A Hall-Higman-Shult Type Theorem for Arbitrary Finite Groups

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Abstract. We shall extend a fixed point theorem of Shult to arbitrary finite groups. This will have applications to the study of group automorphisms.

1. Introduction

We shall prove the following:

Theorem A. *Suppose that:*

- R *is a group of prime order* r *that acts on the* r'*-group* G.
- V *is a faithful irreducible* RG*-module over a field of characteristic* p.
- $C_V(R) = 0$.

Then either $[G, R] = 1$ *or* r *is a Fermat prime and* $[G, R]$ *is a nonabelian special* 2*-group.*

The configuration described in Theorem A was first considered by Shult. His [10, Theorem 3.1, p.702] amounts to proving Theorem A when G has prime power order. Shult's work is a nonmodular analogue of the celebrated work of Hall and Higman [7]. In the first edition of his book *Finite Group Theory*, Aschbacher extended Shult's results, essentially proving Theorem A when G is soluble [1, (36.4), p.194].

Shult used his result to study automorphisms of soluble finite groups. In [2] and [3], Theorem A will be applied to study automorphisms of arbitrary finite groups. Aschbacher's motivation was a new proof of the Soluble Signalizer Functor Theorem. If it is ever possible to prove the General Signalizer Functor Theorem outside of the inductive framework of the Classification of Finite Simple Groups, then Theorem A may have a role to play.

A number of further remarks are in order.

- The exceptional case, when $[G, R] \neq 1$ does occur.
- The proof of Theorem A presented here does not use the Classification of Finite Simple Groups. The heaviest tool used is Glauberman's Z^*-Theorem, which uses Modular Representation Theory. In a forthcoming article, we shall examine the exceptional case in greater detail. This will lead to a more elementary, though longer proof of Theorem A.
- A numerological proof of Theorem A is possible using the Classification. In §7 we will outline our argument. This will give insight as to why Theorem A is true.
- In the case that r is not a Fermat prime, Theorem A follows easily from Shult's result.

Several problems for further study will suggest themselves to anyone who studies the proof of Theorem A. Indeed, one result of a general nature has already emerged – a criterion for a weakly closed subgroup to be strongly closed, Theorem 4.1.

2. Preliminaries

Henceforth, group will mean finite group. We recall some elementary facts about coprime action. If the group R acts on the group G and p is a prime then

$\text{И}_G(R, p) =$ the set of R-invariant p-subgroups of G
$\text{И}^*_G(R, p) =$ the set of maximal members of $\text{И}_G(R, p)$ under inclusion.

We say that R acts *coprimely* on G if R acts as a group of automorphisms on G; R and G have coprime orders; and at least one of R or G is soluble. The following results are well known, see [1] for example.

Theorem 2.1 (Coprime Action). *Suppose that R acts coprimely on G and let p be a prime. Then:*

(a) $\text{И}^*_G(R, p) \subseteq \mathrm{Syl}_p(G)$, so G possesses R-invariant Sylow p-subgroups.
(b) $C_G(R)$ acts transitively by conjugation on $\text{И}^*_G(R, p)$.
(c) If $P \in \text{И}^*_G(R, p)$ then $C_P(R) \in \mathrm{Syl}_p(C_G(R))$.
(d) $G = C_G(R)[G, R]$. In particular, $[G, R] = [G, R, R]$.
(e) If G is abelian then $G = C_G(R) \times [G, R]$.
(f) If K is an R-invariant normal subgroup of G and $\overline{G} = G/K$ then $C_{\overline{G}}(R) = \overline{C_G(R)}$.
(g) Any two elements of $C_G(R)$ that are conjugate in G are already conjugate in $C_G(R)$.
(h) If R is elementary abelian then

$$G = \langle\, C_G(R_0) \mid R_0 \text{ is a hyperplane of } R \,\rangle.$$

Lemma 2.2. *Suppose that R acts coprimely on G and that H is an R-invariant subgroup of G with the property:*

> *whenever $p \in \pi(|G : C_G(R)|)$ there exists $P \in \mathrm{\mathcal{V}}^*_G(R,p)$ with $[P,R] \leq H$.*

Then $G = C_G(R)H$.

Proof. It suffices to show that $|G : C_G(R)|_p = |H : C_H(R)|_p$ for each prime p. If $p \in \pi(|G : C_G(R)|)$ then choose $P \in \mathrm{\mathcal{V}}^*_G(R,p)$ with $[P,R] \leq H$. If $p \notin \pi(|G : C_G(R)|)$ then choose $P \in \mathrm{Syl}_p(C_G(R))$, so $P \in \mathrm{\mathcal{V}}^*_G(R,p)$ and $[P,R] \leq H$ in this case also.

By Coprime Action, $C_P(R) \in \mathrm{Syl}_p(C_G(R))$ and $P = C_P(R)[P,R]$ so then

$$|G : C_G(R)|_p = |P : C_P(R)| = |[P,R] : C_{[P,R]}(R)|.$$

Choose Q with $[P,R] \leq Q \in \mathrm{\mathcal{V}}^*_H(R,p)$. Again,

$$|H : C_H(R)|_p = |[Q,R] : C_{[Q,R]}(R)|.$$

By Coprime Action, some $C_G(R)$-conjugate of Q is contained in P. As $[P,R,R] = [P,R]$, it follows that $[Q,R] = [P,R]$. Then $|H : C_H(R)|_p = |G : C_G(R)|_p$, which completes the proof.

Lemma 2.3. *Suppose that R acts coprimely on G and that M is an R-invariant subgroup of G. Then $|C_G(R) : C_M(R)|$ divides $|G : M|$.*

Proof. Let p be a prime. By Coprime Action, there exists an R-invariant Sylow p-subgroup P of G with $P \cap M \in \mathrm{Syl}_p(M)$. By Coprime Action, $C_P(R) \in \mathrm{Syl}_p(C_G(R))$ and $C_{P\cap M}(R) \in \mathrm{Syl}_p(C_M(R))$. Then

$$|C_G(R) : C_M(R)|_p = |C_P(R) : C_{P\cap M}(R)| = |C_P(R)(P\cap M)|/|P\cap M|$$

is a power of p bounded by $|P : P \cap M| = |G : M|_p$ and hence divides $|G : M|_p$. This completes the proof.

Given a chain

$$V_0 \trianglelefteq V_1 \trianglelefteq \cdots \trianglelefteq V_n \qquad (\mathcal{C})$$

of subgroups of the group V and a group G acting on V then G *stabilizes* $(\mathcal{C})$ if G normalizes each V_i and acts trivially on each quotient V_i/V_{i-1}.

Lemma 2.4. *Suppose that G stabilizes the chain $1 \trianglelefteq U \trianglelefteq V$, where V is a p-group. Then $G/C_G(V)$ is an abelian p-group. If V is elementary abelian then so is $G/C_G(V)$.*

Proof. As $[V,G,G] = 1$, the Three Subgroups Lemma implies $[G,G]$ centralizes V. Using the identity $[a,bc] = [a,c][a,b]^c$ we have $[v,g^n] = [v,g]^n$ for all $v \in V, g \in G$ and $n \in \mathbb{N}$. Thus g^q centralizes V, where q is the exponent of V. The result follows.

Suppose that G, S and A are groups with $A \leq S \leq G$. If, for all $g \in G$,

$$A^g \leq S \quad \text{implies} \quad A^g = A$$

then we say A *is weakly closed in* S *with respect to* G. If, for all $g \in G$,

$$A^g \cap S \leq A$$

then we say A *is strongly closed in* S *with respect to* G. The following is well known and has a straightforward proof.

Lemma 2.5. *Let G be a group, p a prime, $S \in \mathrm{Syl}_p(G)$ and $A \leq S$. Then the following are equivalent:*

(a) A is weakly closed in S with respect to G.
(b) A is weakly closed in $N_G(A)$ with respect to G.
(c) Whenever B is a conjugate of A such that A and B normalize one another then $A = B$.

Lemma 2.6 [1, (31.15), p.159]. *Let P be a p-subgroup of the soluble group G. Then $O_{p'}(C_G(P)) \leq O_{p'}(G)$.*

Theorem 2.7. *Let p be an odd prime. There exists a mapping K that assigns to each p-group $P \neq 1$ a characteristic subgroup $K(P) \neq 1$ that has the following property:*

if G is a group with abelian Sylow 2-subgroups, $C_G(O_p(G)) \leq O_p(G)$ and $P \in \mathrm{Syl}_p(G)$ then

$$K(P) = K(O_p(G)) \trianglelefteq G.$$

Proof. Put $K(P) = K^\infty(P)K_\infty(P)$ where K^∞ and K_∞ are the functors defined by Glauberman [11, p.226]. By [11, (4.12)(3), p.203] and [11, (5.8)(3), p.237] both $K^\infty(P)$ and $K_\infty(P)$ are normal in G. Then $K(P) \leq O_p(G)$ and [11, (5.4)(ii), p.228] implies $K(P) = K(O_p(G))$.

Theorem 2.8. *Let $L \neq 1$ be a group and X a faithful $GF(2)L$-module. Let T and T^* be 2-subgroups of L. Set*

$$\Phi = C_X(T), \quad \Phi^* = C_X(T^*),$$

$$\Gamma = \{\, \Phi^l, \Phi^{*l} \mid l \in L \,\} \quad \textit{and} \quad q = |\Phi| > 1.$$

Suppose that

(a) $L = \langle\, T, T^* \,\rangle$.
(b) $X = \Phi \oplus \Phi^*$.
(c) $|\Phi^*| = |\Phi|$.
(d) Distinct members of Γ have trivial intersection.
(e) T stabilizes $0 < \Phi < X$ and T^* stabilizes $0 < \Phi^* < X$.
(f) $|T| \geq q$ and $|T^*| \geq q$.

Then $q^2 - 1$ *divides* $|L|, |T| = q$ *and* $N_L(T)$ *acts transitively on* $T^{\#}$.

Proof. Let $M = N_L(\Phi)$ and let T_1 consist of those elements of M that stabilize $0 < \Phi < X$. Then $T \leq T_1 \trianglelefteq M$. Consider the action of T_1 on $\Gamma - \{\,\Phi\,\}$. Using (b), $N_{T_1}(\Phi^*) = 1$ so $|\Gamma - \{\,\Phi\,\}| \geq |T_1| \geq |T| \geq q$. On the other hand, (b) and (d) imply $|\Gamma| \leq q+1$. We deduce that $|T_1| = |T| = q, T \trianglelefteq M, |\Gamma| = q+1$ and that T is regular on $\Gamma - \{\,\Phi\,\}$. Similarly, T^* is regular on $\Gamma - \{\,\Phi^*\,\}$ so L is transitive on Γ and then $q+1$ divides L.

From the above, $M = N_L(T)$ and for all $g \in L$,

$$T^g \cap M \neq 1 \quad \text{implies} \quad g \in M.$$

Lemma 2.4 implies that T is elementary abelian. A simple argument involving involutions, e.g. Lemma 5.3(b), implies that M is transitive on $T^{\#}$. In particular, $q-1$ divides $|M|$ and then q^2-1 divides $|L|$.

Remarks. – One can show directly that L is transitive on $X^{\#}$.
– A theorem of Glauberman [4, Theorem 2, p.5] can be used to show that $L \cong SL_2(q)$ and that X is a natural module for L.

The following is a variant of Thompson's $P \times Q$-Lemma [1, (24.1), p.112].

Lemma 2.9. *Suppose that* $R \times X \times Y$ *acts on the group* $T = [T, R]$, *that* R, Y *are* $2'$*-groups and* X, T *are* 2*-groups. If* $[\Phi(T), R] = 1$ *and* $[C_T(X), R, Y] = 1$ *then* $[T, Y] = 1$.

Proof. Set $\overline{T} = T/\Phi(T)$. Since $T = [T, R]$ we have $C_{\overline{T}}(R) = 1$ by Coprime Action(e). Let $Q = [C_T(X), R]$. It follows from the Three Subgroups Lemma that $\overline{Q} = C_{\overline{T}}(X)$. The $P \times Q$-Lemma forces $[\overline{T}, Y] = 1$. Then $[T, Y] = 1$ by a well known result of Burnside.

Lemma 2.10. *Suppose the group* G *acts* 2*-transitively on a set* Ω. *Let* M *be the stabilizer of a point and* p *a prime. If* M *does not contain a Sylow* p*-subgroup of* G *then* $O_p(M) \leq O_p(G)$.

Proof. Choose $P \in \mathrm{Syl}_p(M)$. Now $P \notin \mathrm{Syl}_p(G)$ so there exists $g \in N_G(P) - M$. Then $O_p(M) \leq P = P^g \leq M^g$. By 2-transitivity, $N_G(M) = M$ so $M^g \neq M$. Moreover, M acts transitively by conjugation on $M^G - \{\,M\,\}$. It follows that $O_p(M)$ is contained in every conjugate of M, so $O_p(M) \leq O_p(M_G) \leq O_p(G)$.

Lemma 2.11 [8, Theorem 5.4, p.277]. *Suppose that* T *is a* 2*-group of class at most* 2 *and exponent at most* 4. *Suppose that* $\{\, x \in T \mid x^2 = 1 \,\}$ *is a subgroup* N *of* T. *Then* $|T| \leq |N|^3$.

Lemma 2.12. *Suppose the group* X *acts on the group* Y *and that the group* XY *acts on the set* Ω. *Suppose that* X *or* Y *is soluble, that* X *and* Y *have coprime orders and that* Y *is transitive on* Ω. *Then* X *fixes an element of* Ω.

Proof. Set $G = XY$ and choose $\alpha \in \Omega$. Then $G = G_\alpha Y$. Using the Schur-Zassenhaus Theorem we see that G_α contains a complement to Y, which is then conjugate to X.

3. Some special actions

Throughout this section, we assume the following:

Hypothesis 3.1.
- *The group $R \times A$ acts on the group T.*
- *V is a faithful RAT-module over an algebraically closed field of characteristic p.*
- *R has prime order r.*
- *T has prime power order and is an $\{ r, p \}'$-group.*
- *$T = [T, R] \neq 1$.*
- *A is an r'-group.*
- *$C_V(R) = 0$.*

Note that $r \neq p$ because $C_V(R) = 0$.

Lemma 3.2. *(a) r is a Fermat prime, so $r = 2^n + 1$ for some n.*
(b) T is a nonabelian special 2-group and

$$T' = Z(T) = \Phi(T) = C_T(R).$$

(c) R centralizes every R-invariant abelian subgroup of T.
(d) The homogeneous components for T on V are the same as the homogeneous components for $\Phi(T)$ on V. In particular, $C_V(\Phi(T)) = C_V(T)$ and the homogeneous components are normalized by R.
(e) If T is extraspecial, for instance if T is homogeneous on V, then $T \cong 2_-^{1+2n}$, R is irreducible on $T/\Phi(T)$ and $[T, A] = 1$.

Proof. (a),(b),(c). By [1, (36.2), p.193] r is a Fermat prime, T is a 2-group and R centralizes every R-invariant abelian subgroup of T. Then T is nonabelian. By [1, (24.7), p.114], T is special and $Z(T) = C_T(R)$.

(d). Since $\Phi(T) = Z(T)$, Schur's Lemma implies that $\Phi(T)$ acts homogeneously on any homogeneous component for T. Hence it suffices to show that if U is a homogeneous component for $\Phi(T)$ then U is contained in a homogeneous component for T.

Now RT normalizes U because $[\Phi(T), RT] = 1$. Set $\overline{T} = T/C_T(U)$. Then R acts on $\overline{T}$, U is a faithful $R\overline{T}$-module and $C_U(R) = 0$. If $\overline{T} = 1$ then $U \leq C_V(T)$ so U is contained in a homogeneous component for T. Suppose that $\overline{T} \neq 1$. Applying (b) to $\overline{T}, U$ in the role of T, V it follows that $\overline{T}$ is a nonabelian special 2-group. Now $\Phi(\overline{T}) = \overline{\Phi(T)}$ so $\Phi(\overline{T})$ is abelian and acts homogeneously on U. Hence $\Phi(\overline{T})$ is cyclic so $\overline{T}$ is extraspecial. By [1, (34.9), p.180] an extraspecial 2-group has only one faithful irreducible representation over an algebraically closed field. Thus U is homogeneous as a T-module.

(e). Assume now that T is extraspecial. [1, (36.1), p.192] implies that $T \cong 2^{1+2n}$. Let $\overline{T} = T/\Phi(T)$ and recall that we may regard $\overline{T}$ as a $GF(2)$-orthogonal space. Then $\overline{T}$ has either $2^{n-1}(2^n - 1)$ or $2^{n-1}(2^n + 1)$ nonsingular vectors depending on whether the Witt index of $\overline{T}$ is n or $n - 1$

respectively. Now $\overline{T} = [\overline{T}, R]$ and $\overline{T}$ is abelian, so by Coprime Action(e), $C_{\overline{T}}(R) = 0$. Thus the nonsingular vectors fall into orbits of size $2^n + 1$ under the action of R. Consequently $\overline{T}$ has Witt index $n - 1$, so $T \cong 2_-^{1+2n}$.

Every nontrivial R-invariant subspace of $\overline{T}$ must contain at least $2^n + 1$ vectors and hence have dimension larger than n. Since $\dim(\overline{T}) = 2n$, it follows that R is irreducible on $\overline{T}$.

It remains to prove that $[T, A] = 1$. We may assume that A is a q-group for some prime q. We claim that $C_{\overline{T}}(A) \neq 0$. This is clear if $q = 2$ because $\overline{T}$ is a 2-group, so suppose $q \neq 2$. Now A permutes the 2^{n-1} orbits of R on the nonsingular vectors of $\overline{T}$. As $q \neq 2$ it follows that there is an A-invariant orbit. Applying Lemma 2.12, with A and R in the roles of X and Y respectively, we obtain $C_{\overline{T}}(A) \neq 0$ in this case also. Now R is irreducible on $\overline{T}$ and $[R, A] = 1$ whence $C_{\overline{T}}(A) = \overline{T}$. Then $[T, A] \leq \Phi(T)$. Also $[\Phi(T), R] = 1$ so it follows from the Three Subgroups Lemma that $[T, A] = 1$.

Corollary 3.3. *If A acts trivially on $\Phi(T)$ then A acts trivially on T.*

Proof. Now $[\Phi(T), RA] = 1$ so Lemma 3.2(d) implies that RA normalizes the homogeneous components for T. Hence we may assume that T is homogeneous on V. Apply Lemma 3.2(e).

Lemma 3.4. *(a) Set $\overline{T} = T/\Phi(T)$. Then $C_{\overline{T}}(A) = \overline{[C_T(A), R]}$.*
(b) If A acts trivially on $T/\Phi(T)$ then A acts trivially on T.

Proof. Let Q be the inverse image of $C_{\overline{T}}(A)$ in T. Then $[Q, A, R] \leq [\Phi(T), R] = 1$. Also $[A, R] = 1$ so $[A, R, Q] = 1$. The Three Subgroups Lemma forces $[R, Q, A] = 1$. Thus $[Q, R] \leq C_T(A) \leq Q$. Using Coprime Action(d) we have $[Q, R] = [C_T(A), R]$ and also $Q = C_Q(R)[Q, R]$. Now $C_Q(R) \leq C_T(R) = \Phi(T)$ whence $\overline{Q} = \overline{[C_T(A), R]}$. This proves (a). Suppose that A acts trivially on $\overline{T}$. Then $\overline{T} = C_{\overline{T}}(A) = \overline{[C_T(A), R]}$, hence $T = \Phi(T)C_T(A)$. This implies $T = C_T(A)$, proving (b).

Lemma 3.5. *Let $Z = [\Phi(T), A]$ and suppose that Z inverts V, so $Z \cong \mathbb{Z}_2$. Let $\overline{T} = T/\Phi(T)$. Then*

$$1 < [\overline{T}, A] < \overline{T}$$

and R acts irreducibly on both factors of this chain. Moreover, $[\overline{T}, A, A] = 1$ and $[\overline{T}, A] = C_{\overline{T}}(A)$.

Proof. Let U be an irreducible RAT-submodule and let $K = C_T(U)$. Observe that

$$[\Phi(T) \cap K, A] \leq Z \cap K = 1.$$

Now $K = C_K(R)[K, R]$. As $C_K(R) \leq C_T(R) = \Phi(T)$ we see that $[C_K(R), A] = 1$. Also $\Phi([K, R]) \leq \Phi(T)$ so $[\Phi([K, R]), A] = 1$. Thus

$[K, A] = 1$ by Corollary 3.3 applied to $[K, R]$ in the role of T. Since $K \trianglelefteq AT$ it follows that $[K, [T, A]] = 1$.

Suppose that $U < V$. Set $\widetilde{V} = V/U$ and $\widetilde{T} = T/C_T(\widetilde{V})$. Now $\Phi(\widetilde{T}) = \widetilde{\Phi(T)}$ so by induction, $C_{\widetilde{T}/\Phi(\widetilde{T})}(A) = [\widetilde{T}/\Phi(\widetilde{T}), A]$. Thus $\widetilde{K} \leq [\widetilde{T}, A]\Phi(\widetilde{T})$. Now $[K, [T, A]] = 1$ and $\Phi(\widetilde{T}) = Z(\widetilde{T})$ so $\widetilde{K}' = 1$. Then $K' \leq C_T(\widetilde{V}) \cap C_T(U) = 1$ and Lemma 3.2(c) forces $K \leq \Phi(T)$. Hence $\Phi(T/K) = \Phi(T)/K$ so $T/\Phi(T) \cong (T/K)/\Phi(T/K)$ and the conclusion follows by induction. Hence we may assume that RAT is irreducible on V.

Now A stabilizes the chain $1 \leq Z \leq \Phi(T)$ so Lemma 2.4 implies that A induces an elementary abelian 2-group on $\Phi(T)$. Corollary 3.3 implies that A induces an elementary abelian 2-group on T. Then $[\overline{T}, A] < \overline{T}$. Also, $[\Phi(T), A] = Z \neq 1$ so by Lemma 3.4(b) we have

$$1 < [\overline{T}, A] < \overline{T}. \tag{$*$}$$

The conclusions $[\overline{T}, A, A] = 1$ and $[\overline{T}, A] = C_{\overline{T}}(A)$ follow once we establish that R is irreducible on each factor of $(*)$. To do this, choose $a \in A$ with a nontrivial on $\Phi(T)$ and set $A_0 = \langle\, a \,\rangle$. Now $Z = [\Phi(T), A] = [\Phi(T), A_0]$ and $[\overline{T}, A_0] \leq [\overline{T}, A]$. Hence we may assume that $A = A_0$, so that $A = \langle\, a \,\rangle$.

Let W be a homogeneous component for RT on V. Since A induces an elementary abelian 2-group on T we have $[T, a^2] = 1$ so $Wa^2 = W$. The irreducibility of RAT on V forces $V = W + Wa$. Lemma 3.2(e), applied to the action of RT on W, implies that R has exactly one noncentral chief factor on $T/C_T(W)$. It follows that R has at most two noncentral chief factors on T. Now $C_T(R) = \Phi(T)$ so $C_{\overline{T}}(R) = 1$. We deduce that R is irreducible on each factor of $(*)$. The proof is complete.

Lemma 3.6. *Let $\mathbb{Z}_2 \cong Z \leq \Phi(T)$ and suppose that Z inverts V. Set $\overline{T} = T/Z$. Then either*

(a) T is extraspecial and $Z = \Phi(T)$, or
(b) $C_{\overline{T}}(R) = \overline{\Phi(T)} = \Phi(\overline{T}) = \overline{T}' = Z(\overline{T})$, so $\overline{T}$ is a nonabelian special 2-group.

Proof. By Coprime Action(f) and Lemma 3.2

$$C_{\overline{T}}(R) = \overline{C_T(R)} = \overline{\Phi(T)} = \Phi(\overline{T}) = \overline{T}' \leq Z(\overline{T}).$$

Let Q be the inverse image of $Z(\overline{T})$ in T and let $V_1, \ldots, V_m$ be the homogeneous components for T on V. Note that $Z \cap C_T(V_i) = 1$ because Z inverts V. Suppose that $Q \leq C_T(V_i)\Phi(T)$ for some i. Then $[Q, T] \leq Z \cap C_T(V_i) = 1$ so $Q \leq Z(T) = \Phi(T)$ and (b) holds. Hence we may suppose that $Q \not\leq C_T(V_i)\Phi(T)$ for all i.

By Lemma 3.2(e), R is irreducible on the Frattini quotient of $T/C_T(V_i)$ whence $T = QC_T(V_i)$. Also $[Q, C_T(V_i)] \leq Z \cap C_T(V_i) = 1$. Thus $T = Q * C_T(V_i)$. Then for any i, j we have

$$[C_T(V_i), T] = [C_T(V_i), Q * C_T(V_j)] \leq C_T(V_j).$$

Fixing i and letting j vary we obtain $[C_T(V_i), T] = 1$. Then $C_T(V_i) \leq Z(T) = \Phi(T)$ so as $T = Q * C_T(V_i)$ we have $T = Q$. Now $[Q, T] \leq Z$ so (a) holds.

4. TI-subgroups

The material in this section is heavily influenced by the work of Timmesfeld [12]. We shall comment on this point in more detail later. Recall that a subgroup Φ of a group G is a *TI-subgroup* if, for all $g \in G$,

$$\Phi \cap \Phi^g \neq 1 \quad \text{implies} \quad \Phi = \Phi^g.$$

Our goal is to prove the following:

Theorem 4.1. *Let G be a group with $O_2(G) = 1$ and suppose Φ, M, S and U are subgroups satisfying:*

- *$\Phi \neq 1$ is an elementary abelian 2-group.*
- *$M = N_G(\Phi)$ and $S \in \mathrm{Syl}_2(M)$.*
- *$U \leq N_M(S)$ and $\Phi \leq Z(U)$.*

Suppose also that:

- *Φ is weakly closed in M with respect to G.*
- *Φ is a TI-subgroup.*
- *M is the unique maximal 2-local subgroup of G that contains U.*

Then Φ is strongly closed in M with respect to G, so that for all $g \in G$,

$$\Phi^g \cap M \neq 1 \quad \textit{implies} \quad g \in M.$$

In particular, $S \in \mathrm{Syl}_2(G)$ and M controls G-fusion in S.

Theorem 4.1 may be viewed as a *pushing up* theorem. The uniqueness and weak closure hypotheses are statements about certain 2-local subgroups. The conclusion that M controls G-fusion in S is a statement about all 2-local subgroups.

We consider the following:

Hypothesis 4.2.

- *Φ is an elementary abelian 2-subgroup of the group G.*
- *Φ is a TI-subgroup of G.*
- *$\Phi^* \in \Phi^G - \{\Phi\}$ satisfies $N_{\Phi^*}(\Phi) \neq 1$.*

We adopt the notation

$$\Phi_0^* = N_{\Phi^*}(\Phi) \quad \textit{and} \quad \Phi_0 = N_{\Phi}(\Phi^*).$$

Theorem 4.1 is a straightforward consequence of the following, which is a slight extension of a result of Timmesfeld [12, (3.8), p.252].

Theorem 4.3. *Assume Hypothesis 4.2 and that Φ is weakly closed in $N_G(\Phi)$ with respect to G. Let U be a subgroup of $N_G(\Phi_0)$ such that $\langle \Phi_0^{*U} \rangle$ is a 2-group. Then $\langle \Phi_0^{*U} \rangle$ is abelian.*

Timmesfeld invokes a deep result of Aschbacher to determine the possibilities for the group $\langle \Phi, \Phi^* \rangle$. He then reaches the conclusion of Theorem 4.3 with a case by case analysis. By contrast, the proof of Theorem 4.3 presented here is elementary and avoids the use of deep classification theorems. The reader will however notice numerous similarities in the arguments we employ and those developed by Timmesfeld [12].

Lemma 4.4. *Assume Hypothesis 4.2. Then:*

(a) $|\Phi_0| = |\Phi_0^*|$ and we have symmetry between Φ and Φ^*. In particular, if $\Phi^* \leq N_G(\Phi)$ then $[\Phi^*, \Phi] = 1$.
(b) Φ_0^* stabilizes $1 \leq \Phi_0 \leq \Phi$ and Φ_0 stabilizes $1 \leq \Phi_0^* \leq \Phi^*$.
(c) $\langle \Phi_0, \Phi_0^* \rangle = \Phi_0 \times \Phi_0^* \trianglelefteq \langle \Phi, \Phi^* \rangle$. In particular, $\Phi_0 \times \Phi_0^*$ may be regarded as a module for $\langle \Phi, \Phi^* \rangle$.
(d) Φ stabilizes $1 < \Phi_0 < \Phi_0 \times \Phi_0^*$ and Φ^* stabilizes $1 < \Phi_0^* < \Phi_0 \times \Phi_0^*$.

Proof. We have $[\Phi_0, \Phi_0^*] \leq \Phi \cap \Phi^* = 1$ so

$$\langle \Phi_0, \Phi_0^* \rangle = \Phi_0 \times \Phi_0^*.$$

Let $1 \neq a \in \Phi_0^*$. Now a is an involution that acts on the elementary abelian 2-group Φ so $C_\Phi(a) \neq 1$ and $[\Phi, a, a] = 1$. Since Φ^* is TI we obtain

$$[\Phi, a] \leq C_\Phi(a) \leq \Phi \cap N_G(\Phi^*) = \Phi_0.$$

Then $\Phi_0 \neq 1$ and there is symmetry between Φ and Φ^*. Now a was arbitrary so $[\Phi, \Phi_0^*] \leq \Phi_0$. Then (b), (c) and (d) follow.

Suppose that $|\Phi_0| < |\Phi_0^*|$. Consider the action of Φ on $\Phi_0 \times \Phi_0^*$. Then for each $x \in \Phi$, $\Phi_0^* \cap \Phi_0^{*x} \neq 1$, so $\Phi \leq N_G(\Phi_0^*)$ because Φ^* is TI. Hence $\Phi = \Phi_0$ so $|\Phi| < |\Phi_0^*| \leq |\Phi^*|$, a contradiction. Thus $|\Phi_0| \geq |\Phi_0^*|$ and (a) follows by symmetry.

Proof of Theorem 4.3. Assume false. We may suppose that $\Phi_0^* \leq U$. Choose $u \in U$ with $|\langle \Phi_0^*, \Phi_0^{*u} \rangle|$ minimal subject to the condition

$$[\Phi_0^*, \Phi_0^{*u}] \neq 1.$$

Claim 1. *If Φ_1 and Φ_2 are distinct conjugates of Φ then*

$$\Phi_1 \cap C_G(\Phi_2) = 1.$$

Proof. Suppose there exists $1 \neq g \in \Phi_1 \cap C_G(\Phi_2)$. Then since Φ_1 is TI we have $\Phi_2 \leq C_G(g) \leq N_G(\Phi_1)$, contrary to the weak closure of Φ_1.

Claim 2. *(a)* $\langle\, \Phi_0^{*h} \mid h \in \Phi_0^{*u} \,\rangle$ *is elementary abelian.*
(b) $1 \neq [\Phi_0^*, \Phi_0^{*u}] \leq C_G(\Phi)$.
(c) $N_{\Phi_0^{*u}}(\Phi^*) = 1$.

Proof. Set $S = \langle\, \Phi_0^*, \Phi_0^{*u} \,\rangle$. Since S is a nonabelian 2-group we have $\langle\, \Phi_0^{*S} \,\rangle < S$. The choice of u implies $\langle\, \Phi_0^{*S} \,\rangle$ is abelian. This proves (a).

Since Φ_0^* stabilizes the chain $1 \leq \Phi_0 \leq \Phi$ and since $u \in N_G(\Phi_0) \leq N_G(\Phi)$ it follows that Φ_0^{*u} also stabilizes this chain. Thus $[\Phi_0^*, \Phi_0^{*u}] \leq C_G(\Phi)$ by Lemma 2.4, proving (b).

Suppose there exists $1 \neq a \in N_{\Phi_0^{*u}}(\Phi^*)$. Using (b) and Claim 1 we have $[\Phi_0^*, a] \leq C_{\Phi^*}(\Phi) = 1$. Then $\Phi_0^* \leq C_G(a) \leq N_G(\Phi^{*u})$. Note that $\Phi^{*u} \neq \Phi$ because $u \in N_G(\Phi_0) \leq N_G(\Phi)$. Using (b) and Claim 1 we have $1 \neq [\Phi_0^*, \Phi_0^{*u}] \leq C_{\Phi^{*u}}(\Phi) = 1$. This contradiction proves (c) and completes the proof of Claim 2.

Consider the chain

$$1 < \Phi_0 < \Phi_0 \times \Phi_0^*. \tag{$\mathcal{C}$}$$

Claim 3. Φ_0^{*u} *stabilizes* $(\mathcal{C})$.

Proof. Choose $1 \neq a \in \Phi_0^{*u}$ and set $K = \langle\, \Phi_0^*, \Phi_0^{*a} \,\rangle$. Claim 2(a) and (c) imply that

$$K = \Phi_0^* \times \Phi_0^{*a}.$$

We will show that $\langle\, \Phi^*, \Phi^{*a} \,\rangle$ normalizes K. Using Lemma 4.4(b), $[\Phi_0, \Phi^*, \Phi_0^{*a}] \leq [\Phi_0^*, \Phi_0^{*a}] = 1$. Also $[\Phi_0^{*a}, \Phi_0, \Phi^*] = [[\Phi_0^*, \Phi_0]^a, \Phi^*] = [1, \Phi^*] = 1$ because $a \in U \leq N_G(\Phi_0)$. The Three Subgroups Lemma forces $[\Phi^*, \Phi_0^{*a}, \Phi_0] = 1$. Now $\Phi_0^{*a} \leq C_G(\Phi_0^*) \leq N_G(\Phi^*)$ whence

$$[\Phi^*, \Phi_0^{*a}] \leq \Phi^* \cap C_G(\Phi_0) \leq N_{\Phi^*}(\Phi) = \Phi_0^*.$$

Thus $\Phi^* \leq N_G(K)$. Now a is an involution so K is a-invariant and then $\langle\, \Phi^*, \Phi^{*a} \,\rangle \leq N_G(K)$.

Note that $\langle\, \Phi^*, \Phi^{*a} \,\rangle$ is a-invariant. By considering an a-invariant Sylow 2-subgroup of $\langle\, \Phi^*, \Phi^{*a} \,\rangle$ and by using the weak closure of Φ^*, we may choose $g \in \langle\, \Phi^*, \Phi^{*a} \,\rangle$ such that a normalizes Φ^{*g}. Claim 2(c) implies that a does not normalize Φ_0^* so $\Phi_0^* \cap \Phi_0^{*g} = 1$. Then as $|\Phi_0^{*a}| = |\Phi_0^{*g}|$ we obtain

$$K = \Phi_0^* \times \Phi_0^{*g}. \tag{1}$$

Also, since the involution a acts on the 2-group Φ_0^{*g} we have

$$C_{\Phi_0^{*g}}(a) \neq 1.$$

Recall that $K = \Phi_0^* \times \Phi_0^{*a}$. Then visibly

$$C_K(a) = [K, a] = [\Phi_0^*, a] \leq C_K(\Phi),$$

the inclusion following from Claim 2(b). Thus

$$1 \neq C_{\Phi_0^{*g}}(a) \leq \Phi^{*g} \cap C_G(\Phi).$$

Claim 1 forces $\Phi^{*g} = \Phi$. Then $\Phi_0^{*g} \leq C_\Phi(\Phi_0^*) \leq N_\Phi(\Phi^*) = \Phi_0$ so as $|\Phi_0^*| = |\Phi_0|$ we have $\Phi_0^{*g} = \Phi_0$. From (1) we obtain $K = \Phi_0 \times \Phi_0^*$. In particular, $\Phi_0 \times \Phi_0^*$ is a-invariant. Moreover

$$[K, a] \leq C_K(\Phi) = \Phi_0 \times C_{\Phi_0^*}(\Phi).$$

By Claim 1, $C_{\Phi_0^*}(\Phi) = 1$ so $[K, a] \leq \Phi_0$. Also $[\Phi_0, a] = 1$ since $[\Phi_0, \Phi_0^*] = 1$, $u \in N_G(\Phi_0)$ and $a \in \Phi_0^{*u}$. This completes the proof of Claim 3.

We are now in a position to derive a contradiction. Lemma 4.4 and Claim 3 imply that Φ and Φ_0^{*u} stabilize $(\mathcal{C})$. Since $|\Phi_0| = |\Phi_0^*| = |\Phi_0^{*u}|$ and since Φ^* is TI, it follows from Claim 2(c) that

$$\Phi_0 \times \Phi_0^* = \Phi_0 \cup \bigcup_{t \in \Phi_0^{*u}} \Phi_0^{*t}.$$

Choose $x \in \Phi - \Phi_0$. Such a choice is possible by the weak closure of Φ. Then $\Phi_0^{*x} \neq \Phi_0^*$. By the above, $\Phi_0^{*xy} = \Phi_0^*$ for some $y \in \Phi_0^{*u}$. Then $xy \in N_G(\Phi_0^*) \leq N_G(\Phi^*)$. Also, as x and y stabilize $(\mathcal{C})$ we have $[\Phi_0^*, xy] \leq \Phi_0^* \cap \Phi_0 = 1$ and $[\Phi_0, xy] = 1$. Thus $xy \in C_G(\Phi_0 \times \Phi_0^*)$.

Now Φ^* acts on $\Phi_0 \times \Phi_0^*$ so $[\Phi^*, xy] \leq C_G(\Phi_0 \times \Phi_0^*)$. Thus

$$[\Phi^*, xy] \leq \Phi^* \cap C_G(\Phi_0) \leq N_{\Phi^*}(\Phi) = \Phi_0^*,$$

whence xy stabilizes the chain $1 \leq \Phi_0^* \leq \Phi^*$. Lemma 2.4 yields

$$(xy)^2 \in C_G(\Phi^*).$$

Since x and y are involutions, $(xy)^2 = [x, y]$. Note that $y \in \Phi_0^{*u} \leq U \leq N_G(\Phi)$. As $x \in \Phi$ we obtain

$$[x, y] \in \Phi \cap C_G(\Phi^*).$$

Claim 1 forces $[x, y] = 1$. Then

$$x \in N_\Phi(\Phi^{*u}) = (N_\Phi(\Phi^*))^u = \Phi_0^u = \Phi_0,$$

contrary to the choice of x. The proof is complete.

Proof of Theorem 4.1. Suppose that $g \in G - M$ and that $\Phi^g \cap M \neq 1$. Set $\Phi^* = \Phi^g$. Hypothesis 4.2 is satisfied and we adopt the notation defined there. Without loss of generality, $\Phi_0^* \leq S$. Let $M^* = M^g$. Set

$$W = \langle\, \Phi_0^{*U} \,\rangle \leq M.$$

Now W is a 2-group because $U \leq N_M(S)$. Theorem 4.3 implies that W is abelian. As Φ^* is a TI-subgroup we have $W \leq C_G(\Phi_0^*) \leq M^*$ and so $\Phi^* W$ is a 2-group.

Since $U \leq N_G(W)$ the uniqueness hypothesis implies $N_G(W) \leq M$. In particular,

$$N_{\Phi^*}(W) \leq \Phi^* \cap M = \Phi_0^* \leq W.$$

But $\Phi^* W$ is a 2-group so this forces $\Phi^* \leq W \leq M$, contradicting the weak closure of Φ. Thus no such g exists and the strong closure of Φ is established.

By the weak closure of Φ, $N_G(S) \leq M$, so $S \in \mathrm{Syl}_2(G)$. Also $\Phi \trianglelefteq S$ so $\Phi \cap Z(S) \neq 1$. Suppose A is any subgroup with $Z(S) \leq A \leq S$. If $g \in N_G(A)$ then $1 \neq (\Phi \cap Z(S))^g \leq \Phi^g \cap A \leq \Phi^g \cap M$ so the previous paragraph forces $g \in M$. Thus $N_G(A) \leq M$. Alperin's Fusion Theorem implies that M controls G-fusion in S.

5. Strong embedding

Recall that a proper subgroup M of a group G is *strongly embedded* in G if M has even order and $M \cap M^g$ has even order implies $g \in M$. The theory of strong embedding and related ideas is developed fully in [6]. We require only a special case, consequently much shorter proofs are possible. It is emphasized that a large portion of this section is a presentation of material from [6].

Theorem 5.1. *Let G be a group, Φ an elementary abelian 2-subgroup of G, set $M = N_G(\Phi)$ and $\Omega = \Phi^G$. Consider the conjugation action of G on Ω.*

Suppose that Φ is not normal in G, that Φ is noncyclic and that for all $g \in G$,

$$\Phi^g \cap M \neq 1 \quad \textit{implies} \quad g \in M.$$

Let $g \in G - M$, set $D = M \cap M^g$ and $m = |\Phi| - 1$. Then:

(a) G is 2-transitive on Ω.
(b) Every involution of G is conjugate to an element of $D\Phi$.
(c) Suppose that $D = O_{2'}(D) \times O_2(D)$. Then $|\Omega| \equiv 2 \bmod m$.

Remarks.

- (a) is a special case of a result of Aschbacher and Bender. We follow closely the proof given in [6]. Our more restrictive hypothesis allows us to truncate the argument at an early stage.
- Suppose that G is simple. A result of Aschbacher implies that M is strongly embedded in G and then a result of Bender implies that D is a Frobenius complement in M. This implies that D is semiregular on $\Omega - \{\, \Phi, \Phi^g \,\}$ and regular, by conjugation, on $\Phi^{\#}$. Then (c) follows. For our purposes, it is precisely the congruence (c) that we need. A short proof is possible, re-using arguments from the proof of (a).

Recall that if t is an involution in a group X then t is *isolated* in X if t is the only X-conjugate of t that commutes with t.

Suppose that the involution t acts on the group D and that t is isolated in $D\langle t \rangle$, for example if D has odd order. Set

$$I_D(t) = \{\, d \in D \mid d \text{ has odd order and } d^t = d^{-1} \,\}.$$

As t is isolated, any two conjugates of t generate a dihedral group of twice odd order. Hence the maps $C_D(t)x \mapsto [x,t]$ and $u \mapsto C_D(t)u$ are injections from $D/C_D(t)$ into $I_D(t)$ and $I_D(t)$ into $D/C_D(t)$, respectively, so

$$I_D(t) \text{ is a transversal to } C_D(t) \text{ in } D.$$

Hypothesis 5.2.
– *G is a group with subgroups Φ and M.*
– *$\Phi \trianglelefteq M < G$.*
– *Φ has even order.*
– *For all $g \in G$, $\Phi^g \cap M$ has even order implies $g \in M$.*

We adopt the notation:

– *$\mathcal{Z} = \{\, z \in G \mid z$ is an involution that is conjugate to an element of $\Phi \,\}$.*
– *For each $X \subseteq G$ set $\mathcal{Z}_X = \mathcal{Z} \cap X$.*
– *$\Omega = \Phi^G$, the set of conjugates of Φ. We regard G as a permutation group, acting on Ω by conjugation.*
– *For each $X \subseteq G$, set $\Omega_X = \mathrm{Fix}_\Omega(X)$.*
– *Let $m = |\mathcal{Z}_M|$.*
– *For each prime p, m_p is the largest power of p that divides m.*

Trivially we have:

– *Each element of $\mathcal{Z}$ fixes a unique element of Ω.*
– *$\mathcal{Z}_M = \mathcal{Z}_\Phi$.*
– *$C_G(z) \leq M$ for all $z \in \mathcal{Z}_M$.*

Lemma 5.3. *Assume Hypothesis 5.2.*

(a) Any pair of distinct elements of Ω is interchanged by an element of $\mathcal{Z}$.
(b) $\mathcal{Z}$ is a single conjugacy class of 2-central involutions and $\mathcal{Z}_M$ is a single M-conjugacy class.
(c) Let $t \in \mathcal{Z}_{G-M}$ and $z \in \mathcal{Z}_M$. Set $D = M \cap M^t$. Then D is t-invariant and t is isolated in $D\langle\, t\,\rangle$. Moreover, $I_D(t)$ is a transversal to $C_M(z)$ in M and

$$m = |\mathcal{Z}_M| = |I_D(t)| = |D : C_D(z)| = |D : C_D(t)|.$$

In particular, D acts transitively on $\mathcal{Z}_M$.

Proof. We claim:

Let $a, b \in \mathcal{Z}$, suppose ab has even order and that the involution in $\langle\, ab\,\rangle$ is contained in M. Then $\langle\, a, b\,\rangle \leq M$. $(*)$

Indeed, consider the dihedral group $F = \langle\, a, b\,\rangle$ and let u be the involution in $\langle\, ab\,\rangle$. Then $u \in Z(F)$ and au is F-conjugate to a or b. Hence $au \in \Phi^x$ for some $x \in G$ and then $a \in \mathcal{Z} \cap C_G(au) \subseteq \mathcal{Z} \cap M^x \subseteq \Phi^x$. Then $u \in \Phi^x$ so $u \in \mathcal{Z}$. As $u \in M$ we obtain $\langle\, a, b\,\rangle \leq C_G(u) \leq M$.

Let $a \in \mathcal{Z}_M$ and $b \in \mathcal{Z}_{G-M}$. Then ab has odd order and $b = a^c$ for some $c \in \mathcal{Z}_{\langle\, a,b\,\rangle}$. $(**)$

Indeed, if ab has even order then the involution in $\langle\, ab\,\rangle$ centralizes a and so is contained in M. But then $(*)$ contradicts $b \notin M$. To prove the remaining assertion in $(**)$, choose $g \in \langle\, ab\,\rangle$ with $b = a^g$ and put $c = ag$.

Observe that (a) follows from $(**)$. We prove (c). Since $t^2 = 1$, D is t-invariant. Let $d \in D$. Now $tt^d \in D \le M$ and $t \notin M$ so $(*)$ implies tt^d has odd order. Hence t is isolated in $D\langle\, t\,\rangle$.

Let $w \in \mathcal{Z}_M$. By $(**)$, $w^s = z^t$ for some $s \in \mathcal{Z}$. Then $w = z^{ts} \in \Phi^{ts} \cap M$ so $ts \in M$. By $(*)$, ts has odd order. Since t inverts ts, we have $ts \in I_D(t)$. In particular, D is transitive on $\mathcal{Z}_M$ and $M = C_M(z)I_D(t)$.

Let $d, e \in I_D(t)$ and suppose $z^d = z^e$. Now $dt, et \in \mathcal{Z}$ and

$$(dt)(et) = (dt)(et)^{-1} = de^{-1} \in C_G(z) \le M.$$

Now $dt \notin M$ so $(*)$ implies de^{-1} has odd order and is inverted by dt. We have $\langle\, dt, z\,\rangle \le N_G(\langle\, de^{-1}\,\rangle)$. By $(**)$, dt and z are conjugate in $\langle\, dt, z\,\rangle$. Thus z inverts de^{-1} also. As z centralizes de^{-1} we deduce that $d = e$. Thus $u \mapsto C_M(z)u$ is an injection from $I_D(t)$ into $M/C_M(z)$, so as $M = C_M(z)I_D(t)$, $I_D(t)$ is a transversal to $C_M(z)$ in M and to $C_D(z)$ in D. Since t is isolated in $D\langle\, t\,\rangle$, $I_D(t)$ is a transversal to $C_D(t)$ in D. This proves (c).

By (c), $\mathcal{Z}_M$ is a single M-conjugacy class. Then by definition, $\mathcal{Z}$ is a single G-conjugacy class. As z fixes a unique point of Ω, M contains a Sylow 2-subgroup of G. Since t is isolated in $D\langle\, t\,\rangle$ it follows that t centralizes a Sylow 2-subgroup of D. Then $|D : C_D(t)|$ is odd so (c) implies m is odd. Now $m = |M : C_M(z)|$ so z is 2-central. This proves (b).

In order to proceed further, we must use Glauberman's Z^*-Theorem [9, Theorem 7.1, p.131]. Namely, if t is an isolated involution in the group X then $t \in Z^*(X)$, the inverse image of $Z(X/O_{2'}(X))$ in X.

Lemma 5.4. *Suppose t is an isolated involution in the group X. Let p be an odd prime. Then:*

(a) $\mathit{И}^*_X(t,p) \subseteq \mathrm{Syl}_p(X)$, *so X possesses a t-invariant Sylow p-subgroup.*
(b) If $P \in \mathit{И}^*_X(t,p)$ then $C_P(t) \in \mathrm{Syl}_p(C_X(t))$.

Proof. By the Z^*-Theorem, $t \in Z^*(X)$ so $X = C_X(t)O_{2'}(X)$. Let $P \in \mathit{И}^*_X(t,p)$, choose P_0 with $C_P(t) \le P_0 \in \mathrm{Syl}_p(C_X(t))$ and set $Y = P_0O_{2'}(X) \le X$. By Coprime Action, $P = C_P(t)[P,t]$, so as $[P,t] \le O_{2'}(X)$ we have $P \le Y$. Also $|Y|_p = |X|_p$ so $\mathrm{Syl}_p(Y) \subseteq \mathrm{Syl}_p(X)$. Apply Coprime Action to the action of t on the $2'$-group Y.

Lemma 5.5. *Assume Hypothesis 5.2. Let (β,γ) be a pair of distinct elements of Ω that is interchanged by $t \in \mathcal{Z}$. Let p be an odd prime and set $D = G_{\beta\gamma}$. Then:*

(a) D possesses a t-invariant Sylow p-subgroup.
(b) If P is a t-invariant p-subgroup of D then $|I_P(t)| \le m_p$, with equality if $P \in \mathrm{Syl}_p(D)$.

Proof. Lemma 5.3(c) and the Z^*-Theorem imply $t \in Z^*(D\langle t \rangle)$, so (a) follows from Lemma 5.4. To prove (b), choose P^* with $P \le P^* \in \mathcal{U}^*_X(t,p)$. By Lemma 5.4 we have $P^* \in \mathrm{Syl}_p(D)$ and $C_{P^*}(t) \in \mathrm{Syl}_p(C_D(t))$. Using Lemma 5.3(c), $m_p = |D : C_D(t)|_p = |P^* : C_{P^*}(t)| = |I_{P^*}(t)|$. Since $I_P(t) \subseteq I_{P^*}(t)$, (b) follows.

Hypothesis 5.6.
– *Hypothesis 5.2.*
– *Φ is a noncyclic elementary abelian 2-group.*

Lemma 5.7. *Assume Hypothesis 5.6. Let p be an odd prime, P a p-subgroup of G and set $N = N_G(P)$. Suppose that every pair of distinct elements of Ω_P is interchanged by an element of $\mathcal{Z}_N$. Then:*

$$|\Omega_P| \le 2 \quad \textit{or} \quad [P, \mathcal{Z}_N] = 1.$$

Proof. Suppose $s \in \mathcal{Z}_N$ and $[P,s] \ne 1$. Without loss, $s \in \Phi$. If $\Phi \in \Omega_P$ then $P \le N_G(\Phi) = M$ so $[P,s] \le P \cap \Phi = 1$ because Φ is a 2-group. Thus $\Phi \notin \Omega_P$. Recall that s has a unique fixed point on Ω, namely Φ. Then s is fixed point free on Ω_P.

Suppose that $\mathcal{Z} \cap C_N(s) \ne \{ s \}$. Then there exists $A \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ with $s \in A \le N$ and $A^\# \subseteq \mathcal{Z}$. Hence $A \le C_G(s) \le M$ and by Coprime Action(h),

$$P = \langle C_P(a) \mid a \in A^\# \rangle \le M,$$

contradicting $\Phi \notin \Omega_P$. We deduce that

$$\mathcal{Z} \cap C_N(s) = \{ s \}.$$

It follows that s is isolated in N and that $\mathcal{Z}_N = s^N$. Glauberman's Z^*-Theorem implies that $s \in Z^*(N)$. Let $L = \langle s \rangle O_{2'}(N)$. Then $\mathcal{Z}_N \subseteq L$ whence L is transitive on Ω_P. Choose $\alpha \in \Omega_P$. Now $L_\alpha \le O_{2'}(N)$ because s is fixed point free on Ω_P. By hypothesis, each coset of L_α that is not equal to L_α contains an element of $\mathcal{Z}_N$. This forces $L_\alpha = O_{2'}(N)$. Hence $|\Omega_P| = |L : L_\alpha| = 2$.

Lemma 5.8. *Assume Hypothesis 5.6. Let p be an odd prime, P a p-subgroup of G and suppose*

$$|I_P(u)| = m_p \ne 1$$

for some $u \in \mathcal{Z} \cap N_G(P)$. Then $|\Omega_P| \le 2$.

Proof. Assume false and let P be a maximal counterexample. Set $N = N_G(P)$. If u is not isolated in N then the same argument as in the proof of Lemma 5.7 proves that $[P,u] = 1$, contrary to $|I_P(u)| \ne 1$. Hence u is isolated in N.

We claim that any pair (β, γ) of distinct elements of Ω_P is interchanged by an element of $\mathcal{Z}_N$. By Lemma 5.3(a) there exists $t \in \mathcal{Z}$ that interchanges

(β, γ). Using Lemma 5.5(a) and conjugating t by a suitable element of $G_{\beta\gamma}$, there exists Q with

$$P \leq Q = Q^t \in \mathrm{Syl}_p(G_{\beta\gamma}).$$

If $P = Q$ then $t \in N$. Hence we may assume that $P < Q$. Lemma 5.5(b) implies $|I_Q(t)| = m_p$ so the maximal choice of P implies $\Omega_Q = \{\, \beta, \gamma \,\}$. Now $|\Omega| \equiv |\Omega_Q| \equiv 2 \bmod p$ and p is odd so it follows that every p-subgroup of G has at least two fixed points.

Let $A = N_Q(P) > P$ and choose B with $A \leq B \in \mathrm{Syl}_p(N)$. Now u is isolated in N so u normalizes a Sylow p-subgroup of N by Lemma 5.4. Conjugating u by a suitable element of N we may assume $u \in N_G(B)$. Recall that u has a unique fixed point on Ω and that $|\Omega_B| \geq 2$. Hence there exists $\delta \in \Omega_B$ with $\delta \neq \delta^u$. Then B is a u-invariant subgroup of $G_{\delta\delta^u}$. Lemma 5.5(b) implies $|I_B(u)| \leq m_p$. Now $P \leq B$ and $|I_P(u)| = m_p$ so $I_B(u) = I_P(u)$. Also $B = C_B(u)I_B(u)$ so $[B, u] \leq \langle\, I_B(u) \,\rangle \leq P$.

Now $P \leq A \leq B$ so A is u-invariant and $I_A(u) = I_P(u)$. Thus $|I_A(u)| = m_p$. Recall that $A = N_Q(P) > P$. The maximal choice of P forces $\Omega_A = \{\, \beta, \gamma \,\}$. Since u normalizes A and has a unique fixed point, it follows that u interchanges (β, γ). This completes the proof of the claim. Now apply Lemma 5.7.

Lemma 5.9. *Assume Hypothesis 5.6. Let p be a prime such that $m_p \neq 1$, let (α, β) be a pair of distinct elements of Ω that is interchanged by $t \in \mathcal{Z}$ and let P be a t-invariant Sylow p-subgroup of $G_{\alpha\beta}$. Then:*

(a) $\Omega_P = \{\, \alpha, \beta \,\}$ *and* $P \in \mathrm{Syl}_p(G)$.
(b) $N_G(G_{\alpha\beta}) = G_{\alpha\beta}\langle\, t \,\rangle$.

Proof. Lemmas 5.5(b) and 5.8 imply $\Omega_P = \{\, \alpha, \beta \,\}$. Since p is odd, a Sylow p-subgroup of $N_G(P)$ must fix α and β, so $P \in \mathrm{Syl}_p(N_G(P))$ and then $P \in \mathrm{Syl}_p(G)$. This proves (a) and (b) follows from a Frattini Argument.

The following is elementary.

Lemma 5.10 (Bender's Criterion). *Suppose the group X acts on the set Δ, with $|\Delta| > 2$. Let p be an odd prime. Suppose that $\Delta_P = \{\, \alpha, \beta \,\}$ whenever α and β are distinct members of Δ and $P \in \mathrm{Syl}_p(X_{\alpha\beta})$. Then X is 2-transitive on Δ.*

Proof of Theorem 5.1(a). Hypothesis 5.6 is satisfied. Note that $\mathcal{Z}_M = \Phi^\#$ so $m = |\Phi| - 1$. Since Φ is noncyclic there exists a prime p with $m_p \neq 1$. Apply Lemma 5.9 and Bender's Criterion.

Proof of Theorem 5.1(b). It suffices to take an involution $s \in G - M$ and show that s is conjugate into $D\Phi$. Set $E = M \cap M^s$. By Lemma 5.3(a), $M^s = M^t$ for some $t \in \mathcal{Z}$. Using Lemma 5.9(b) we have $s \in N_G(E) = E\langle\, t \,\rangle$. Now t is isolated in $E\langle\, t \,\rangle$ so it is 2-central in $E\langle\, t \,\rangle$. Conjugating s

by a suitable element of E we may suppose that $[s,t]=1$. Choose $\varPhi^* \in \varOmega$ with $t \in \varPhi^*$ and set $M^* = N_G(\varPhi^*)$. Then $s \in C_G(t) \cap Et \subseteq M^* \cap M\varPhi^* = (M^* \cap M)\varPhi^*$. Apply 2-transitivity.

Lemma 5.11. *Assume Hypothesis 5.6, let D be a 2-point stabilizer and suppose that $D = O_{2'}(D) \times O_2(D)$. Let p be a prime with $m_p \neq 1$ and let P be a p-subgroup of G with $|\varOmega_P| \geq 3$. Then*

$$m_p \quad \textit{divides} \quad |D|_p/|P|.$$

Proof. We may suppose that P is maximal subject to $|\varOmega_P| \geq 3$. Set $N = N_G(P)$ and $\varDelta = \varOmega_P$. Let α, β be distinct members of $\varDelta$. Lemma 5.9(a) implies $P \notin \mathrm{Syl}_p(G_{\alpha\beta})$. Thus $P \notin \mathrm{Syl}_p(N_{\alpha\beta})$ and if $Q \in \mathrm{Syl}_p(N_{\alpha\beta})$ then $\varDelta_Q = \{\, \alpha, \beta \,\}$ by the maximality of P. Bender's Criterion implies that N is 2-transitive on $\varDelta$.

Let n and z interchange (α,β) with $n \in N$ a 2-element and $z \in \mathcal{Z}$. Then

$$z \in G_{\alpha\beta} n = (O_{2'}(G_{\alpha\beta}) \times O_2(G_{\alpha\beta}))\, n.$$

Conjugating z by a suitable element of $G_{\alpha\beta}$, we may suppose that $z \in O_2(G_{\alpha\beta})n$. Now $P \leq O_{2'}(G_{\alpha\beta})$ whence $z \in N$. Since α and β were arbitrary, Lemma 5.7 implies $[P, \mathcal{Z}_N] = 1$. In particular, $[P,z]=1$.

By 2-transitivity we may suppose that $D = G_{\alpha\beta}$. Now z interchanges (α,β) so Lemma 5.3(c), with z, G_α and G_β in the roles of t, M and M^t respectively, implies that $m = |D : C_D(z)|$. As $P \leq C_D(z)$, the result follows.

Proof of Theorem 5.1(c). Let p be a prime with $m_p \neq 1$ and let $\varDelta$ be an orbit for the action of D on $\varOmega - \{\, \varPhi, \varPhi^g \,\}$. Choose $\gamma \in \varDelta$, so $|\varDelta| = |D : D_\gamma|$. Let $P \in \mathrm{Syl}_p(D_\gamma)$. Then $|\varOmega_P| \geq 3$ so Lemma 5.11 implies m_p divides $|D|_p/|P| = |\varDelta|_p$. We deduce that m divides $|\varDelta|$ and then that m divides $|\varOmega - \{\, \varPhi, \varPhi^g \,\}|$. The proof is complete.

6. The minimal counterexample

For the remainder of this paper we assume Theorem A to be false and let G be a minimal counterexample. Henceforth we adopt the notation defined in the statement of Theorem A. We may suppose that the field of definition for V is algebraically closed. This section establishes basic properties of G.

Lemma 6.1. *(a) p and r are distinct odd primes.*
(b) r is a Fermat prime, so $r = 2^n + 1$ for some n.
(c) $\pi(|G : C_G(R)|) = \{\, 2, p \,\}$.

Proof. Since R is an r-group and $C_V(R) = 0$ it follows that $p \neq r$. Let $\sigma = \pi(|G : C_G(R)|)$. Applying Lemma 3.2(a,b) to R-invariant Sylow subgroups of G we see that $\sigma \subseteq \{\, 2, p \,\}$ and that $\sigma \subseteq \{\, p \,\}$ if r is not a Fermat prime.

If $\sigma = \emptyset$ then $[G, R] = 1$, contrary to the fact that G is a counterexample to Theorem A. Suppose that $\sigma = \{ q \}$ for some q. Choose $Q \in \mathcal{U}^*_G(R, q)$. Then $G = C_G(R)Q$ and then $[G, R] = [Q, R] \neq 1$. Recall that $[G, R] \trianglelefteq G$. Now $O_p(G) = 1$ by the irreducibility of RG, so $q = 2$ and $p \neq 2$. Lemma 3.2(b) implies that $[G, R]$ is a nonabelian special 2-group, again contrary to the fact that G is a counterexample. We deduce that $\sigma = \{ 2, p \}$, that $p \neq 2$ and that r is a Fermat prime.

Lemma 6.2. *Let Q be an R-invariant abelian 2-subgroup of G. Then $[Q, R] = 1$.*

Proof. By Coprime Action(d) we may assume $Q = [Q, R]$. Apply Lemma 3.2(c).

Lemma 6.3. *Suppose that H is a proper R-invariant subgroup of G. Set $K = [H, R] \trianglelefteq H$. Let $Q \in \mathcal{U}^*_K(R, 2)$. Then:*

(a) $K = QO_p(K)$, in particular, K is a $\{ 2, p \}$-subgroup.
(b) $Q = [Q, R]$. If $Q \neq 1$ then Q is a nonabelian special 2-group and

$$Q' = Z(Q) = \Phi(Q) = C_Q(R).$$

(c) $H = N_H(Q)O_p(H) = (C_H(R) \cap N_H(Q))QO_p(H)$.
(d) If $P \in \mathcal{U}^*_H(R, p)$ then

$$P = N_P(Q)O_p(H) = (C_P(R) \cap N_P(Q))O_p(H)$$

and PQ is a subgroup.

Proof. The minimality of G implies that K induces a 2-group on each RK-composition factor of V. Now $O_p(RK)$ is the largest subgroup of RK that acts trivially on every RK-composition factor of V, so (a) follows. Since $K = [K, R]$ we have $Q = [Q, R]$. Now $p \neq 2$ so (b) follows from Lemma 3.2(b). Also $O_p(K) \leq O_p(H)$ so $QO_p(H) \trianglelefteq H$. A Frattini argument and Coprime Action(d) prove (c) and the first part of (d). Now $O_p(K) \leq P$ so $PQ = PK$, which is a subgroup. The proof is complete.

Lemma 6.4. *(a) $G = [G, R] = O^2(G)$.*
(b) $F(G) = Z(G), [Z(G), R] = 1$ and $Z(G)$ is a cyclic p'-group that acts semiregularly on $V^\#$.
(c) If $P \in \mathcal{U}^*_G(R, p)$ and $S \in \mathcal{U}^*_G(R, 2)$ then $G = \langle\, [P, R], [S, R] \,\rangle$.

Proof. The irreducibility of RG on V implies that $O_p(G) = 1$. Then $O_p([G, R]) = 1$. Suppose $G \neq [G, R]$. Then $[G, R]$ is proper so applying Lemma 6.3 to $[G, R]$ in the role of H, and recalling $[G, R] = [G, R, R]$ by Coprime Action(d), we conclude from Lemma 6.3(a) that $[G, R]$ is a 2-group. But now Lemmas 6.1(b) and 6.3(a,b) contradict the choice of G as a counterexample. Thus $G = [G, R]$. Lemma 3.2(c) yields $[Z(G), R] = 1$ and so the irreducibility of RG implies that $Z(G)$ is a cyclic p'-group that acts semiregularly on $V^\#$.

Suppose, for a contradiction, that $[F(G), R] \neq 1$. Now $O_p(G) = 1$ so Lemma 6.1(c) implies $[O_2(G), R] \neq 1$. Then $O_2(G) \not\leq Z(G)$. Let $H = C_G(O_2(G)) < G$ and $K = [H, R] \trianglelefteq \trianglelefteq G$. Then $O_p(K) \leq O_p(G) = 1$ so Lemma 6.3(a) implies that K is a 2-group. Thus $K \leq O_2(G) \leq C_G(K)$ so K is abelian. Lemma 6.2 implies $[K, R] = 1$. By Coprime Action(d), $K = [K, R]$ whence $[H, R] = 1$. Now $H \trianglelefteq G = [G, R]$ so $H \leq Z(G)$. In particular, H is a p'-group.

Choose $P \in \text{И}^*_G(R, p)$. Lemma 6.1(c) implies that R is faithful on P. As H is a p'-group, RP is faithful on $O_2(G)$. Then [1, (36.4), p.194] implies that $C_V(R) \neq 0$, a contradiction. We deduce that $[F(G), R] = 1$. Since $G = [G, R]$, this forces $F(G) \leq Z(G)$, completing the proof of (b).

Suppose that $O^2(G) < G$. Now $O_p(O^2(G)) = 1$ so Lemma 6.3(a) yields $[O^2(G), R] \leq O_2(O^2(G)) \leq O_2(G)$. Consequently $[O^2(G), R] = [O^2(G), R, R] = 1$ by (b). But then $\pi(|G : C_G(R)|) \subseteq \{2\}$, contrary to Lemma 6.1(c). Thus $O^2(G) = G$, which completes the proof of (a).

Assume the hypothesis of (c) and set $H = \langle [P, R], [S, R] \rangle$. Lemmas 2.2 and 6.1(c) imply that $G = C_G(R)H$. Then $G = [G, R] = [H, R] \leq H$, proving (c).

7. Notation and outline

We define the notation that will be used throughout the remainder of this paper and give an outline of the proof of Theorem A. Let

$$\mathcal{M}_R = \{ M < G \mid M \text{ is maximal subject to being } R\text{-invariant} \}.$$

For each $X < G$ let

$$\mathcal{M}_R(X) = \{ M \in \mathcal{M}_R \mid X \leq M \}.$$

We once and for all fix $S \in \text{И}^*_G(R, 2)$ and set

$$T = [S, R] \trianglelefteq S.$$

Lemma 6.1(c) implies that $T \neq 1$. Lemma 3.2(b) implies that T is a non-abelian special 2-group and that

$$1 \neq T' = Z(T) = \Phi(T) = C_T(R).$$

Set

$$\Phi = \Phi(T).$$

Note that Φ is elementary abelian because T is special. Let

$$\overline{G} = G/Z(G).$$

Lemma 6.4 implies that $F(\overline{G}) = 1$ and that $Z(G)$ is cyclic. Let

$$Z = Z(G) \cap \Phi.$$

Then $Z = 1$ or $Z \cong \mathbb{Z}_2$. In the latter case, Z inverts V.

If $H \in \mathcal{M}_R(T)$ then $H = C_H(R)[H,R]$ and Lemma 6.3(a) implies

$$[H,R] = TO_p([H,R]).$$

We distinguish two cases:

The even case $O_p([H,R]) = 1$ for all $H \in \mathcal{M}_R(T)$.
The odd case $O_p([H,R]) \neq 1$ for some $H \in \mathcal{M}_R(T)$.

The fact that $G = O^2(G)$ implies that considerable G-fusion must take place in S. On the other hand, Lemma 3.2(e) indicates that R causes so much fusion that there is room for no more. Thus we often derive a contradiction by arguing that $G \neq O^2(G)$ or that r divides $|G|$. A good example is Lemma 7.2 below. There follows a more detailed outline of the proof of Theorem A.

The first aim is to establish

- $\mathcal{M}_R(T) = \{ N_G(\Phi) \}$.
- $\overline{\Phi}$ is a TI-subgroup of $\overline{G}$.
- Φ is weakly closed in $N_G(\Phi)$ with respect to G.

The odd case is relatively straightforward. Proving weak closure in the even case is more subtle. Assuming the contrary, we show that G has a section isomorphic to $SL_2(q)$, where $q = |\overline{\Phi}|$. A numerical argument shows that r divides $q^2 - 1$, contrary to the fact that G is an r'-group. A significant difficulty arises if $Z \neq 1$ since there may be elements of G that centralize $\overline{\Phi}$ but not Φ.

Next we set

$$\Omega = \Phi^G, M = N_G(\Phi), G_0 = C_G(R), \Omega_0 = \Phi^{G_0} \text{ and } M_0 = C_M(R).$$

We regard G and G_0 as permutation groups on Ω and Ω_0 respectively. Invoking Theorems 4.1 and 5.1 it follows that G is 2-transitive on Ω. Letting D be a 2-point stabilizer, we obtain

$$|G : M| = 1 + |M : D|. \tag{1}$$

If $|\Omega_0| = 1$ then $C_G(R) \leq M$ and so $|G : M| \equiv 1 \bmod r$. But then (1) implies r divides $|G|$, a contradiction. Thus $|\Omega_0| > 1$ and another application of Theorem 5.1 implies that G_0 is 2-transitive on Ω_0. We obtain

$$|G_0 : M_0| = 1 + |M_0 : D_0|. \tag{2}$$

Manipulations involving (1) and (2) force $q^2 - 1$ to divide $|G|$ and r to divide $q^2 - 1$. In fact, using a result of Burnside, we could pursue the analysis further to show that $G_0 \cong SL_2(q)$. In any event, we have contradicted the fact that G is an r'-group and completed the proof of Theorem A.

Lemma 7.1. *(a) T is weakly closed in $N_G(T)$ with respect to G.*
(b) $N_{\overline{G}}(\overline{T}) = \overline{N_G(T)}$ and $\overline{T}$ is weakly closed in $N_{\overline{G}}(\overline{T})$ with respect to $\overline{G}$.

Proof. Suppose that $T \leq Q \leq S$. Then $[Q, R] \leq [S, R] = T \leq Q$ so Q is R-invariant. Set $H = N_G(Q)$. Then $T \in \mathrm{Syl}_2([H, R])$ by Lemma 6.3(a) so $T = Q \cap [H, R] \trianglelefteq H$ whence $H \leq N_G(T)$. Then (a) follows from Alperin's Fusion Theorem and (b) follows readily.

Lemma 7.2. $\overline{\Phi} \neq 1$.

Proof. Assume false. Then $\Phi = Z \cong \mathbb{Z}_2$. Let $H = N_G(T)$, so that $H = C_H(R)[H, R]$. Now $\Phi \leq Z(G)$ so Corollary 3.3 implies that $[C_H(R), T] = 1$. In particular, as $S = C_S(R)T$ and $T' = Z$ we have $\overline{T} \leq Z(\overline{S})$. Then Lemma 7.1 implies that $\overline{H}$ controls $\overline{G}$-fusion in $\overline{S}$. Now $\overline{G} = O^2(\overline{G})$ by Lemma 6.4 so the Focal Subgroup Theorem yields

$$\overline{H} = O^2(\overline{H}).$$

However, $H = C_H(R)[H, R]$, $[H, R] = TO_p([H, R])$, $[C_H(R), T] = 1$ and $C_H(R) \cap T = Z$. Thus $1 \neq \overline{T}$ is a homomorphic image of $\overline{H}$, contrary to the above.

8. The even case

The following will be proved:

Theorem 8.1. *Assume the even case. Set $M = N_G(\Phi)$. Then $M = N_G(T)$ and*

(a) $\mathcal{M}_R(T) = \{ M \}$.
(b) Φ is weakly closed in M with respect to G.
(c) $\overline{\Phi}$ is an elementary abelian TI-subgroup of $\overline{G}$.

Throughout this section, we assume the even case and set

$$M = N_G(\Phi).$$

Note that $M < G$ by Lemmas 6.4(b) and 7.2.

Lemma 8.2. *(a) $M = N_G(T) = C_M(R)T$ and $[M, R] = T$.*
(b) $\mathcal{M}_R(T) = \{ M \}$.
(c) $\overline{\Phi}$ is an elementary abelian TI-subgroup of $\overline{G}$.

Proof. Let $H \in \mathcal{M}_R(T)$. Since we are in the even case, $O_p([H, R]) = 1$. Lemma 6.3 implies $[H, R] = T \trianglelefteq H$. Then $H \leq M$ so $H = M$. This proves (a) and (b). To prove (c), let $g \in G$ and suppose that $\Phi \cap \Phi^g \not\leq Z$. Recall that $[RT, \Phi] = 1$ so using (b) we have

$$T^g \leq C_G(\Phi \cap \Phi^g) \leq M = N_G(T).$$

Lemma 7.1 forces $T^g = T$ so $\Phi^g = \Phi$. The proof is complete.

Recall that $\overline{G} = G/Z(G)$. Elements of G that centralize $\overline{\Phi}$ but not Φ require special handling. To this end, we define

$$N = C_G(\overline{\Phi}) \quad \text{and} \quad Q = [T, N].$$

Now $T \leq N$ so Lemma 8.2(a) implies $N \leq M$. Then $N \trianglelefteq M$ and so $\Phi = T' \leq Q \trianglelefteq M$. Hence $M = N_G(Q)$.

Also recall that $Z = \Phi \cap Z(G)$ and that if $Z \neq 1$ then $Z \cong \mathbb{Z}_2$ and Z inverts V.

Lemma 8.3. *Suppose that* $[\Phi, N] \neq 1$. *Then:*

(a) If X is a 2-subgroup of M or if $X \leq N$ then X stabilizes the chain

$$\Phi \leq Q \leq T.$$

(b) If $X \leq N$ and $[\Phi, X] \neq 1$ then $[\Phi, X] = Z$ and $Q = [T, X]\Phi$.
(c) $[C_G(\overline{Q}), R] \leq Q$.

Proof. Since $N \leq N_G(\Phi)$ we obtain $[\Phi, N] = Z \cong \mathbb{Z}_2$. This proves the first assertion of (b). Let $\widetilde{T} = T/\Phi$. Consider the action of RM on $\widetilde{T}$. Note that T acts trivially because $\Phi = T'$. Moreover, $M = C_M(R)T$ so it follows that $[\widetilde{T}, X]$ is R-invariant for any $X \leq M$.

Now $N = C_N(R)T$ hence $[\Phi, C_N(R)] = Z$. Also $\widetilde{Q} = [\widetilde{T}, N] = [\widetilde{T}, C_N(R)]$. Lemma 3.5 implies that

$$1 < \widetilde{Q} < \widetilde{T}, \tag{$*$}$$

$[\widetilde{Q}, N] = 1$, $\widetilde{Q} = C_{\widetilde{T}}(N)$ and that R acts irreducibly on both factors of $(*)$.

(a). Since Q is the full inverse image of $\widetilde{Q}$ it suffices to show that X stabilizes $(*)$. If $X \leq N$, this is clear. Suppose that X is a 2-subgroup of M. As $M = TC_M(R)$ by Lemma 8.2, and as R is irreducible on both factors in $(*)$, X is trivial on these factors and $[\widetilde{T}, X] = 1$ or $\widetilde{Q}$.

(b). As just observed, either $[\widetilde{T}, X] = 1$ or $[\widetilde{T}, X] = \widetilde{Q}$. Suppose that $[\widetilde{T}, X] = 1$. Then $[T, X] \leq \Phi = Z(T)$ so T stabilizes $1 \leq \Phi \leq \Phi X$. This forces $[\Phi X, T'] = 1$. But $T' = \Phi$ and $[\Phi, X] \neq 1$. We deduce that $[\widetilde{T}, X] = \widetilde{Q}$. Then (b) follows by taking inverse images.

(c). Let $L = [C_G(\overline{Q}), R]$. Since $T \leq N_G(\overline{Q})$, Lemma 8.2 implies $C_G(\overline{Q}) \leq M$. Thus $L \leq [M, R] = T$. Now R is irreducible on $\widetilde{Q} = [\widetilde{T}, N]$ so either $[\widetilde{L}, N] = 1$ or $[\widetilde{L}, N] = \widetilde{Q}$. If $[\widetilde{L}, N] = 1$ then as $\widetilde{Q} = C_{\widetilde{T}}(N)$ we have $\widetilde{L} \leq \widetilde{Q}$, which proves (c) in this case.

Suppose, for a contradiction, that $[\widetilde{L}, N] = \widetilde{Q}$. Observe that $[Q, L, N] \leq [Z(G), N] = 1$. Using (a) and $L \leq T$ we have $[N, Q, L] \leq [\Phi, L] = 1$. The Three Subgroups Lemma forces $[L, N, Q] = 1$. Since $[\widetilde{L}, N] = \widetilde{Q}$ we have $[L, N]\Phi = Q$. Then $[Q, Q] = 1$. Lemma 6.2 implies that $[Q, R] = 1$. Then $Q \leq C_T(R) = \Phi$ so $\widetilde{Q} = 1$. This contradicts $(*)$ and completes the proof of (c).

Lemma 8.4. *Suppose* $[\Phi, N] \neq 1$. *Then* Q *is weakly closed in* M *with respect to* G.

Proof. Assume false. By Lemma 2.5, there exists $x \in G$ such that Q and Q^x normalize one another but $Q \neq Q^x$. Then $Q^x \leq N_G(Q) = M$. Lemma 8.3(a) implies that $[T, Q^x] \leq Q$, so $T \leq N_G(QQ^x)$, and also that $[Q, Q^x] \leq \Phi$. By symmetry,

$$\langle T, T^x \rangle \leq N_G(QQ^x) \quad \text{and} \quad [Q, Q^x] \leq \Phi \cap \Phi^x.$$

Now $x \notin M$ so $\Phi \cap \Phi^x \leq Z(G)$, by Lemma 8.2(c). In particular, $Q^x \leq C_G(\overline{Q})$, so Lemma 8.3(c) yields

$$[Q^x, R] \leq Q.$$

Also, Q is R-invariant so we deduce that QQ^x is R-invariant. But $\mathcal{M}_R(T) = \{ M \}$ so we obtain

$$T^x \leq N_G(QQ^x) \leq M = N_G(T).$$

This contradicts the weak closure of T, Lemma 7.1. The proof is complete.

Lemma 8.5. *Let* $x \in G$ *and suppose that*

$$[\overline{\Phi}, \overline{\Phi}^x] = 1.$$

Then $[\Phi, \Phi^x] = 1$.

Proof. Assume false. Now $\Phi^x \leq N$ so $[\Phi, N] \neq 1$. Also, $M = N_G(\Phi)$ so $x \notin M$. Lemma 8.3(b) and the fact that $\overline{\Phi}$ is TI in $\overline{G}$ imply that $Z = [\Phi, \Phi^x] \leq \Phi \cap \Phi^x \leq \Phi \cap Z(G) = Z$. Then $\mathbb{Z}_2 \cong Z = \Phi \cap \Phi^x$ and $\Phi\Phi^x$ is a subgroup.

Consider the chain

$$\Phi < \Phi\Phi^x. \tag{$*$}$$

By Lemma 8.3(a), $[Q, \Phi^x] \leq \Phi$ so Q stabilizes $(*)$. Also Φ^x stabilizes $(*)$ because $[\Phi, \Phi^x] = Z \leq \Phi$. Lemma 8.3(b) implies that $Q = [T, \Phi^x]\Phi$ so $\langle \Phi^{xT} \rangle$ stabilizes $(*)$. Moreover,

$$Q \leq \langle \Phi^{xT} \rangle \Phi. \tag{$**$}$$

We claim that $\overline{\langle \Phi^{xT} \rangle}$ is abelian. Indeed, let $t \in T$ and consider the action of Φ^{xt} on $(*)$. Now $\Phi\Phi^x$ is nonabelian so it possesses a nonidentity element that does not have order two. However, Φ is elementary, whence

$$\Phi \cup \bigcup_{g \in \Phi^{xt}} \Phi^{xg} \quad \subsetneq \quad \Phi\Phi^x.$$

Now $\overline{\Phi}$ is TI in $\overline{G}$ so the intersection of distinct conjugates of Φ is equal to Z. Thus the left hand side has cardinality $|\Phi| + |\Phi^{xt} : N_{\Phi^{xt}}(\Phi^x)|(|\Phi^x| - 2)$

and the right hand side has cardinality $\frac{1}{2}|\Phi|^2$. Thus $|N_{\Phi^{xt}}(\Phi^x)| > 2$ and we may choose $h \in N_{\Phi^{xt}}(\Phi^x)$ with $h \notin Z(G)$.

Since Φ^{xt} stabilizes $(*)$ we have $[h, \Phi^x] \leq \Phi \cap \Phi^x = Z$ so as $\overline{\Phi}^{xt}$ is TI in $\overline{G}$ we obtain $\Phi^x \leq N_G(\Phi^{xt})$. Then $[\Phi^x, \Phi^{xt}] \leq \Phi \cap \Phi^{xt} = Z$. As t was arbitrary, the claim is established.

The claim implies that $\overline{\langle \Phi^{xT} \rangle \Phi}$ is abelian. In particular, $\langle \Phi^{xT} \rangle \Phi \leq N^x$. From $(**)$ we have $Q \leq N^x \leq M^x$. As $x \notin M = N_G(Q)$, we have contradicted the weak closure of Q, Lemma 8.4. The proof is complete.

Corollary 8.6. *Let $x \in G$ and suppose that*

$$\overline{\langle \Phi^{xR} \rangle}$$

is abelian. Then $[\Phi^x, R] = 1$.

Proof. Every RG-conjugate of Φ is in fact a G-conjugate because $[\Phi, R] = 1$. Lemma 8.5 implies that $\langle \Phi^{xR} \rangle$ is an R-invariant abelian 2-group. Apply Lemma 6.2.

Lemma 8.7. *Φ is weakly closed in M with respect to G.*

Proof. Assume false. Then there exists $x \in G$ such that Φ and Φ^x normalize one another but $\Phi \neq \Phi^x$. Then $T \neq T^x$. Moreover, if $[\Phi, N] \neq 1$ then $Q \neq Q^x$ because $N_G(Q) = M$.

Since $\overline{\Phi}$ is TI in $\overline{G}$ we have $[\overline{\Phi}, \overline{\Phi^x}] = 1$. Lemma 8.5 implies that $[\Phi, \Phi^x] = 1$. Now $C_G(\Phi) = (C_G(R) \cap C_G(\Phi))T$. Corollary 3.3 implies $[C_G(R) \cap C_G(\Phi), T] = 1$, whence

$$C_G(\Phi) = (C_G(R) \cap C_G(T)) * T. \qquad (*)$$

In particular, $[T, \Phi^x] \leq T' = \Phi$.

Let $X = \Phi\Phi^x$, so then

$$\overline{X} = \overline{\Phi} \times \overline{\Phi^x},$$

and T normalizes X. In fact, T stabilizes the chain $1 < \Phi < X$. By symmetry, $T^x \leq N_G(X)$ and T^x stabilizes $1 < \Phi^x < X$. Set

$$L = \langle T, T^x \rangle \leq N_G(X) \quad \text{and} \quad \widetilde{L} = L/C_L(\overline{X}).$$

Since $\mathcal{M}_R(T) = \{ M \}$, the weak closure of T implies that L is not contained in any proper R-invariant subgroup of G.

Claim 1. *$C_{\overline{L}}(\overline{X})$ is an abelian 2-group.*

Proof. Suppose that $C_L(\overline{X}) > C_L(X)$. Interchanging Φ and Φ^x if necessary, we may assume $[\Phi, C_L(\overline{X})] \neq 1$. Then $C_L(\overline{X}) \leq N$ and $[\Phi, N] \neq 1$. Lemma 8.3(b) yields

$$Q = [T, C_L(\overline{X})]\Phi \leq C_L(\overline{X}) \leq C_G(\overline{\Phi^x}) \leq M^x.$$

The weak closure of Q, Lemma 8.4, forces $Q = Q^x$. This is not the case so we deduce that $C_L(\overline{X}) = C_L(X)$.

Let $I = \langle c^2 \mid c \in C_L(X) \rangle$. Since $[\Phi(T), R] = 1$ it follows from $(*)$ that $I \leq C_G(R) \cap C_G(T)$. Similarly, $I \leq C_G(T^x)$. Thus I is R-invariant and $L \leq C_G(I)$. Since L is not contained in any proper R-invariant subgroup, we have $I \leq Z(G)$. The claim follows.

Set

$$q = |\overline{\Phi}|,$$

so $q > 1$ by Lemma 7.2.

Claim 2. $|\widetilde{T}| \geq q$.

Proof. If $\overline{\langle \Phi^{xR} \rangle}$ is abelian then Corollary 8.6 forces $[\Phi^x, R] = 1$ whence L is contained in the R-invariant subgroup $N_G(X)$. This is not the case. We deduce that $\overline{\langle \Phi^{xR} \rangle}$ is nonabelian.

Choose $\rho \in R$ such that $[\overline{\Phi^x}, \overline{\Phi^{x\rho}}] \neq 1$. Now $\Phi^x \leq M$ and $[M, R] = T$ so $T\Phi^x$ is R-invariant. As $\Phi^x \leq C_L(X)$ we obtain $\widetilde{\Phi^{x\rho}} \leq \widetilde{T\Phi^x} = \widetilde{T}$. Thus it suffices to show that $|\widetilde{\Phi^{x\rho}}| \geq q$. Note that $\Phi^{x\rho}$ stabilizes

$$1 < \Phi < X$$

because T does.

Let $h \in \Phi^{x\rho} \cap C_L(X)$ and suppose that $h \notin Z(G)$. Then $X \leq C_G(h) \leq N_G(\Phi^{x\rho})$ and so $[X, \Phi^{x\rho}] \leq \Phi \cap \Phi^{x\rho} \leq Z(G)$. But $\Phi^x \leq X$ so we have contradicted $[\overline{\Phi^x}, \overline{\Phi^{x\rho}}] \neq 1$. Thus $\Phi^{x\rho} \cap C_L(X) \leq Z(G)$ and as $q = |\overline{\Phi}| = |\Phi/\Phi \cap Z(G)|$, the claim follows.

We also have $|\widetilde{T^x}| \geq q$. Theorem 2.8 implies that $q^2 - 1$ divides $|\widetilde{L}|$, that $|\widetilde{T}| = q$ and that $N_{\widetilde{L}}(\widetilde{T})$ acts transitively on $\widetilde{T}^{\#}$. (In fact, $\widetilde{L} \cong SL_2(q)$.) Let

$$D = N_L(T) \quad \text{and} \quad A = C_T(\overline{X}).$$

The weak closure of T implies that $\widetilde{D} = N_{\widetilde{L}}(\widetilde{T})$. Then D normalizes A and acts transitively on $(T/A)^{\#}$. Note that $T \leq D \leq M$ and $[M, R] = T$ so D is R-invariant. If A is R-invariant then R acts on $(T/A)^{\#}$ so as D is transitive on $(T/A)^{\#}$, Lemma 2.12 implies that R has a fixed point on $(T/A)^{\#}$. This is absurd because $C_T(R) = \Phi \leq A$. We deduce that A is not R-invariant.

Let ρ be a generator for R and set $B = A^\rho \neq A$. Now B is D-invariant because D is R-invariant. The transitivity of D on $(T/A)^\#$ forces $T = AB$. Recall that $\overline{G} = G/Z(G)$ and that $[Z(G), R] = 1$. Then $T \cap Z(G) = T \cap Z(G) \cap C_G(R) = \Phi \cap Z(G) = Z$. Now $\overline{\Phi} \neq 1$ so Lemma 3.6 implies $Z(\overline{T}) = \overline{\Phi}$. Also, $\overline{A}$ and $\overline{B}$ are abelian by Claim 1 so as $\overline{T} = \overline{AB}$ we have $\overline{A} \cap \overline{B} \leq \overline{\Phi}$. Since $\Phi \leq A \cap B$ we deduce that

$$A \cap B = \Phi.$$

Now $q = |\widetilde{T}| = |T/A| = |AB/A| = |A : \Phi|$. Similarly, $q = |B : \Phi|$. Then

$$|T : \Phi| = |A : \Phi||B : \Phi| = q^2.$$

Since $R \cong \mathbb{Z}_r$ and $C_T(R) = \Phi$ it follows that r divides $q^2 - 1$. But $q^2 - 1$ divides $|\widetilde{L}|$ so we have contradicted the fact that G is an r'-group. The proof of the weak closure of Φ is complete.

Theorem 8.1 now follows form Lemmas 8.2 and 8.7.

9. The odd case

The following will be proved:

Theorem 9.1. *Assume the odd case. Set $M = N_G(\Phi)$. Then:*

(a) $\mathcal{M}_R(T) = \{\, M \,\}$.
(b) Φ is weakly closed in M with respect to G.
(c) $\overline{\Phi}$ is an elementary abelian TI-subgroup of $\overline{G}$.

Throughout this section, we assume the odd case.

Lemma 9.2. *One of the following holds:*

(a) $m(\overline{\Phi}) \geq 3$.
(b) $m(\overline{\Phi}) = 2$, $p = 3$ *and if* $P \in \mathcal{V}_G(R, p)$ *is chosen maximal subject to* $TP = PT \neq G$ *then* $C_G(R) \cap N_P(T)$ *acts transitively on* $\overline{\Phi}^\#$.

Proof. Choose $P \in \mathcal{V}_G(R, p)$ maximal subject to $TP = PT \neq G$. Then $P \neq 1$ because we are in the odd case. Set $H = PT$ and $P_1 = C_P(R) \cap N_P(T)$. Lemma 6.3(d) yields

$$P = P_1 O_p(H).$$

Recall that $p \neq 2$. Consider the action of P_1 on $\overline{\Phi}$ and suppose that (a) and (b) fail. Then $[\overline{\Phi}, P_1] = 1$ whence $[\Phi, P_1] = 1$. Corollary 3.3 implies that $[T, P_1] = 1$. We deduce that $T \leq N_G(P)$. As $O_p(G) = 1$, $G \neq N_G(P)$. Lemma 6.3(d) and the maximal choice of P imply that $P \in \mathrm{Syl}_p(N_G(P))$. Then $P \in \mathrm{Syl}_p(G)$. But now Lemma 6.4(c) implies that $G = \langle\, P, T \,\rangle$, contradicting $O_p(G) = 1$. Thus one of (a) or (b) holds.

Let
$$\mathcal{L} = \{\, P \in \mathit{\Lambda\!\!/}_G(R,p) \mid TC_G(T) \leq N_G(P) \,\}$$
and let
$$\mathcal{L}^*$$
be the set of maximal members of $\mathcal{L}$ under inclusion.

Lemma 9.3. *Let $H \in \mathcal{M}_R(TC_G(T))$. Then every member of $\mathcal{L}$ that is contained in H is contained in $O_p(H)$. Moreover, if $O_p(H) \neq 1$ then $O_p(H) \in \mathcal{L}^*$.*

Proof. Let $K = [H,R]$ so $K = TO_p(K) \trianglelefteq H$ by Lemma 6.3(a). By the Frattini Argument, $H = N_H(T)O_p(K)$ whence
$$O_p(C_G(T))O_p(K) \leq O_p(H).$$
Choose $P \in \mathcal{L}$ with $P \leq H$. Now $P = C_P(T)[P,T]$. We have $C_P(T) \leq O_p(C_G(T))$ because P is $TC_G(T)$-invariant. Also $[P,T] \leq P \cap K \leq O_p(K)$. Hence $P \leq O_p(H)$. Now suppose $O_p(H) \neq 1$. Choose P with $O_p(H) \leq P \in \mathcal{L}^*$. Then $N_P(O_p(H))$ is a member of $\mathcal{L}$ contained in H. Hence $N_P(O_p(H)) \leq O_p(H)$ so $P = O_p(H)$ and the proof is complete.

Corollary 9.4. *If $P, Q \in \mathcal{L}^*$ and $P \cap Q \neq 1$ then $P = Q$.*

Proof. Assume false and consider a counterexample with $I = P \cap Q$ maximal. Then $I < N_P(I) \in \mathcal{L}$ and $I < N_Q(I) \in \mathcal{L}$. Choose H with $N_G(I) \leq H \in \mathcal{M}_R(TC_G(T))$. Lemma 9.3 implies that
$$I < N_P(I) \leq P \cap O_p(H) \quad \text{and} \quad O_p(H) \in \mathcal{L}^*.$$
The maximality of I forces $P = O_p(H)$. Similarly, $Q = O_p(H)$, so $P = Q$, a contradiction.

Lemma 9.5. *$\mathcal{L}$ contains nontrivial members.*

Proof. Since we are in the odd case, $O_p([H,R]) \neq 1$ for some $H \in \mathcal{M}_R(T)$. Set $K = [H,R]$. Now $K = TO_p(K)$ and $K = [K,R]$ so $[O_p(K),R] \neq 1$. By Lemma 9.2, $m(\overline{\Phi}) \geq 2$ so
$$O_p(K) = \langle\, O_p(K) \cap C_G(a) \mid a \in \Phi - Z(G) \,\rangle.$$
Thus there exists $a \in \Phi - Z(G)$ with
$$1 \neq [O_p(K) \cap C_G(a), R].$$
Since $[C_G(a),R] = TO_p([C_G(a),R])$ we have $1 \neq O_p([C_G(a),R]) \leq O_p(C_G(a)) \in \mathcal{L}$.

Lemma 9.6. *$\mathcal{L}$ possesses a unique maximal member.*

Proof. Choose $P, Q \in \mathcal{L}^*$. We must show that $P = Q$. Let Φ_0 be a complement to Z in Φ. Note that $C_G(a)$ is contained in a member of $\mathcal{M}_R(TC_G(T))$ for all $a \in \Phi_0^\#$. By Lemma 9.2, $m(\Phi_0) \geq 2$.

Suppose that $m(\Phi_0) \geq 3$. By Coprime Action there is a hyperplane Φ_1 of Φ_0 with $C_P(\Phi_1) \neq 1$. Now $m(\Phi_1) \geq 2$ so $C_Q(a) \neq 1$ for some $a \in \Phi_1^\#$. Choose H with $C_G(a) \leq H \in \mathcal{M}_R(TC_G(T))$. Now $C_P(a)$ and $C_Q(a)$ are nontrivial members of $\mathcal{L}$ that are contained in H. Lemma 9.3 and Corollary 9.4 imply $P = Q$. Hence we may suppose that $m(\Phi_0) = 2$.

Choose $\widetilde{P}$ maximal subject to $P \leq \widetilde{P} \in \mathcal{V}_G(R, p)$ and $\widetilde{P}T = T\widetilde{P} \neq G$. Set $\widetilde{P}_0 = C_G(R) \cap N_{\widetilde{P}}(T)$. By Maschke's Theorem we may choose Φ_0 so that it is $\widetilde{P}_0$-invariant. Lemma 9.2 implies that $\widetilde{P}_0$ is transitive on $\Phi_0^\#$.

Now $\widetilde{P}_0$ normalizes both $TC_G(T)$ and R hence $\langle P^{\widetilde{P}_0} \rangle$ is $RTC_G(T)$-invariant. Since $\langle P^{\widetilde{P}_0} \rangle$ is a p-group it follows that $\langle P^{\widetilde{P}_0} \rangle \in \mathcal{L}$. Now $P \in \mathcal{L}^*$ so we deduce that P is $\widetilde{P}_0$-invariant. Recall that $m(\Phi_0) = 2$. Hence $C_P(a) \neq 1$ for some $a \in \Phi_0^\#$. As P is $\widetilde{P}_0$-invariant and $\widetilde{P}_0$ is transitive on $\Phi_0^\#$, we have $C_P(a) \neq 1$ for all $a \in \Phi_0^\#$. Now choose $a \in \Phi_0^\#$ with $C_Q(a) \neq 1$. As in the case $m(\Phi_0) \geq 3$, it follows that $P = Q$. The proof is complete.

For the remainder of this section we fix P such that

$$\mathcal{L}^* = \{ P \}.$$

Lemma 9.5 implies $P \neq 1$. Set

$$M = N_G(P).$$

Lemma 9.3 implies $P = O_p(M)$.

Theorem 9.7. *$M = N_G(T)P$ and $\mathcal{M}_R(T) = \{ M \}$.*

Proof. Let $N = N_G(T)$. By Lemma 6.3(c), $N = C_N(R)TO_p(N)$. Now $TO_p(N) \leq M$ because $TC_G(T) \leq M$. Also, $C_N(R)$ normalizes both R and $TC_G(T)$ so $C_N(R)$ permutes $\mathcal{L}^* = \{ P \}$. Thus $N \leq M$. By Lemma 6.3(c), $M = NP$.

Now suppose that $H \in \mathcal{M}_R(T)$. By Lemma 6.3(c), $H = N_H(T)O_p(H)$. If $C_G(T) \leq H$ then $O_p(H) \in \mathcal{L}$ whence $O_p(H) \leq P$ and $H \leq M$. In particular, $C_G(a) \leq M$ for all $a \in \Phi - Z(G)$. Returning to arbitrary $H \in \mathcal{M}_R(T)$ we have

$$O_p(H) = \langle C_{O_p(H)}(a) \mid a \in \Phi - Z(G) \rangle \leq M$$

because $m(\overline{\Phi}) \geq 2$. Hence $H \leq M$. We deduce that $\mathcal{M}_R(T) = \{ M \}$.

Lemma 9.8. *Let $g \in G$ and suppose that $m(\overline{\Phi^g \cap M}) \geq 2$. Then $g \in M$.*

Proof. Now $PT \trianglelefteq M$ so $P\Phi \trianglelefteq M$ and we may conjugate Φ^g by a suitable member of P to suppose that $\Phi^g \cap M$ normalizes Φ. Consider the action of $\overline{\Phi^g \cap M}$ on $\overline{\Phi}$ and recall that $m(\overline{\Phi}) \geq 2$. If $m(\overline{\Phi}) = 2$ then as $m(\overline{\Phi^g \cap M}) \geq 2$, there exists $\overline{a} \in \overline{\Phi^g \cap M}^{\#}$ with $[\overline{a}, \overline{\Phi}] = 1$. Theorem 9.7 implies that $C_{\overline{G}}(\overline{a}) \leq \overline{M}^{\overline{g}}$. Then $\overline{\Phi} \leq \overline{M}^{\overline{g}}$. In particular, $m(\overline{\Phi \cap M^g}) \geq 2$. Suppose that $m(\overline{\Phi}) \geq 3$. Choose $\overline{a} \in \overline{\Phi^g \cap M}^{\#}$. Since $\overline{a}$ induces an involution on the elementary abelian 2-group $\overline{\Phi}$ we have $m(C_{\overline{\Phi}}(\overline{a})) \geq \frac{1}{2}m(\overline{\Phi}) \geq \frac{3}{2}$. Thus $m(C_{\overline{\Phi}}(\overline{a})) \geq 2$. As before, $C_{\overline{G}}(\overline{a}) \leq \overline{M}^{\overline{g}}$ so $m(\overline{\Phi \cap M^g}) \geq 2$ in this case also. Thus we have the symmetrical configuration

$$m(\overline{\Phi^g \cap M}) \geq 2 \quad \text{and} \quad m(\overline{\Phi \cap M^g}) \geq 2.$$

Using Coprime Action and Theorem 9.7 we have

$$P^g = \langle\, C_{P^g}(a) \mid a \in \Phi \cap M^g - Z(G) \,\rangle \leq M.$$

Let

$$D = (\Phi^g \cap M)P^g TP.$$

Note that D is a soluble $\{\,2, p\,\}$-subgroup of M because $\Phi^g \cap M$ normalizes P^g and $TP \trianglelefteq M$. Let $a \in \Phi^g \cap M - Z(G)$. Now $C_G(a) \leq M^g$ so using Lemma 2.6 we obtain

$$C_{P^g}(a) \leq O_{2'}(C_G(a)) \cap D \leq O_{2'}(C_D(a)) \leq O_{2'}(D) = O_p(D).$$

Since $m(\overline{\Phi^g \cap M}) \geq 2$ we have

$$P^g = \langle\, C_{P^g}(a) \mid a \in \Phi^g \cap M - Z(G) \,\rangle \leq O_p(D).$$

Note that $P \trianglelefteq D$ because $D \leq M$, whence

$$O_p(D) = PP^g$$

and $T \leq N_G(PP^g)$.

Now $[D, R] \leq [M, R] \leq TO_p(M) = TP \leq D$ so D and hence $O_p(D)$, are R-invariant. Theorem 9.7 forces $N_G(PP^g) \leq M$. By symmetry, $T^g \leq N_G(PP^g)$ so $T^g \leq M$. Using the weak closure of T, Lemma 7.1, there exists $m \in M$ with $T^{gm} = T$. Then $gm \in N_G(T) \leq M$ so $g \in M$, completing the proof.

Corollary 9.9. *Assume* $m(\overline{\Phi}) \geq 3$. *Let* $g \in G$ *and suppose that* $\overline{\Phi^g \cap M} \neq 1$. *Then* $g \in M$.

Proof. As in the proof of Lemma 9.8 we may suppose that $\Phi^g \cap M$ normalizes Φ. Choose $\overline{a} \in \overline{\Phi^g \cap M}^{\#}$. As in Lemma 9.8, $m(C_{\overline{\Phi}}(\overline{a})) \geq 2$, $C_{\overline{G}}(\overline{a}) \leq \overline{M}^{\overline{g}}$ and so $m(\overline{\Phi \cap M^g}) \geq 2$. Lemma 9.8 implies $g \in M$.

Lemma 9.10. $N_{\overline{M}}(\overline{\Phi})$ *is transitive on* $\overline{\Phi}^{\#}$.

Proof. By Lemma 9.2 we may suppose that $m(\overline{\Phi}) \geq 3$. Now $TP \trianglelefteq M$ so $\Phi P \trianglelefteq M$. Corollary 9.9 implies that Hypothesis 5.2 is satisfied with $\overline{\Phi P}$ in the role of Φ. Lemma 5.3(b) implies that $\overline{M}$ acts transitively on the involutions of $\overline{\Phi P}$. Since $\overline{M} = N_{\overline{M}}(\overline{\Phi})\overline{P}$, the conclusion follows.

Lemma 9.11. *$\Phi \trianglelefteq M$, so that $M = N_G(\Phi)$.*

Proof. Assume false. We will argue that M contains a Sylow p-subgroup of G, contrary to Lemma 6.4(c). Note that $\Phi P \trianglelefteq M$. By Lemma 9.10, either $C_\Phi(P) = Z$ or $C_\Phi(P) = \Phi$. In the latter case, $\Phi = O_2(\Phi P) \trianglelefteq M$. Hence we have $C_\Phi(P) = Z$. Choose $P^* \in \mathsf{N}^*_M(R, p)$ and set

$$H = TP^* \quad \text{and} \quad L = \Phi P^*.$$

Both H and L are subgroups because $P = O_p(M) \leq P^*$, $TP \trianglelefteq M$ and $\Phi P \trianglelefteq M$.

We claim that $O_p(H) = O_p(L)$. Clearly $O_p(H) \leq O_p(L)$. Let $Y = TO_p(L) \subseteq H$. Then Y is a subgroup because $O_p(L)$ normalizes TP and $P \leq O_p(L)$. By Lemma 6.3(d), with Y in the role of H,

$$O_p(L) = (C_G(R) \cap N_{O_p(L)}(T))O_p(Y).$$

Now $[N_{O_p(L)}(T), \Phi] \leq \Phi \cap O_p(L) = 1$ so Corollary 3.3 implies that $[C_G(R) \cap N_{O_p(L)}(T), T] = 1$. Then $T \leq N_G(O_p(L))$ so $O_p(L) \leq O_p(H)$ and the claim is established.

We apply Theorem 2.7, to L. Now $P \leq O_p(L)$ and $C_\Phi(P) = Z$ whence $C_L(O_p(L)) \leq O_p(L)Z$. Note that L has abelian Sylow 2-subgroups. Theorem 2.7 yields $1 \neq K(P^*) = K(O_p(L))$. Actually, we must consider L/Z, but this causes no difficulty because $Z \leq Z(G)$.

Since $O_p(L) = O_p(H)$ we have $K(P^*) = K(O_p(H)) \trianglelefteq H$, whence $T \leq N_G(K(P^*))$. Theorem 9.7 implies $N_G(K(P^*)) \leq M$. In particular, $N_G(P^*) \leq M$. But $P^* \in \mathsf{N}^*_M(R, p)$ so this forces $P^* \in \mathrm{Syl}_p(G)$ and then Lemma 6.4(c) supplies a contradiction. The proof is complete.

Proof of Theorem 9.1. (a) follows from Lemma 9.11 and Theorem 9.7. Now $m(\overline{\Phi}) \geq 2$ by Lemma 9.2 so (b) follows from Lemma 9.8. To prove (c), let $g \in G$ and suppose that $\Phi \cap \Phi^g \not\leq Z(G)$. Now $[\Phi, RT] = 1$ so using (a) we have $\Phi^g \leq C_G(\Phi \cap \Phi^g) \leq M$. Then $\Phi^g = \Phi$ by (b).

10. The final contradiction

Henceforth we adopt the following notation:

$$\Omega = \Phi^G, M = N_G(\Phi), G_0 = C_G(R), \Omega_0 = \Phi^{G_0} \text{ and } M_0 = C_M(R).$$

We regard G and G_0 as permutation groups on Ω and Ω_0 respectively. Recall that

$$\overline{G} = G/Z(G)$$

that $\overline{\Phi} \neq 1$ by Lemma 7.2, and that $\Phi \leq G_0$.

Lemma 10.1. *(a)* $\mathcal{M}_R(T) = \{ M \}$.
(b) $\overline{\Phi}$ *is a* TI*-subgroup.*
(c) Φ *is weakly closed in* M *with respect to* G.
(d) For all $g \in G$, $\overline{\Phi}^g \cap \overline{M} \neq 1$ *implies* $g \in M$.
(e) If H *is an* R*-invariant subgroup of* G *and* $\overline{\Phi} \cap \overline{H} \neq 1$ *then* M *contains a member of* $\mathcal{V}^*_H(R,2)$.
(f) $\overline{M}$ *controls* $\overline{G}$*-fusion in* $\overline{S}$ *and* $\overline{M} = O^2(\overline{M})$.
(g) $\overline{\Phi}$ *is noncyclic.*

Proof. (a), (b) and (c) follow from Theorems 8.1 and 9.1. (d) follows from Theorem 4.1 with $R\overline{G}, \overline{\Phi}, R\overline{M}, \overline{S}$ and $R\overline{T}$ in the roles of G, Φ, M, S and U respectively. To prove (e), we may suppose that H is a 2-group. Recall that $S \in \mathcal{V}^*_G(R,2)$ and $T = [T,R] \trianglelefteq S$. By Coprime Action, $H^g \leq S$ for some $g \in C_G(R)$. Then $g \in M$ by (d), hence $H \leq M$ and we are done. Theorem 4.1 implies that $\overline{M}$ controls $\overline{G}$-fusion in $\overline{S}$. Now $\overline{G} = O^2(\overline{G})$ by Lemma 6.4 so (f) follows from the Focal Subgroup Theorem.

To prove (g) suppose that $\overline{\Phi}$ is cyclic. Then $[\overline{\Phi}, M] = 1$ so M induces a 2-group on Φ. As $\overline{M} = O^2(\overline{M})$ we have $[\Phi, M] = 1$. Now $M = M_0[M,R]$ and $[M,R] = TO_p([M,R])$ by Lemma 6.3. Corollary 3.3 implies $[M_0, T] = 1$. But then $\overline{T}$ is an image of $\overline{M}$, contradicting $\overline{M} = O^2(\overline{M})$. This completes the proof.

Lemma 10.2. *(a)* G *is* 2*-transitive on* Ω.
(b) $T = [M,R] \trianglelefteq M$ *and* $M = M_0T$.

Proof. (a). This follows from Theorem 5.1(a) and Lemma 10.1(d,g) with $\overline{G}, \overline{M}$ and $\overline{\Phi}$ in the roles of G, M and Φ.

(b). By Lemma 6.4, $O_p(G) = 1$ and M does not contain a Sylow p-subgroup of G. Using Lemma 2.10 we have $O_p([M,R]) \leq O_p(M) \leq O_p(G) = 1$. Since $[M,R] = TO_p([M,R])$, we are done.

Lemma 10.3. $|\Omega_0| > 1$ *and* G_0 *is* 2*-transitive on* Ω_0.

Proof. Choose $g \in G - M$ and set $D = M \cap M^g$. Since G is 2-transitive on Ω we have

$$|G : M| = 1 + |M : D|. \qquad (*)$$

Recall that $G_0 = C_G(R)$. If $G_0 \leq M$ then $N_{GR}(R) = RG_0 \leq RM$ and $R \in \mathrm{Syl}_r(RG)$, so RM is the unique point of RG/RM fixed by R. Thus $|G : M| \equiv 1 \bmod r$. Then $(*)$ implies r divides $|M : D|$, which is absurd because G is an r'-group. Thus $G_0 \not\leq M$ and another application of Theorem 5.1 completes the proof.

Lemma 10.4. *Let* $\mathbb{Z}_2 \cong \overline{X} \leq \overline{M_0}$. *Then* $C_{\overline{G}}(\overline{X}) \leq \overline{M}$.

Proof. Assume false. Let H be the inverse image of $C_{\overline{G}}(\overline{X})$ in G. Then $H \not\leq M$. Recall that $Z(G) \leq G_0$ by Lemma 6.4(b). The full inverse image of $\overline{X}$ is a nilpotent subgroup of G_0 so there is a 2-subgroup $X \leq G_0$ that maps onto $\overline{X}$ and satisfies $X \trianglelefteq H$. Also, RH stabilizes the chain $1 \trianglelefteq Z(G) \cap X \trianglelefteq X$ so Lemma 2.4 implies that RH induces an abelian 2-group on X.

Now $1 \neq C_{\overline{\Phi}}(\overline{X}) \leq \overline{\Phi} \cap \overline{H}$. Lemma 10.1(e) implies that M contains a member of $\mathcal{U}^*_H(R, 2)$. Conjugating X by a suitable element of M_0, we may suppose that

$$S \cap H \in \mathcal{U}^*_H(R, 2).$$

Set

$$K = [H, R] \quad \text{and} \quad Q = S \cap K \in \mathcal{U}^*_K(R, 2).$$

Since RH induces an abelian 2-group on X we have $[X, K] = 1$. By Lemma 6.3(b), $Q = [Q, R]$, so as $T = [S, R]$ we have $Q \leq C_T(X)$. Moreover, $[C_T(X), R] \leq Q$, whence

$$Q = [C_T(X), R].$$

Claim 1. $m(\overline{\Phi(Q)}) \leq 1$.

Proof. Assume false. By Lemma 6.3(b), $H = N_H(\Phi(Q))O_p(K)$. Also $\Phi(Q) \leq \Phi(T) = \Phi$ and $\mathcal{M}_R(T) = \{ M \}$. Since $\overline{\Phi(Q)}$ is noncyclic it follows from Coprime Action(h) and Lemma 10.1(d) that $O_p(K) \leq M$. As $\overline{\Phi}$ is TI we have $N_G(\Phi(Q)) \leq M$, so $H \leq M$, a contradiction.

Claim 2. *$M \cap H$ induces a 2-group on Φ.*

Proof. Lemma 10.2(b) implies that $T = [M, R] \trianglelefteq M$ whence $M \cap H = (M_0 \cap H)[M \cap H, R] \leq (M_0 \cap H)T$. As $\Phi = Z(T)$ it suffices to prove that $M_0 \cap H$ induces a 2-group on Φ. Let $Y \leq M_0 \cap H$ have odd order. Then $[X, Y] = 1$ since H induces a 2-group on X. Now $Q = T \cap K$ so Y normalizes Q. By Claim 1, $[\Phi(Q), Y] = 1$ so Corollary 3.3 forces $[Q, Y] = 1$. Applying Lemma 2.9 to the action of $R \times X \times Y$ on T we obtain $[T, Y] = 1$. Then $[\Phi, Y] = 1$ and the claim is established.

Let

$$\Phi_1 = \Phi \cap H.$$

Note that $\Phi_1 \trianglelefteq M \cap H$ because $\Phi \trianglelefteq M$. By Lemma 10.1(d), for all $\overline{g} \in \overline{H}$, $\overline{\Phi_1}^{\overline{g}} \cap (\overline{M} \cap \overline{H}) \neq 1$ implies $\overline{g} \in \overline{M} \cap \overline{H}$. Moreover, $\overline{M} \cap \overline{H} < \overline{H}$ because $H \not\leq M$. Lemma 5.3(b) implies that $\overline{M} \cap \overline{H}$ acts transitively on $\overline{\Phi_1}^{\#}$. Then Claim 2 forces $m(\overline{\Phi_1}) \leq 1$.

Now $\overline{\Phi_1} = C_{\overline{\Phi}}(\overline{X})$ so as $\mathbb{Z}_2 \cong \overline{X} \leq \overline{M} = N_{\overline{G}}(\overline{\Phi})$ we have

$$m(\overline{\Phi_1}) \geq \tfrac{1}{2} m(\overline{\Phi}).$$

Also $m(\overline{\Phi}) \geq 2$ by Lemma 10.1(g). It follows that $m(\overline{\Phi}) = 2$ and that $\overline{X}$ acts nontrivially on $\overline{\Phi}$. Now Aut$(\overline{\Phi}) \cong$ Sym(3) whence $\mathbb{Z}_2$ is an image of $\overline{M}$. But $\overline{M} = O^2(\overline{M})$ by Lemma 10.1(f). This contradiction completes the proof of Lemma 10.4.

Corollary 10.5. *$\overline{M_0}$ is strongly embedded in $\overline{G_0}$, so that $\overline{M_0} \cap \overline{M_0}^{\overline{g}}$ has odd order for all $\overline{g} \in \overline{G_0} - \overline{M_0}$.*

Proof. By Coprime Action, $C_{\overline{S}}(R) \in \mathrm{Syl}_2(\overline{G_0})$. Moreover $\Phi \leq G_0$ so Lemma 10.1(d) implies $N_{\overline{G_0}}(C_{\overline{S}}(R)) \leq \overline{M_0}$. Apply Lemma 10.4.

Choose $u \in G_0 - M_0$ with u conjugate to an element of Φ. Set

$$D = M \cap M^u, \quad D_0 = C_D(R) \quad \text{and} \quad Q = [D, R] \trianglelefteq D.$$

Note that D is R-invariant because $u \in G_0$. Recall that

$$Z = Z(G) \cap \Phi = Z(G) \cap T$$

and that either $Z = 1$ or $Z \cong \mathbb{Z}_2$ and Z inverts V.

Lemma 10.6. *$\overline{D_0}$ has odd order, $[D_0, Q] = 1$ and $\overline{D} = \overline{D_0} \times \overline{Q}$. If $Q \neq 1$ then $Q \leq T$, Q is extraspecial with $\Phi(Q) = Z \cong \mathbb{Z}_2$ and R acts irreducibly on $Q/\Phi(Q)$.*

Proof. Corollary 10.5 implies $\overline{D_0}$ has odd order. Suppose that $Q \neq 1$. Now $Q = [D, R] \leq [M, R] = T$ so $\Phi(Q) \leq \Phi(T) = \Phi$. Since u is an involution, D is u-invariant. Then $u \in N_G(\Phi(Q))$. As $\overline{\Phi}$ is TI in $\overline{G}$ and $u \notin M$ it follows that $\Phi(Q) \leq Z(G) \cap \Phi = Z$. Now $Q = [Q, R] \neq 1$ so Q is a nonabelian special 2-group by Lemma 3.2(b). Then $\Phi(Q) = Z$ and Q is extraspecial. Lemma 3.2(e) implies that $[D_0, Q] = 1$ and that R is irreducible on $Q/\Phi(Q)$.

Set

$$q = |\overline{\Phi}| \quad \text{and} \quad \alpha = \frac{|T : \Phi|}{|Q : \Phi(Q)|} - 1.$$

Now $\overline{\Phi} \cap \overline{D_0} = 1$ because $\overline{D_0}$ has odd order, so

$$|M_0 : D_0| = \beta q$$

for some $\beta \in \mathbb{N}$.

Lemma 10.7. *(a) $\alpha \in \mathbb{N}$ and r divides α.*
(b) $(1 + \beta q)(q - 1)$ divides both $|G|$ and α.

Proof. (a). By Lemma 3.2(b), $\Phi(Q) = C_Q(R) = Q \cap C_T(R) = Q \cap \Phi$. Thus $\alpha = |T/\Phi : Q\Phi/\Phi| - 1 \in \mathbb{Z}$. If $\alpha = 0$ then $T = Q\Phi$ so $T = Q$ and then $\Phi = \Phi(T) = \Phi(Q) \leq Z$, contradicting Lemma 10.1(g). Thus $\alpha \in \mathbb{N}$. As $C_T(R) = \Phi$ and $C_Q(R) = \Phi(Q)$ we have $|T : \Phi| \equiv |Q : \Phi(Q)| \equiv 1 \bmod r$. Hence $\alpha \equiv 0 \bmod r$.

(b). By 2-transitivity,

$$|G : M| = 1 + |M : D| \quad \text{and} \quad |G_0 : M_0| = 1 + |M_0 : D_0|.$$

The quotient $|G : M|/|G_0 : M_0|$ is an integer by Lemma 2.3, so dividing and subtracting 1 yields

$$\frac{|M_0 : D_0|}{1 + |M_0 : D_0|}\left(\frac{|M : D|}{|M_0 : D_0|} - 1\right) \in \mathbb{Z}. \qquad (*)$$

Now $M = M_0T, M_0 \cap T = \Phi, D = D_0Q$ and $D_0 \cap Q = \Phi(Q) = Q \cap \Phi$. Thus

$$\alpha = \frac{|M : M_0|}{|D : D_0|} - 1 = \frac{|M : D|}{|M_0 : D_0|} - 1 = \frac{|\Omega| - 1}{|\Omega_0| - 1} - 1. \qquad (**)$$

It follows from $(*)$ and $(**)$ that $1+\beta q$ divides α. Also, $1+\beta q = |G_0 : M_0|$ so $1 + \beta q$ divides $|G|$.

Put $m = q - 1$. By Lemma 10.6, $\overline{D} = O_{2'}(\overline{D}) \times O_2(\overline{D})$ so two applications of Theorem 5.1(c) yield

$$|\Omega| \equiv |\Omega_0| \equiv 2 \bmod m.$$

This and $(**)$ imply $\alpha \equiv 0 \bmod m$. By Lemma 5.3(b), M is transitive on $\overline{\Phi}^{\#}$ so m divides $|G|$. Also, $1 + \beta q = |\Omega_0| \equiv 2 \bmod m$ so as m is odd, $1 + \beta q$ and m are coprime. This proves (b).

Lemma 10.8. $\alpha \le q^2 - 1$.

Proof. Theorem 5.1(b) implies that all involutions of $\overline{G}$ are conjugate into $\overline{D\Phi}$, and hence into $\overline{Q\Phi}$ by Lemma 10.6. Since $\overline{M}$ controls $\overline{G}$-fusion in $\overline{S}$ it follows that all involutions in $\overline{T}$ are $\overline{M}$-conjugate into $\overline{Q\Phi}$.

Suppose that $Q = 1$. Then all involutions of $\overline{T}$ are contained in $\overline{\Phi}$ so Lemma 2.11 implies $|\overline{T}| \le |\overline{\Phi}|^3$, whence $\alpha \le q^2 - 1$. Hence we may assume that $Q \ne 1$.

Let U be a homogeneous component for the action of T on V and set $T_1 = C_T(U)$. Lemma 3.2(d) implies that U and T_1 are R-invariant. Using Lemma 10.6,

$$(Q\Phi \cap T_1)' \le \Phi(Q) \cap T_1 = Z \cap T_1 = 1.$$

because Z inverts V. Then $Q\Phi \cap T_1$ is R-invariant and abelian so Lemma 6.2 forces $Q\Phi \cap T_1 \le C_T(R) = \Phi$. In fact, $(Q\Phi)^g \cap T_1 \le \Phi$ for all $g \in M$ since $T \trianglelefteq M$ and so Ug^{-1} is also a homogeneous component for T. We deduce that all involutions of $\overline{T_1}$ are contained in $\overline{\Phi}$. Lemma 2.11 implies $|\overline{T_1}| \le |\overline{\Phi}|^3$.

From the previous paragraph, $Q \cap T_1 \le \Phi \cap Q \cap T_1 = Z \cap T_1 = 1$ and R is irreducible on the Frattini quotient of T/T_1 by Lemma 3.2(d). Thus $T = QT_1$. Note that $|\overline{T_1}| = |T_1|$ because $T_1 \cap Z = 1$ and recall that $\Phi(Q) = Z \cong \mathbb{Z}_2$. Then

$$\alpha = \frac{|T : \Phi|}{|Q : \Phi(Q)|} - 1 = \frac{|\overline{T_1}|}{|\overline{\Phi}|} - 1 \le |\overline{\Phi}|^2 - 1.$$

The proof is complete.

Lemmas 10.7 and 10.8 imply

$$(1 + \beta q)(q - 1) \leq \alpha \leq q^2 - 1.$$

We must have equality, and then α divides $|G|$. But r divides α and G is an r'-group. This final contradiction completes the proof of Theorem A.

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