A characterisation of p-soluble groups

Paul Flavell

The School of Mathematics and Statistics
The University of Birmingham
Birmingham B15 2TT
United Kingdom
e-mail: p.j.flavell@bham.ac.uk

If p is a prime then a finite group is p-soluble if each of its composition factors is either a p-group or has order coprime to p. For example, soluble groups are p-soluble. However, there are many insoluble groups that are p-soluble. We shall prove the following result.

Theorem *Let G be a finite group and p a prime. Then G is p-soluble if and only if $\langle P, g\rangle$ is p-soluble whenever $g \in G$ and P is a Sylow p-subgroup of G.*

Let G be a minimal counter-example to the above theorem. Since subgroups and quotients of p-soluble groups are themselves p-soluble, it follows that G is a simple group in which every proper subgroup is p-soluble and in which $\langle P, g\rangle \neq G$ for all $g \in G$ and $P \in \mathrm{Syl}_p(G)$. The argument divides into two cases depending on whether a Sylow p-subgroup contains a noncyclic abelian subgroup or not. These possibilities correspond to whether a certain graph, with vertex set $\mathrm{Syl}_p(G)$, is connected or not. In sections 2 and 3 we show that there are no groups satisfying the above conditions. The classification of finite simple groups is not used.

The author is indebted to Dr. Christopher Parker for many helpful discussions during this work.

1 Preliminaries

All groups considered are finite. If g is an element of, and P a subgroup of a group G we let $\langle g^P \rangle = \langle g^x \mid x \in P \rangle$ and we note that $P \leq N(\langle g^P \rangle)$.

Lemma 1.1 *Let p be a prime and H a p-soluble group.*

(i) *If $\mathcal{O}_{p'}(H) = 1$ then $C_H(\mathcal{O}_p(H)) \leq \mathcal{O}_p(H)$.*

(ii) *If q is a prime and K is a $\{p, q\}$-subgroup of H then K is contained in a Hall $\{p, q\}$-subgroup of H.*

(iii) *If $P \in \mathrm{Syl}_p(H)$ and Q is a p'-subgroup of H that is normalized by P then $Q \leq \mathcal{O}_{p'}(H)$.*

(iv) *If T is a p-subgroup of H then $\mathcal{O}_{p'}(C_H(T)) \leq \mathcal{O}_{p'}(H)$.*

Proof (i) is [1, Theorem 6.3.3, p.228].
(ii) This is due to Čunihin. See [2, Cor. D5.3].
(iii) Pass to $H/\mathcal{O}_{p'}(H)$ and apply (i).
(iv) This is well known. Pass to $H/\mathcal{O}_{p'}(H)$ then use Thompson's $P \times Q$-Lemma [1, Theorem 5.3.4, p.179] and (i).

Remark We only need (ii) when H has cyclic or generalized quaternion Sylow p-subgroups and K already contains a Sylow p-subgroup of H. In this case, the proof of (ii) is easier.

Lemma 1.2 ([1, Theorem 6.2.4, p.225]) *Let P be a p-subgroup and D a p'-subgroup of a group G. Suppose that $P \leq N(D)$ and that P contains a noncyclic abelian subgroup. Then $D = \langle C_D(T) \mid 1 \neq T \leq P \rangle$.*

Lemma 1.3 *Let G be a group and p a prime. Define a graph Γ with vertex set $\mathrm{Syl}_p(G)$ and in which two distinct vertices P and Q are joined by an edge if and only if*

$$P \cap Q \neq 1, N_P(P \cap Q) \in \mathrm{Syl}_p(N(P \cap Q)) \text{ and } P^n = Q \text{ for some } n \in N(P \cap Q).$$

Then:

(i) *G acts by conjugation as a transitive group of automorphisms of Γ.*

(ii) *If $P, Q \in \Gamma$ and $P \cap Q \neq 1$ then P and Q are in the same connected component of Γ.*

Proof (i) is obvious.
(ii) We proceed by induction on $|P : P \cap Q|$. If $|P : P \cap Q| = 1$ then $P = Q$, hence result. Now suppose $|P : P \cap Q| \neq 1$. Let $T = P \cap Q$. Then $T < P$ so $T < N_P(T)$ and also $T < N_Q(T)$.

Choose $R \in \mathrm{Syl}_p(G)$ such that $N_P(T) \leq N_R(T) \in \mathrm{Syl}_p(N(T))$ and choose $n \in N(T)$ such that $N_Q(T) \leq N_R(T)^n$. We have $T < N_P(T) \leq P \cap R$ so by induction, P and R are in the same component. Also, $T < N_Q(T) \leq Q \cap R^n$ so again Q and R^n are in the same component. Observe that $T \leq R \cap R^n$. If $T < R \cap R^n$ then by induction, R and R^n are in the same component. If $T = R \cap R^n$ then by the definition of Γ, $\{R, R^n\}$ is an edge. In both cases, we see that P and Q are in the same component.

2 The connected case

In this section we shall prove:

Theorem 2.1 *There are no groups which satisfy the following hypotheses:*

(i) *G is a simple group and p is a prime divisor of $|G|$.*

(ii) *Every proper subgroup of G is p-soluble.*

(iii) *$\langle P, g \rangle \neq G$ for all $g \in G$ and $P \in \mathrm{Syl}_p(G)$.*

(iv) *A Sylow p-subgroup of G contains a noncyclic abelian subgroup.*

In the following sequence of lemmas, we assume the hypotheses of Theorem 2.1. Let Γ be the graph defined in 1.3. For each $P \in \mathrm{Syl}_p(G)$ define

$$\Lambda(P) = \{g \in G \mid \langle g^P \rangle \text{ is a } p'\text{-subgroup}\}.$$

Remark If G really were p-soluble then 1.1(iii) would imply $\Lambda(P) = \mathcal{O}_{p'}(G)$. In particular, $\Lambda(P)$ would be a p'-subgroup defined independently of P. We shall use a connectivity argument to establish the same conclusion for the groups under consideration.

Lemma 2.2 *Let $P \in \Gamma$ and $1 \neq T \leq P$ be such that $N_P(T) \in \mathrm{Syl}_p(N(T))$. If $g \in \Lambda(P)$ then there exist $x \in \Lambda(P)$ and $c \in \mathcal{O}_{p'}(C(T))$ such that $g = cx$ and $x \in \langle T, T^x \rangle$.*

Proof Set $H = \langle P, g \rangle$. Now $g \in \Lambda(P)$ so $H = P\mathcal{O}_{p'}(H)$ and $g \in \mathcal{O}_{p'}(H)$. Set $K = \langle T, T^g \rangle \leq H$. Then $K\mathcal{O}_{p'}(H) = T\mathcal{O}_{p'}(H)$ whence $K = T(K \cap \mathcal{O}_{p'}(H))$. Then $T, T^g \in \mathrm{Syl}_p(K)$ so $T^g = T^x$ for some $x \in \mathcal{O}_{p'}(H) \cap K$. Then $x \in \Lambda(P)$. Let $c = gx^{-1} \in N(T) \cap \mathcal{O}_{p'}(H) = C_{\mathcal{O}_{p'}(H)}(T)$. Now $\mathcal{O}_{p'}(H)$ is normalized by P so $C_{\mathcal{O}_{p'}(H)}(T)$ is normalized by $N_P(T)$. But $N_P(T) \in \mathrm{Syl}_p(N(T))$ so 1.1(iv) implies $C_{\mathcal{O}_{p'}(H)}(T) \leq \mathcal{O}_{p'}(N(T)) = \mathcal{O}_{p'}(C(T))$. Then $c \in \mathcal{O}_{p'}(C(T)), g = cx$ and $x \in \langle T, T^x \rangle$.

Lemma 2.3 *If $P, Q \in \Gamma$ and $P \cap Q \neq 1$ then $\Lambda(P) = \Lambda(Q)$.*

Proof In view of 1.3, we may assume $\{P, Q\}$ is an edge of Γ. Let $T = P \cap Q$ and choose $n \in N(T)$ such that $P^n = Q$.

Let $g \in \Lambda(P)$ and choose c, x in accordance with 2.2. Then $x \in \langle T, T^x \rangle = \langle T, T^{nx} \rangle \leq \langle P, nx \rangle = H$. But $\langle x^P \rangle$ is a p'-group so $x \in \mathcal{O}_{p'}(H)$ by 1.1(iii). Now $n = (nx)x^{-1} \in H$ so $Q = P^n \leq H$ and hence $\langle x^Q \rangle$ is a p'-group. Next we have $x \in \langle T, T^x \rangle = \langle T, T^{cx} \rangle \leq \langle Q, cx \rangle = K$ and again $x \in \mathcal{O}_{p'}(K)$. Also $c = (cx)x^{-1} \in K$. Using 1.1(iv) we have $c \in \mathcal{O}_{p'}(C(T)) \cap K \leq \mathcal{O}_{p'}(C_K(T)) \leq \mathcal{O}_{p'}(K)$ whence $g = cx \in \mathcal{O}_{p'}(K)$ and then $g \in \Lambda(Q)$. Thus $\Lambda(P) \subseteq \Lambda(Q)$. Similarly $\Lambda(Q) \subseteq \Lambda(P)$, hence result.

Lemma 2.4 *Let $P \in \Gamma$. Then $\Lambda(P)$ is a p'-subgroup.*

Proof Let $L = \langle \mathcal{O}_{p'}(C(T)) \mid 1 \neq T \leq P \rangle$. We will show that $\Lambda(P) = L$. Then $\Lambda(P)$ will be a subgroup and so by its definition, it will be a p'-subgroup.

First we show that if $1 \neq T \leq P$ and $h \in \mathcal{O}_{p'}(C(T))$ then $h\Lambda(P) \subseteq \Lambda(P)$. Choose $Q \in \mathrm{Syl}_p(G)$ such that $N_Q(T) \in \mathrm{Syl}_p(N(T))$. Let $g \in \Lambda(Q)$. By 2.2

there exist $x \in \Lambda(Q)$ and $c \in \mathcal{O}_{p'}(C(T))$ such that $g = cx$ and $x \in \langle T, T^x \rangle$. Then

$$x \in \langle T, T^x \rangle = \langle T, T^{hcx} \rangle \leq \langle Q, hcx \rangle = H.$$

Observe that $x \in \mathcal{O}_{p'}(H)$ by 1.1(iii). Using 1.1(iv) we have $hc = (hcx)x^{-1} \in H \cap \mathcal{O}_{p'}(C(T)) \leq \mathcal{O}_{p'}(C_H(T)) \leq \mathcal{O}_{p'}(H)$ whence $hg = hcx \in \mathcal{O}_{p'}(H)$ and $hg \in \Lambda(Q)$. But $1 \neq T \leq P \cap Q$ so $\Lambda(P) = \Lambda(Q)$ by the previous lemma. Hence $h\Lambda(P) \subseteq \Lambda(P)$.

Now $1 \in \Lambda(P)$ so the previous paragraph implies $L \subseteq \Lambda(P)$. Let $g \in \Lambda(P), D = \langle g^P \rangle, 1 \neq T \leq P$ and $d \in C_D(T)$. Choose $Q \in \mathrm{Syl}_p(G)$ such that $N_Q(T) \in \mathrm{Syl}_p(N(T))$. Using 2.3 we have $d \in \Lambda(P) = \Lambda(Q)$ whence $\langle d^{N_Q(T)} \rangle$ is a p'-subgroup of $C(T)$. Lemma 1.1(iii) implies $d \in \mathcal{O}_{p'}(C(T))$. Thus $C_D(T) \leq L$ and 1.2 implies $D \leq L$. We deduce that $\Lambda(P) = L$.

Proof of Theorem 2.1 Fix $P \in \mathrm{Syl}_p(G)$. First we argue that Γ is connected. Let Σ be the connected component that contains P. Lemma 1.3 implies $N(T) \leq N(\Sigma)$ for all $1 \neq T \leq P$. Let $g \in G$ and set $H = \langle P, g \rangle$. Then 1.2 implies $\mathcal{O}_{p'}(H) \leq N(\Sigma)$. Let D be the inverse image of $\mathcal{O}_p(H/\mathcal{O}_{p'}(H))$ in H and set $T = P \cap D$. Then $D \trianglelefteq H, T \in \mathrm{Syl}_p(D)$ and $D = \mathcal{O}_{p'}(H)T$. Lemma 1.1(i) implies $T \neq 1$ so using the Frattini Argument, we see that $H = \mathcal{O}_{p'}(H)N_H(T) \leq N(\Sigma)$. We deduce that $G = N(\Sigma)$ and since G acts transitively on Γ, it follows that Γ is connected.

Now $\Lambda(P^g) = \Lambda(P)^g$ for all $g \in G$ so the connectivity of Γ together with 2.3 and 2.4 imply that $\Lambda(P)$ is a normal p'-subgroup of G. The simplicity of G forces $\Lambda(P) = 1$.

Let $g \in G$ and set $H = \langle P, g \rangle$. Then $\Lambda(P) = 1$ forces $\mathcal{O}_{p'}(H) = 1$ and then 1.1(i) implies $Z(P) \leq \mathcal{O}_p(H)$. Now $\mathcal{O}_p(H) \leq P$ so it follows that $[Z(P), Z(P)^g] = 1$. We deduce that the normal closure of $Z(P)$ in G is abelian. This contradicts the simplicity of G and proves 2.1.

3 The disconnected case

In this section we shall prove:

Theorem 3.1 *There are no groups which satisfy the following hypotheses:*

(i) *G is a simple group, p is a prime divisor of $|G|$ and $P \in \mathrm{Syl}_p(G)$.*

(ii) *Every proper subgroup of G is p-soluble.*

(iii) *$\langle P, g\rangle \neq G$ for all $g \in G$.*

(iv) *P contains no noncyclic abelian subgroup.*

Remark [1, Theorem 5.4.10, p.199] implies that P is either cyclic or generalized quaternion. By a deep theorem of Brauer and Suzuki, there are no simple groups with generalized quaternion Sylow 2-subgroups. However, in Lemma 3.2 we sketch a short fusion argument that allows us to deal with this case in the same way as the cyclic case.

In the following sequence of lemmas, we assume the hypotheses of Theorem 3.1.

Lemma 3.2 (i) *$N(P)/PC(P)$ is a nontrivial cyclic p'-group.*

(ii) *If $P \leq H < G$ then $H = \mathcal{O}_{p'}(H)N_H(P)$.*

Proof Since $P \in \mathrm{Syl}_p(N(P))$ it follows that $N(P)/PC(P)$ is a p'-group. We have already remarked that P is either cyclic or generalized quaternion. Then every p'-subgroup of $\mathrm{Aut}(P)$ is cyclic and consequently $N(P)/PC(P)$ is cyclic. Suppose P is cyclic. Burnside's Normal p-Complement Theorem implies that $N(P)/PC(P) \neq 1$. Suppose that $P \leq H < G$. Set $\overline{H} = H/\mathcal{O}_{p'}(H)$. Then $\mathcal{O}_p(\overline{H}) \leq \overline{P}$ so that 1.1(i) implies $\mathcal{O}_p(\overline{H}) = \overline{P}$. Thus $\mathcal{O}_{p'}(H)P \trianglelefteq H$ and the Frattini Argument yields $H = \mathcal{O}_{p'}(H)N_H(P)$.

Now suppose that P is generalized quaternion, so $p = 2$. Set $M = N(Z(P))$. Since $Z(P)$ is the unique subgroup of order two in P we have $N(T) \leq M$ for all $1 \neq T \leq P$. The simplicity of G and a theorem of Alperin [1, Theorem 7.4.1, p.251] imply

$$P = P \cap G' = \langle [T, N(T)] \mid 1 \neq T \leq P\rangle \leq M'.$$

In particular, M does not have any nontrivial quotients that are 2-groups. Applying 1.1(i) to $\overline{M} = M/\mathcal{O}_{2'}(M)$ we see that $\overline{M}/\mathcal{O}_2(\overline{M})$ is isomorphic to a subgroup of $\mathrm{Out}(\mathcal{O}_2(\overline{M}))$ that contains a nontrivial element of odd order. But $\mathcal{O}_2(\overline{M})$ is cyclic or generalized quaternion whence $\mathcal{O}_2(\overline{M}) \cong Q_8$ and $\mathrm{Out}(\mathcal{O}_2(\overline{M})) \cong S_3$. Consequently $\overline{M}/\mathcal{O}_2(\overline{M}) \cong \mathbf{Z}_3, \mathcal{O}_2(\overline{M}) = \overline{P}, P \cong Q_8, \mathcal{O}_{2'}(M)P \trianglelefteq M = \mathcal{O}_{2'}(M)N(P)$ and finally $N(P)/PC(P) \cong \mathbf{Z}_3$. This proves (i). To prove (ii) apply 1.1(i) to $H/\mathcal{O}_{p'}(H)$ and use the fact that $P \cong Q_8$.

Remark If G really were p-soluble then $G = \mathcal{O}_{p'}(G)N(P)$ and there would be a natural homomorphism $G \longrightarrow N(P)/PC(P)$. We will define a map Δ that corresponds to this homomorphism. It does not seem possible to show directly that Δ is a homomorphism. However, we can show that Δ behaves sufficiently well to enable the use of a transfer argument.

Let
$$\delta \ : \ N(P) \longrightarrow N(P)/PC(P)$$
be the natural homomorphism.

Lemma 3.3 *Let $g \in G$, then:*

(i) *There exist $x \in G$ and $n \in N(P)$ such that $g = nx, x \in \langle P, P^x \rangle$ and $\langle x^P \rangle$ is a p'-group.*

(ii) *If $y \in G$ and $m \in N(P)$ are such that $g = my$ and $\langle y^P \rangle$ is a p'-group then $\delta(m) = \delta(n)$.*

Proof (i) Set $H = \langle P, P^g \rangle$. By 3.2(ii) there exists $x \in \mathcal{O}_{p'}(H)$ such that $P^g = P^x$. Let $n = gx^{-1}$. This proves (i).
(ii) Set $K = \langle P, y \rangle$. Observe that $x \in \langle P, P^x \rangle = \langle P, P^y \rangle \leq K$. Since $\langle x^P \rangle$ and $\langle y^P \rangle$ are p'-groups, 1.1(iii) implies $x, y \in \mathcal{O}_{p'}(K)$. But $nx = my$ so $m^{-1}n = yx^{-1} \in N(P) \cap \mathcal{O}_{p'}(K) \leq C(P)$ whence $\delta(m) = \delta(n)$.

For each $g \in G$, choose n and x in accordance with 3.3(i) and define
$$\Delta(g) = \delta(n).$$
By 3.3(ii), Δ is a well defined map $G \longrightarrow N(P)/PC(P)$.

Lemma 3.4 (i) *If $n \in N(P)$ then $\Delta(n) = \delta(n)$.*

(ii) *If $g \in G$ and $m \in N(P)$ then $\Delta(mg) = \Delta(m)\Delta(g)$.*

(iii) *If $P \leq H < G$ then Δ_H, the restriction of Δ to H, is a homomorphism. Moreover, $\mathcal{O}_{p'}(H) = \{g \in H \mid g$ is a p'-element and $\Delta(g) = 1\}$.*

(iv) *If $g \in G$ and $m \in N(P)$ then $\Delta(m^g) = \Delta(m)$.*

Proof (i) This follows from the definition of Δ.
(ii) Choose $n \in N(P)$ and $x \in G$ in accordance with 3.3(i). Observe that $mg = (mn)x, mn \in N(P)$ and $\langle x^P \rangle$ is a p'-group. Then 3.3(ii) implies $\Delta(mg) = \delta(mn) = \delta(m)\delta(n) = \Delta(m)\Delta(g)$.
(iii) By 3.2(ii) we have $H = N_H(P)\mathcal{O}_{p'}(H)$. Let $g, h \in H$, then $g = nx$ and $h = my$ for some $n, m \in N_H(P)$ and $x, y \in \mathcal{O}_{p'}(H)$. Now $gh = (nm)(x^m y), nm \in N(P)$ and $x^m y \in \mathcal{O}_{p'}(H)$. Then 3.3(ii) implies $\Delta(gh) = \delta(nm) = \delta(n)\delta(m) = \Delta(g)\Delta(h)$, so Δ_H is a homomorphism.

If $g \in \mathcal{O}_{p'}(H)$, then $n \in N_H(P) \cap \mathcal{O}_{p'}(H) \leq C(P)$ so $1 = \delta(n) = \Delta(g)$. Suppose $g \in H$ is a p'-element and that $\Delta(g) = 1$. Then $\delta(n) = 1$ whence $n \in PC_H(P)$. Lemma 1.1(iii) implies $C_H(P) \leq P\mathcal{O}_{p'}(H)$ so $g \in P\mathcal{O}_{p'}(H)$. Since g is a p'-element we deduce that $g \in \mathcal{O}_{p'}(H)$. This proves (iii).
(iv) Again choose n and x in accordance with 3.3(i) so that $g = nx$. Let $l = m^n \in N(P)$. Now

$$x \in \langle P, P^x \rangle = \langle P, P^{lx} \rangle \leq \langle P, lx \rangle = H,$$

so $x, l \in H$. Using (iii) and the fact that $N(P)/PC(P)$ is abelian, we see that $\Delta(l^x) = \Delta(l)$. Then $\Delta(m^g) = \Delta(m^{nx}) = \Delta(l^x) = \Delta(l) = \Delta(m^n) = \Delta(m)$.

Remark Recall that if $A, B, C \leq G$ then A permutes with B if and only if $AB = BA$, if and only if AB is a subgroup. Moreover, if A and B permute with C then so does $\langle A, B \rangle$ whence $\langle A, B \rangle C$ is a subgroup.

Lemma 3.5 *Let $q \neq p$ be a prime divisor of $|N(P)/PC(P)|$. Then there exists $Q \in \mathrm{Syl}_q(G)$ such that $PQ = QP$ and $N_Q(P) \in \mathrm{Syl}_q(N(P))$. In particular, PQ is a Hall $\{p, q\}$-subgroup of G.*

Proof Assume false. Since a Sylow q-subgroup of $N(P)$ permutes with P we may choose a q-subgroup $Q \leq G$ maximal subject to $PQ = QP$ and to $N_Q(P) \in \mathrm{Syl}_q(N(P))$. By assumption, $Q \notin \mathrm{Syl}_q(G)$. Note that PQ is a $\{p,q\}$-subgroup and that $\mathcal{O}_{p'}(PQ) = \mathcal{O}_q(PQ) \leq Q$. Let $\overline{N} = \mathcal{O}_q(N(P)/PC(P))$. Lemma 3.2(i) implies that $\overline{N}$ is cyclic and contains every q-element of $N(P)/PC(P)$. Moreover, as $N_Q(P) \in \mathrm{Syl}_q(N(P))$ we have

$$\overline{N} = \delta(N_Q(P)) = \Delta(N_Q(P)).$$

Now $Q \notin \mathrm{Syl}_q(G)$ and so $N(Q) - Q$ contains q-elements. If g is such an element then 3.4(iii) applied to $\langle P, g\rangle$ implies that $\Delta(g)$ is a q-element, whence $\Delta(g) \in \overline{N}$. Since $\overline{N}$ is cyclic, there exists $m \in N_Q(P)$ such that $\Delta(m)\Delta(g)$ generates $\overline{N}$. Observe that mg is also a q-element of $N(Q) - Q$ and 3.4(ii) implies that $\Delta(mg)$ generates $\overline{N}$.

By the preceding argument, we may choose a q-element $g \in N(Q) - Q$ such that $\overline{N} = \langle \Delta(g)\rangle$. Set $H = \langle P, g\rangle$. Then HQ is a subgroup since P and $\langle g\rangle$ permute with Q. Now $\langle Q, g\rangle$ is a q-subgroup of HQ and so $Q \notin \mathrm{Syl}_q(HQ)$. The maximality of Q and 1.1(ii) imply

$$G = HQ = QH. \qquad (*)$$

Next we show that $C_Q(P) = 1$. Let

$$W = \langle C_Q(P)^x \mid x \in G \text{ and } C_Q(P)^x \leq Q\rangle.$$

Clearly $N(Q) \leq N(W)$. Now $\Delta(C_Q(P)) = \delta(C_Q(P)) = 1$ so 3.4(iv),(iii) imply $W \leq \mathcal{O}_{p'}(PQ) = \mathcal{O}_q(PQ) \leq Q$ and consequently $PQ \leq N(W)$. But $g \in N(Q)$ so $(*)$ implies $G \leq N(W)$ forcing $1 = W = C_Q(P)$.

The next objective is to prove that $H \cap Q = N_Q(P)$. Let $K = PQ \cap H$. Then $P \leq K$ so $K = P(Q \cap H), Q \cap H \in \mathrm{Syl}_q(K)$ and $\mathcal{O}_q(K) \leq Q \cap H$. Using 1.1(iii) we have $\mathcal{O}_q(K) \leq Q \cap \mathcal{O}_{p'}(H) \leq K \cap \mathcal{O}_{p'}(H) \leq \mathcal{O}_q(K)$ whence

$$\mathcal{O}_q(K) = Q \cap \mathcal{O}_{p'}(H).$$

But P normalizes $\mathcal{O}_q(K)$ whilst g normalizes Q and $\mathcal{O}_{p'}(H)$. Then $\mathcal{O}_q(K) \trianglelefteq H$, $(*)$ implies $\langle \mathcal{O}_q(K)^G\rangle \leq Q$, the simplicity of G forces $\mathcal{O}_q(K) = 1$ and 3.2(ii) yields $P \trianglelefteq K$. Thus $H \cap Q \leq N_Q(P)$.

By 3.3(iii), Δ_H is a homomorphism. Now $\Delta_H(P) = \delta(P) = 1, \langle \Delta_H(g)\rangle = \overline{N}$ and $H = \langle P, g\rangle$, thus $\Delta_H(H) = \overline{N}$. By 3.2(ii) we have $H = \mathcal{O}_{p'}(H)N_H(P)$

and by 3.4(iii) we have $\mathcal{O}_{p'}(H) \leq \mathrm{Ker}(\Delta_H)$. Thus $\Delta_H(N_H(P)) = \overline{N}$ and if we choose $R \in \mathrm{Syl}_q(N_H(P))$ we have $\Delta_H(R) = \overline{N}$. Now $N_Q(P) \in \mathrm{Syl}_q(N(P))$ and so $R^n \leq N_Q(P)$ for some $n \in N(P)$. By $(*)$, there are $h \in H$ and $k \in Q$ such that $n = hk$. Observe that $R^h \leq H \cap Q \leq N_Q(P)$. However, $\Delta(R^h) = \Delta(R) = \overline{N} = \Delta(N_Q(P))$ and since $C_Q(P) = 1$, the restriction of Δ to $N_Q(P)$ is injective. Hence $N_Q(P) = R^h \leq Q \cap H \leq N_Q(P)$ so

$$H \cap Q = N_Q(P).$$

Since $g \in H \cap N(Q)$ it follows that $g \in N(N_Q(P))$. Let $z \in Q$. Then zg is a q-element of $N(Q) - Q$ so as previously, there exists $l \in N_Q(P)$ such that $\overline{N} = \langle \Delta(lzg) \rangle$. But lzg is also a q-element of $N(Q) - Q$ so the preceeding argument, with lzg in place of g, yields $lzg \in N(N_Q(P))$. Since $l, g \in N(N_Q(P))$ we have $z \in N(N_Q(P))$ and we deduce that $N_Q(P) \trianglelefteq Q$. By construction, $N_Q(P) \neq 1$ and hence

$$G = \langle N_Q(P)^G \rangle = \langle N_Q(P)^{QH} \rangle = \langle N_Q(P)^H \rangle \leq H = \langle P, g \rangle,$$

contrary to hypothesis. Hence result.

Proof of Theorem 3.1 By 3.2(i) there exists a prime $q \neq p$ that divides $|N(P)/PC(P)|$. The previous lemma implies G contains a Hall $\{p, q\}$-subgroup $H = PQ$ with $Q \in \mathrm{Syl}_q(G)$ and $N_Q(P) \in \mathrm{Syl}_q(N(P))$.

Let τ be the transfer of G into $N(P)/PC(P)$ relative to H and Δ_H. Let $n \in N_Q(P)$. By [1, Theorem 7.3.3, p.249] there exist $g_1, \ldots g_t \in G$ and $r_1, \ldots, r_t \in \mathbf{N}$ such that $g_i^{-1} n^{r_i} g_i \in H$ for all i, $\sum r_i = |G : H|$ and $\tau(n) = \Delta(\prod g_i^{-1} n^{r_i} g_i)$. The simplicity of G together with 3.4(iii),(iv) imply

$$1 = \tau(n) = \prod \Delta(g_i^{-1} n^{r_i} g_i) = \prod \Delta(n^{r_i}) = \Delta(n)^{|G:H|}.$$

But $\Delta(n)$ is a q-element and $|G : H|$ is coprime to q. Thus $\Delta(n) = 1$ and it follows that $N_Q(P) \leq C(P)$. But $N_Q(P) \in \mathrm{Syl}_q(N(P))$ so $N(P)/PC(P)$ is a q'-group. This contradicts the choice of q and completes the proof of Theorem 3.1.

References

[1] Gorenstein, D. *Finite groups, 2nd edn.* Chelsea Publishing Company, New York, 1980.

[2] Hall, P. *Theorems like Sylow's*, Proc. London Math. Soc., **6**(1956), 286-304.