author's proof of Theorem A. Henceforth, the word *group* will mean *finite group.*

Lemma 1.1 (D. Goldschmidt [2])**.** *Let z be a p-element of the soluble group H. Then*

$$O_{p'}(C_H(z)) \leq O_{p'}(H).$$

Lemma 1.2 (M. B. Powell [1]). *Let d be a p'-element of the group G. If dg is a p'-element for all p'-elements $g \in G$ then $d \in O_{p'}(G)$*

Lemma 1.3. *Let G be a group in which every two elements generate a soluble subgroup. Let z be a p-element of G. Then*

$$O_{p'}(C_G(z)) \leq O_{p'}(G).$$

Proof. Choose $d \in O_{p'}(C_G(z))$, let g be a p'-element of G and set $H = \langle dz, g\rangle$. Since d and z are commuting elements with coprime orders, we have $d, z \in H$. By hypothesis, H is soluble so using Goldschmidt's Lemma we obtain

$$d \in O_{p'}(C_G(z)) \cap H \leq O_{p'}(C_H(z)) \leq O_{p'}(H).$$

Then as g is a p'-element we see that dg is a p'-element. Powell's Lemma forces $d \in O_{p'}(G)$. Q.E.D.

Consequently, if G is a minimal counterexample to Theorem A then we have

$$O_{p'}(C_G(z)) = 1$$

for every p-element z. This argument cannot be applied to the situation in Conjecture B. Thus we have:

Problem 1. *Obtain a generalization of Lemma 1.3 that is applicable to Conjecture B.*

§2. A characterisation of p-soluble groups

As a first step towards proving Conjecture B, the author has established the following:

Theorem C ([4]). *Let P be a Sylow p-subgroup of the group G. Then G is p-soluble if and only if $\langle P, g\rangle$ is p-soluble for all $g \in G$.*

We present an outline of the proof. The following elementary result, which is a precursor of the Goldschmidt Lemma, is the starting point.

Lemma 2.1. *Let P be a Sylow p-subgroup of the p-soluble group G. If D is a p'-subgroup of G that is normalized by P then $D \leq O_{p'}(G)$. In particular, if $d \in G$ then*

$$d \in O_{p'}(G) \quad \text{if and only if} \quad \langle d^P \rangle \quad \text{is a } p'\text{-subgroup.}$$

Suppose now that G is a minimal counterexample to Theorem C. For each $Q \in \mathrm{Syl}_p(G)$ define

$$\Lambda(Q) = \{d \in G \mid \left\langle d^Q \right\rangle \text{ is a } p'\text{-subgroup}\}.$$

Define a graph Γ whose vertices are the Sylow p-subgroups of G and join two distinct vertices Q and R by an edge if and only if

$$\begin{array}{l} Q \cap R \neq 1, N_Q(Q \cap R) \in \mathrm{Syl}_p(N_G(Q \cap R)) \\ \text{and there exists } \ n \in N_G(Q \cap R) \ \text{ such that } \ Q^n = R. \end{array}$$

Firstly it is shown that if $\{Q, R\}$ is an edge of Γ then $\Lambda(Q) = \Lambda(R)$. A connectivity argument is applied to prove that $\Lambda(Q)$ is independent of Q. It is then shown that $\Lambda(Q)$ is a subgroup and hence a normal subgroup of G. Thus

$$\Lambda(P) = O_{p'}(G).$$

However, G is simple since it is a minimal counterexample to Theorem C, so $\Lambda(P) = O_{p'}(G) = 1$.

Now let $g \in G$ and set $H = \langle P, g \rangle$. Then $O_{p'}(H) \leq \Lambda(P) = 1$ so as H is p-soluble we have $Z(P) \leq C_H(O_p(H)) \leq Z(O_p(H))$. Then $Z(P)$ commutes with $Z(P)^g$ and it follows that

$$Z(P) \leq O_p(G),$$

contrary to the simplicity of G.

This argument, when the details are examined, appears to be a generalization of Lemma 1.3. Unfortunately there is one case where it is inapplicable. If P is cyclic of order p then the graph Γ has no edges, so connectivity arguments are useless. This case is more difficult. A transfer argument is used to obtain a contradiction.

§3. The normal closure of a Sylow subgroup

The next step is to replace *p-soluble* by *soluble.*

Conjecture D. *Let P be a Sylow p-subgroup of the group G. Then*

$$P \leq \mathrm{sol}(G) \ \textit{ if and only if } \ \langle P, g \rangle \ \textit{ is soluble for all } \ g \in G.$$

This is much more difficult. Of course, a soluble group is p-soluble so using Theorem C it follows that a minimal counterexample to conjecture D satisfies:

Hypothesis 3.1.

(1) *P is a Sylow p-subgroup of the group G.*
(2) *$\langle P, g\rangle$ is soluble for all $g \in G$.*
(3) *$G = KP$ where $K \trianglelefteq G$ is a p'-subgroup and $|P| = p$.*
(4) *K is a non abelian characteristically simple group and $K = [K, P]$.*
(5) *If H is a proper P-invariant subgroup of K then $[H, P]$ is soluble.*

Thus we have a problem involving coprime action and so the subgroup $C_K(P)$ plays a prominent role. We immediately hit upon a fundamental difficulty: since $[C_K(P), P] = 1$, the fact that G is a minimal counterexample to Conjecture D tells us nothing about $C_K(P)$. Since $K = [K, P]$, we are trying to show that Hypothesis 3.1 implies that K, and hence $C_K(P)$ is soluble. So:

Problem 2. *Why cannot $C_K(P)$ be simple?*

As a final comment we note that the case where a Sylow p-subgroup is cyclic is the difficult case in the proof of Theorem C. Moreover, in the final configuration of the author's proof of Theorem A one has a group G in which the Sylow p-subgroups are cyclic for all $p > 3$. Consequently it seems probable that in any proof of Conjecture B that the configuration described in Hypothesis 3.1, with $P = \langle x\rangle$, will be the most difficult case.

§4. Signalizer functors

Throughout this section we assume Hypothesis 3.1. Fix a prime divisor q of $|C_K(P)|$. A good starting point is to analyze the subgroups $C_K(z)$ for q-elements $1 \neq z \in C_K(P)$. These are proper P-invariant subgroups of K so we know that $[C_K(z), P]$ is soluble. By analogy with Lemma 3.1, we would like to limit the structure of $O_{q'}(C_K(z))$.

We begin with the following extension of Goldschmidt's Lemma to groups that admit a coprime operator group.

Lemma 4.1. *Let $G = PH$ be a group with $P \in \mathrm{Syl}_p(G)$ and $H = O_{p'}(G)$. Suppose that $[H, P]$ is soluble. Let q be a prime and let z be a q-element of $C_H(P)$. Then*

$$(O_{q'}(C_H(z)) \cap O_{q'}(C_H(P)))\,[O_{q'}(C_H(z)), P] \le O_{q'}(H).$$

Note that it is easy to construct examples in which $O_{q'}(C_H(z)) \not\le O_{q'}(H)$.

Returning now to Hypothesis 3.1, for each q-element $1 \neq z \in C_K(P)$ define

$$\theta(z) = (O_{q'}(C_K(z)) \cap O_{q'}(C_K(P)))\,[O_{q'}(C_K(z)), P].$$

Just as in the proof of Lemma 1.3, we would like to be able to argue that $\theta(z) \leq O_{q'}(K)$ and hence deduce that $\theta(z) = 1$. Unfortunately there does not appear to be an easy extension of Powell's Lemma that will suffice.

We turn to ideas from Signalizer Functor Theory. In broad terms the idea is as follows:

(a) Start with some collection $\mathcal{C}$ of subgroups of the group G that ought to be contained in a proper normal subgroup of G.
(b) Show that the members of $\mathcal{C}$ intersect the proper subgroups of G as they 'ought to'.
(c) Using (b), show that $\langle \mathcal{C} \rangle$ is a proper subgroup and use a connectivity argument to force $\mathcal{C} \trianglelefteq G$.

This idea was used in the proof of Theorem C. It is also a basic tool in the classification of simple groups, see [11].

In the situation at hand, $\mathcal{C}$ is the collection of subgroups $\theta(z)$ as z ranges over the q-elements of $C_K(P)$. Turning to (b), let $1 \neq z \in C_K(P)$ be a q-element and let M be a proper P-invariant subgroup of K that contains z. We want to show that

$$\theta(z) \cap M \leq O_{q'}(M).$$

This amounts to showing that $D \leq O_{q'}(M)$ where

$$D = [O_{q'}(C_H(z)), P] \cap M.$$

Now $D = [D, P]C_D(P)$ and by Lemma 4.1 we have $[D, P] \leq O_{q'}(M)$. However, we still have $C_D(P)$ to consider. This lead the author to make the following discovery:

Theorem E ([5]). *Let P be a group of prime order $p > 2$ that acts as a group of automorphisms on the soluble p'-group H. Then*

$$C_{[H,P]}(P) = \left\langle C_{[h,P]}(P) \mid h \in H \right\rangle.$$

The restriction that $p \neq 2$ is essential. Indeed if $p = 2$ then since any pair of involutions generate a dihedral group we have $C_{[h,P]}(P) = 1$ for all $h \in H$.

Using Theorem E and additional arguments, the author has established the following:

Theorem F ([6]). *Assume Hypothesis 3.1 and that $p > 2$. Let z be a q-element of $C_K(P)$ and let M be a proper P-invariant subgroup of K. Then*

$$\theta(z) \cap M \leq O_{q'}(M).$$

Note that we do not require z to be contained in M. An illustration of how Theorem E is used will be given later. At the time of writing, it has not been possible to show that $\theta(z) \leq O_{q'}(K)$. However we have at least a partial solution to Problem 1.

§5. A characterisation of $F_2(G)$.

Although it has not been possible to complete the program outlined in the previous section, the author feels that Theorem E will play a fundamental role in any proof of Conjecture B or D. Indeed the proof of the following special case of Conjecture B uses Theorem E. Recall that $F_2(G)$ is the inverse image of $F(G/F(G))$ in G.

Theorem G ([7]). *Let G be a group and $x \in G$. Then*

$$x \in F_2(G) \quad \textit{if and only if} \quad x \in F_2(\langle x, y \rangle) \quad \textit{for all} \quad y \in G.$$

Later we shall see how Theorems E and G can be used to solve Problem 2.

§6. A conjecture on coprime action

Conjecture H. *Let P be a group of prime order $p > 2$ that acts as a group of automorphisms on the p'-group H. Then*

$$C_{[H,P]}(P) = \left\langle C_{[h,P]}(P) \mid h \in H \right\rangle.$$

Theorem E shows this conjecture to be true when H is soluble. If proved, Conjecture H would have implications for Conjecture B. To see why, suppose that G is a minimal counterexample to Conjecture B and set $P = \langle x \rangle \cong \mathbf{Z}_p$. Assume further that G satisfies Hypothesis 3.1 and that $p > 2$. As we have remarked earlier, this could be the most difficult case in any proof of Conjecture B.

Now let $k \in$K and consider $[k, P]$. Using Sylow's Theorem we may suppose that $k \in [k, P]$. Let $c \in C_K(P)$. Then

$$k \in [k, P] \leq \left\langle P, P^k \right\rangle = \left\langle P, P^{ck} \right\rangle \leq \langle P, ck \rangle.$$

By hypothesis, $\langle P, ck \rangle$ is soluble. As $k \in \langle P, ck \rangle$ we deduce that $\langle [k, P], c \rangle$ is soluble. Consequently $\left\langle C_{[k,P]}(P), c \right\rangle$ is soluble for all $c \in C_K(P)$ and then the minimality of G forces $C_{[k,P]}(P) \leq \mathrm{sol}(C_K(P))$. Recall that $K = [K, P]$. Then the truth of Conjecture H would imply that $C_K(P)$ is soluble.

Next we give an interpretation of Conjecture H. We have

$$H = C_H(P)[H, P]$$

so there is a natural epimorphism

$$H \longrightarrow C_H(P)/\left(C_H(P) \cap [H, P]\right).$$

Set

$$D = \left\langle C_{[h,P]}(P) \mid h \in H \right\rangle \trianglelefteq C_H(P).$$

Define a map

$$\delta : H \longrightarrow C_H(P)/D$$

as follows: let $h \in H$. By Sylow's Theorem there exists $k \in [h, P]$ such that $P^h = P^k$. Thus we can write

$$h = ck$$

with $k \in [h, P]$ and $c \in C_H(P)$. Define

$$\delta(h) = Dc.$$

It is easily verified that δ is well defined.

If Conjecture H is true then δ is a homomorphism and it coincides with the natural epimorphism $H \longrightarrow C_H(P)/\left(C_H(P) \cap [H, P]\right)$. Conversely, if δ is a homomorphism then Conjecture H is true.

§7. Large 2-generated soluble subgroups

When attempting to prove Conjecture B, it seems inevitable that one has to consider modules for a soluble group in which some critical element has a large fixed point subspace. Such modules arose in the proofs of Theorems E and F, a contradiction being obtained by showing that such a module could not exist. There was other information available so it was not necessary to delve too deeply into the structure of modules for soluble groups.

After many false starts, the author has been able to extend these arguments and put them in a more general setting. The following theory emerged.

Theorem I ([8]). *Let G be a soluble group, let P be a subgroup of G with prime order $p > 3$ such that $G = \langle P^G \rangle$. Suppose that V is a faithful irreducible G-module over a field of non zero characteristic. Then*

$$\dim C_V(P) < \frac{1}{2}\dim V.$$

This result appears to be highly non trivial.

Next, let G be a group and P a subgroup of G with prime order $p > 3$. Define

$$\Sigma_G(P) = \{A \leq G \mid A \text{ is soluble and } A = \langle P, P^a \rangle \text{ for some } a \in A\}.$$

This set is partially ordered by inclusion and we let

$$\Sigma^*_G(P)$$

denote the set of maximal elements of $\Sigma_G(P)$.

Using Theorem I it is possible to establish the following fundamental property of members of $\Sigma^*_G(P)$.

Theorem J ([8]). *Let G be a group and P a subgroup of G with prime order $p > 3$. Let $A \in \Sigma^*_G(P)$. Then*

$$F(A)V$$

is nilpotent for every nilpotent subgroup V that is normalized by A.

Corollary K ([8]). *If G is soluble then $\pi(F(A)) \subseteq \pi(F(G))$.*

Thus the members of $\Sigma^*_G(P)$ exert global control over the structure of a soluble group. In fact, one can go much further:

Corollary L ([9]). *Let G be a soluble group, P a subgroup of G with prime order $p > 3$ and suppose that $G = \langle P^G \rangle$. Then there exists $g \in G$ such that $\langle P, P^g \rangle$ has the same Fitting height as G and $g \in \langle P, P^g \rangle$.*

For a group G and a subgroup P of prime order $p > 3$ we let

$$\Sigma^f_G(P)$$

be the set of members of $\Sigma_G(P)$ with maximal Fitting height. If G is soluble we define

$$\psi(G)$$

to be the smallest normal subgroup of G such that $G/\psi(G)$ has Fitting height less than that of G. If $G \neq 1$ then $1 \neq \psi(G) \leq F(G)$.

Corollary M ([9]). *Let G be a soluble group and P a subgroup of G with prime order $p > 3$. If $A \in \Sigma_G^f(P)$ then*

$$\psi(A) \leq F(G).$$

Thus, just by examining the members of $\Sigma_G(P)$, one can write down a subnormal nilpotent subgroup of G. This suggests an obvious strategy for proving Conjecture B, one which involves aiming directly at the Fitting subgroup. The following result provides evidence that this strategy could work and also shows that the theory developed so far is effective in proving generation theorems.

Theorem N. *Let $\mathcal{C}$ be a conjugacy class of the group G and suppose that the members of $\mathcal{C}$ have order prime to 6. Then $\langle \mathcal{C} \rangle$ is soluble if and only if every four members of $\mathcal{C}$ generate a soluble subgroup.*

Proof. Let $x \in \mathcal{C}$. We may suppose that x has prime order $p > 3$. Set $P = \langle x \rangle$ and choose $A \in \Sigma_G^f(P)$. Let $g \in G$ and set $H = \langle A, A^g \rangle$. By hypothesis, H is soluble. Now A and A^g are members of $\Sigma_H^f(P)$ so Corollary M implies that $\langle \psi(A), \psi(A)^g \rangle$ is nilpotent. The Baer-Suzuki Theorem implies that $\psi(A) \leq F(G)$. Now apply induction to $G/F(G)$. Q.E.D.

The results I-M are invalid without the hypothesis that $p > 3$. However it should be a routine matter to extend the theory so that the hypothesis *prime to 6* in Theorem N can be removed, provided that *four* is replaced by some larger number.

This theory can also be used to solve Problem 2, at least if $p > 3$.

Theorem O. *Assume Hypothesis 3.1 and that $p > 3$. Then*

$$F_2(C_K(P)) \neq 1.$$

Proof. By Theorem G there exists $g \in G$ such that $P \not\leq F_2(\langle P, g \rangle)$. Set $H = \langle P, g \rangle$ and $H_0 = \left\langle P^H \right\rangle \trianglelefteq H$, so that H_0 has Fitting height at least 3. Now P is a Sylow subgroup of H_0 so we have $H_0 = \left\langle P^{H_0} \right\rangle$. Corollary L implies that the members of $\Sigma_{H_0}^f(P)$ and hence the members of $\Sigma_G^f(P)$ have Fitting height at least 3.

Choose $A \in \Sigma_G^f(P)$. Let $\psi_2(A)$ denote the inverse image of $\psi(A/\psi(A))$ in A. Then $\psi_2(A)$ has Fitting height 2. As $A = \left\langle P^A \right\rangle$ we have $P \cap \psi_2(A) = 1$ so then $\psi_2(A) \leq K$.

Let $c \in C_K(P)$, choose $a \in A$ such that $A = \langle P, P^a \rangle$, and set $L = \langle A, c \rangle$. Now $a \in A = \langle P, P^a \rangle = \langle P, P^{ca} \rangle \leq \langle P, ca \rangle$ whence $L = \langle P, ca \rangle$ and L is soluble. Let $L_0 = \left\langle P^L \right\rangle \trianglelefteq L$. Then $A \leq L_0 = \left\langle P^{L_0} \right\rangle$ and

Corollary L implies that A has the same Fitting height as L_0. It follows that $\psi_2(A) \le \psi_2(L_0)$ whence $\psi_2(A) \le F_2(L)$. We deduce that

$$C_{\psi_2(A)}(P) \le F_2\left(\langle C_{\psi_2(A)}(P), c\rangle\right)$$

for all $c \in C_K(P)$. Theorem G implies that

$$C_{\psi_2(A)}(P) \le F_2\left(C_K(P)\right).$$

Since $\psi_2(A)$ has Fitting height 2 we have $C_{\psi_2(A)}(P) \ne 1$. This completes the proof of Theorem O. Q.E.D.

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