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Generating Finite Groups with Maximal Subgroups of Maximal Subgroups

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INTRODUCTION

This paper grew out of the following question: Suppose M is a maximal subgroup of a finite simple group G and H is a maximal subgroup of M. Is it possible to find $g \in G$ such that $G = \langle H, g \rangle$?

The answer is yes and it is proved as follows: In a counterexample we have

$$\langle H, g \rangle \cap M = H \qquad \text{for all } g \in G - M.$$

The first stage is to prove that $H_M = 1$ and then to prove that H is a Frobenius complement in G. An application of Frobenius' Theorem contradicts the simplicity of G.

Much of the argument can be made to work without such stringent conditions on G, M, and H. Thus we make the following definition:

DEFINITION. A γ-triple is a triple of groups (G, M, H) with the properties

(i) $H < M < G$;

(ii) $\langle H, g \rangle \cap M = H$ for all $g \in G - M$.

Note that we no longer require M to be maximal in G or H to be maximal in M.

Our original question now generalizes to the problem: Find some sort of structure theorem for γ-triples. The first result we prove is:

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THEOREM A. *Let (G, M, H) be a γ-triple. Then*

$$H_M \trianglelefteq \trianglelefteq G.$$

Then we shall prove:

THEOREM B. *Let (G, M, H) be a γ-triple. Then*

$$H/H_M \text{ is cyclic.}$$

Before we can proceed any further we need a suitable notion of irreducibility. First we make the following observation: if (G, M, H) is a γ-triple, N is a group and $\tilde{G}$ is an extension of N by G (so that $N \trianglelefteq \tilde{G}$ and $\tilde{G}/N \cong G$), let $\tilde{M}$ (resp. $\tilde{H}$) denote the inverse image of M (resp. H) in $\tilde{G}$. Then $(\tilde{G}, \tilde{M}, \tilde{H})$ is also a γ-triple. Conversely, if (G, M, H) is a γ-triple, $N \trianglelefteq G$, and $N \leq H$ then $(G/N, M/N, H/N)$ is also a γ-triple. We are led to the following definition:

DEFINITION. A γ-triple (G, M, H) is *irreducible* if $H_G = 1$, that is, if the only normal subgroup of G contained in H is 1.

The preceding discussion shows that every γ-triple is made out of an irreducible γ-triple and a group.

We shall need a generalization of Frobenius' Theorem due to Wielandt. Thus we make the following definition:

DEFINITION. A W-triple is a triple of groups (G, H, N) with the properties

(i) $N \trianglelefteq H \leq G$;

(ii) $H \cap H^g \leq N$ for all $g \in G - H$.

The fundamental theorem about W-triples is:

WIELANDT'S THEOREM. *Let (G, H, N) be a W-triple. Then G contains a normal subgroup K such that*

$$G = HK \quad \text{and} \quad H \cap K = N.$$

For a proof of Wielandt's Theorem see [3, Exercise 1, p. 347]. This result is a generalization of Frobenius' Theorem as can be seen by putting $N = 1$.

Now we can state our next theorem on γ-triples.

THEOREM C. *Let (G, M, H) be an irreducible γ-triple with $H > H_M$. Then there exists a prime p such that*

(i) *(G, H, H_M) is a W-triple;*

(ii) *$G = HO_p(G)$ and $H \cap O_p(G) = H_M$;*

(iii) *H/H_M is a cyclic p'-group;*

(iv) *G is soluble.*

Note that if $H_M = 1$ then G is a Frobenius group with complement H. If $H = H_M$ then Theorem C is not applicable; however, in this case we have $H \trianglelefteq M$ and Theorem A implies $H \trianglelefteq \trianglelefteq G$.

Then we answer our original question:

COROLLARY D. *Suppose M is a maximal subgroup of a simple group G and H is a maximal subgroup of M. Then there exists $g \in G$ such that $G = \langle H, g \rangle$.*

Finally, we shall construct a number of examples of γ-triples.

1. NOTATION AND QUOTED RESULTS

Throughout this paper, group means finite group, $H \le G$ means H is a subgroup of G, $H < G$ means H is a proper subgroup of G, $H \trianglelefteq G$ means H is a normal subgroup of G, and $H \trianglelefteq \trianglelefteq G$ means H is a subnormal subgroup of G. If $H \le G$ then $H_G =$ the core of H in $G = \cap\{H^g : g \in G\}$. If $H \le G$ and $x \in G$ then $[x, H] = \langle [x, h] : h \in H \rangle$.

THEOREM 1.1 (Wielandt). *Let H be a subgroup of a group G. Then $H \trianglelefteq \trianglelefteq G$ if and only if $H \trianglelefteq \trianglelefteq \langle H, H^g \rangle$ for all $g \in G$* [1, *Theorem* 14.10].

LEMMA 1.2. *Suppose a p'-group Q acts on a p-group P. Then:*

(i) $P = C_P(Q)[P, Q]$ *and* $[P, Q] = [P, Q, Q]$;

(ii) *If T is a Q-invariant normal subgroup of P then*

$$C_{P/T}(Q) = C_P(Q)T/T$$

[2, *Theorems* 5.3.5, 5.3.6, *and* 6.2.2(iv)].

LEMMA 1.3. *Let p, q, and r be distinct primes. Let $Z \cong \mathbb{Z}_r$ act faithfully and irreducibly on an elementary abelian q-group Q and let V be a faithful* GF$(p)ZQ$*-module. Then*

$$C_V(Z) \neq 0.$$

This is a restatement of [2, Theorem 3.4.4].

LEMMA 1.4. *Let an r-group R act on an r'-group G and let q be a prime divisor of $|G|$. Then there exists an R-invariant Sylow q-subgroup of G* [2, *Theorem* 6.2.2(i)].

2. PRELIMINARY LEMMAS

LEMMA 2.1. *Let (G, M, H) be a γ-triple.*

(i) *If (G, M, L) is also a γ-triple then so is $(G, M, H \cap L)$.*

(ii) *If $m \in M$ then (G, M, H^m) is a γ-triple.*

(iii) *(G, M, H_M) is a γ-triple.*

Proof. Let $g \in G - M$. Then

$$\langle H \cap L, g\rangle \cap M \subseteq \langle H, g\rangle \cap M \cap \langle L, g\rangle = H \cap L.$$

This proves (i). As for (ii), if $g \in G - M$ then $g^{m^{-1}} \in G - M$ so

$$\langle H^m, g\rangle \cap M = \left(\langle H, g^{m^{-1}}\rangle \cap M\right)^m = H^m.$$

Part (iii) is a consequence of (i) and (ii).

LEMMA 2.2. *Let H be a subgroup of a group G. Then*:

(i) *$[x, H]$ is normalized by H.*

(ii) *$\langle H, H^x\rangle = [x, H]H = H[x, H]$.*

Proof. Let $h, k \in H$. Then

$$[x, hk] = [x, k][x, h]^k$$

so

$$[x, h]^k \in \langle [x, g] : g \in H\rangle = [x, H].$$

This proves (i) and (ii) follow immediately.

Presumably the following few lemmas on W-triples are known but we cannot find a reference.

LEMMA 2.3. *Let G be a group and $N \trianglelefteq H \leq G$. Then (G, H, N) is a W-triple if and only if*

$$N_G(D) \leq H \qquad \text{for all } D \leq H \text{ with } D \nleq N.$$

Proof. The only if part is obvious; as for the if part, let $g \in G$ and $D = H \cap H^g$ and suppose that $D \nleq N$. Choose a prime p such that a Sylow p-subgroup P of D is not contained in N. Then, by hypothesis, $N_G(P) \leq H$ and thus $N_{H^g}(P) \leq H \cap H^g = D$. This implies that $P \in \mathrm{Syl}_p(H^g)$ and as $P \leq H$ we see that $P \in \mathrm{Syl}_p(H)$. Now P and $P^{g^{-1}}$ are Sylow p-subgroups of H so there exists $h \in H$ such that $P^h = P^{g^{-1}}$. Then $hg \in N_G(P) \leq H$ so $g \in H$ and we deduce that (G, H, N) is a W-triple.

LEMMA 2.4. *Let (G, H, N) be a W-triple, let $D \leq H$ with $D \nleq N$, and let $g \in G - H$. Then*

$$g \in \langle H, D^g \rangle.$$

Proof. Since D is not contained in N there exists a prime p and a Sylow p-subgroup P of D such that $P \nleq N$. Let Q be a Sylow p-subgroup of H that contains P, then $Q \nleq N$ and thus $N_G(Q) \leq H$. This implies that Q is a Sylow p-subgroup of G and hence of $\langle H, D^g \rangle$. By Sylow's Theorem there exists $x \in \langle H, D^g \rangle$ such that $P^{gx} \leq Q$. Thus $P \leq H \cap H^{(gx)^{-1}}$ and since $P \nleq N$ this forces $gx \in H$. Hence $g \in \langle H, D^g \rangle$.

LEMMA 2.5. *Let (G, H, N) be a W-triple and let M be a normal subgroup of G such that*

$$G = HM \qquad \text{and} \qquad H \cap M = N.$$

Then:

(i) *$|H : N|$ and $|M : N|$ are coprime.*

(ii) *If L is another normal subgroup of G such that $G = HL$ and $H \cap L = N$ then $M = L$.*

Proof. Note that the existence of M is guaranteed by Wielandt's Theorem. Suppose p is a prime divisor of $|H : N|$. Let P be a Sylow p-subgroup of H. Then $P \nleq N$ so $N_G(P) \leq H$ and thus $P \in \mathrm{Syl}_p(G)$. Since $M \trianglelefteq G$ it follows that $P \cap M \in \mathrm{Syl}_p(M)$. Now $P \cap M \leq H \cap M \leq N$ so N contains a Sylow p-subgroup of M. We deduce that p cannot divide $|M : N|$. This proves (i).

To prove (ii), let p be a prime divisor of $|M|$ and let $P \in \mathrm{Syl}_p(M)$. If p does not divide $|H : N|$ then, as $G = HL$ and $H \cap L = N$, it follows that L contains a Sylow p-subgroup of G. Since $L \trianglelefteq G$ this implies $P \leq L$. If p does divide $|H : N|$ then as $|M : N|$ is prime to p we see that $P \leq N$ and thus $P \leq L$. We deduce that L contains every Sylow subgroup of M and similarly that M contains every Sylow subgroup of L. Thus $M = L$.

LEMMA 2.6. *Suppose that $H \trianglelefteq \trianglelefteq G$ but that H is not normal in G. Then*

$$N_G(H)\langle H^G \rangle < G.$$

Proof. If $H \trianglelefteq \langle H^G \rangle$ then $N_G(H)\langle H^G \rangle = N_G(H) < G$. Now suppose that H is not normal in $\langle H^G \rangle$. Then, since H is subnormal in $\langle H^G \rangle$, there exists $N \trianglelefteq \langle H^G \rangle$ such that $H \leq N < \langle H^G \rangle$. Choose $g \in G$ such that $H^g \nleq N$. Then $g \notin N_G(H)\langle H^G \rangle$.

LEMMA 2.7. *Let p, q, and r be distinct primes. Let $Z \cong \mathbb{Z}_r$ act faithfully and irreducibly on an elementary abelian q-subgroup Q and let ZQ act on a* *p-group P. Then*

$$[P,Q] \leq \langle C_P(Z)^Q \rangle.$$

Proof. Assume false and consider a counterexample with $|P|$ minimal. Lemma 1.2(i) implies $P = [P,Q]$. Suppose T is a proper nontrivial ZQ-invariant normal subgroup of P. Minimality of $|P|$ implies

$$[P/T,Q] \leq \langle C_{P/T}(Z)^Q \rangle$$

and, using Lemma 1.2(ii), it follows that

$$P = [P,Q] \leq T\langle C_P(Z)^Q \rangle.$$

Let $D = \langle C_P(Z)^Q \rangle$. Then

$$P = [TD,Q] = \langle [T,Q]^D \rangle [D,Q].$$

Minimality of $|P|$ forces $[T,Q] \leq D$, and as D is Q-invariant we see that $P \leq D$, a contradiction.

What we have just done implies $\Phi(P) = 1$. Thus P may be regarded as a $\mathrm{GF}(p)ZQ$-module, and another application of the previous paragraph implies that ZQ acts irreducibly on P. The only proper normal subgroups of ZQ are Q and 1, so as $[P,Q] \neq 1$ we deduce that ZQ is faithful on P. Lemma 1.3 implies that $C_P(Z) \neq 0$. Now $\langle C_P(Z)^{ZQ} \rangle = \langle C_P(Z)^Q \rangle$, so as ZQ is irreducible on P we deduce that

$$P = \langle C_P(Z)^Q \rangle,$$

a contradiction.

3. PROOF OF THEOREMS

Proof of Theorem A. Assume the theorem is false. Using Lemma 2.1(iii) we may suppose that $H \trianglelefteq M$. By Theorem 1.1, there exists $x \in G$ such that H is not subnormal in $\langle H, H^x \rangle$. Choose such an x with $[x,H]$ minimal. A second application of Theorem 1.1 implies there exists $y \in \langle H, H^x \rangle$ such that H is not subnormal in $\langle H, H^y \rangle$.

By Lemma 2.2(ii), there exists $h \in H$ and $z \in [x,H]$ such that $y = hz$. Thus H is not subnormal in $\langle H, H^z \rangle$. Lemma 2.2(i) implies $[z,H] \leq [x,H]$, so choice of x forces $[z,H] = [x,H]$. In particular,

$$z \in [z,H].$$

Since $M > H$ we may choose $m \in M - H$. Since $H \trianglelefteq M$ we have $z \notin M$ and also $mz \notin M$. Observe that

$$z \in [z, H] \leq \langle H, H^z \rangle = \langle H, H^{mz} \rangle \leq \langle H, mz \rangle$$

and then that

$$m \in \langle H, mz \rangle \cap M.$$

Since $mz \notin M$ the definition of a γ-triple yields $m \in H$ contrary to the choice of m. This completes the proof of Theorem A.

LEMMA 3.1. *Let (G, M, H) be a γ-triple. Then (G, H, H_M) is a W-triple.*

Proof. First we prove

$$M \cap H^g \leq H_M \qquad \text{for all } g \in G - M. \qquad (*)$$

Indeed, let $g \in G - M$ and $m \in M$. Then $gm \in G - M$ so

$$(M \cap H^g)^m = M \cap H^{gm} \leq M \cap \langle H, gm \rangle = H.$$

Thus $M \cap H^g \leq H^{m^{-1}}$ for all $m \in M$ and $(*)$ follows.

Now let $x \in G - M$ and set $X = \langle H, x \rangle$. Suppose D is any subgroup of H not contained in H_M. Then $(*)$ forces $N_X(D) \leq H$. Lemma 2.3 now implies that (X, H, H_M) is a W-triple. Next, let E be any subgroup of H not contained in H_M. Fix $y \in G - M$. Then $(\langle H, y \rangle, H, H_M)$ is a W-triple so Lemma 2.4 implies that $y \in \langle H, E^y \rangle$. Let $n \in N_G(E)$. By $(*)$ we have $n \in M$ and thus $ny \in G - M$. Then

$$ny \in \langle H, E^{ny} \rangle = \langle H, E^y \rangle,$$

but as $y \in \langle H, E^y \rangle$ we obtain $n \in \langle H, E^y \rangle$. Thus $n \in \langle H, y \rangle \cap M = H$ and we deduce that $N_G(E) \leq H$. Another application of Lemma 2.3 completes the proof.

LEMMA 3.2. *Let (G, M, H) be a γ-triple with $H > H_M = 1$. Then:*

(i) *G is a Frobenius group with complement H and kernel $O_p(G)$ for some prime p.*

(ii) *H is cyclic.*

Proof. The previous lemma implies that (G, H, H_M) is a W-triple and since $H_M = 1$ we see that G is a Frobenius group with complement H. Let K be the Frobenius kernel of G, so that $G = HK$, $K \trianglelefteq G$, and $H \cap K = 1$. Thompson's Theorem implies that K is nilpotent. Now $M = H(M \cap K)$, and as $M > H$ we have that $M \cap K \neq 1$.

Let p be a prime divisor of $|M \cap K|$ and let $m \in M \cap K$ have order p. If x is any member of $O_{p'}(K)$ then, as K is nilpotent, $[m, x] = 1$. Thus $m \in M \cap \langle mx \rangle \leq M \cap \langle H, mx \rangle$. However, $m \notin H$ so this implies $mx \in M$. We deduce that $O_{p'}(K) \leq M$.

If K is not a p-group, then since K is nilpotent there is a prime $q \neq p$ such that $O_q(K) \neq 1$. Then by the previous paragraph $O_q(K) \leq M$, and another application of the previous paragraph, with q in place of p, yields $O_{q'}(K) \leq M$. Since K is nilpotent we have $K = O_{q'}(K)O_q(K)$ and thus $K \leq M$. Then $M = G$ contrary to (G, M, H) being a γ-triple. We deduce that K is a p-group. Thus (i) is proved.

Next we prove (ii). Since K is a p-group and $K \nleq M$ we have that $N_K(M \cap K) > M \cap K$ and thus $(N_G(M \cap K), M, H)$ is also a γ-triple. So, without loss of generality, we may suppose that $G = N_G(M \cap K)$. We also note that for any $k \in K - M$ we have

$$\langle k^H \rangle \cap M \leq \langle H, k \rangle \cap K \cap M \leq H \cap K = 1. \qquad (*)$$

Since $M \cap K \trianglelefteq G$ and since K is a p-group it follows that $Z(K) \cap M \neq 1$. Let U be a minimal H-invariant subgroup of $Z(K) \cap M$. Choose $g \in K - M$ and let W be a minimal H-invariant subgroup of $\langle g^H \rangle$. The choice of U and W implies that they are elementary abelian. Since $U \leq Z(K)$ it follows that $\langle U, W \rangle$ is elementary abelian also. Using $(*)$ we see that $U \cap W = 1$. Thus

$$\langle U, W \rangle = U \times W.$$

Since $U \times W$ is normalized by H, we may regard it as an H-module over $\mathrm{GF}(p)$. We see that U and W are irreducible H-submodules of $U \times W$.

Let $u \in U^{\#}$ and $w \in W^{\#}$. Set $v = uw$ and $V = \langle v^H \rangle$. From $(*)$ we have $W \cap M = 1$, so as $u \in M$ it follows that $v \notin M$. Another application of $(*)$ implies that $V \cap U = 1$. Since $V \leq U \times W$ and since W is an irreducible H-module, it follows that V is irreducible also. Next we consider the projection maps

$$\pi_U : V \to U \qquad \text{and} \qquad \pi_W : V \to W.$$

These maps are H-homomorphisms and they are nontrivial as $V \nsubseteq U$ and $V \nsubseteq W$. Since U, W, and V are all irreducible, we deduce that π_U and π_W are H-isomorphisms. Thus $\pi_U^{-1}\pi_W$ is a H-isomorphism $U \to W$ that maps u to w and $\pi_W^{-1}\pi_U$ is a H-isomorphism $W \to U$ that maps w to u.

Let $E = \mathrm{End}_H(W)$. The preceding paragraph implies that E is transitive on $W^{\#}$. Thus E is irreducible on W and hence $\mathrm{End}_E(W)$ is a field. Since $H \leq \mathrm{End}_E(W)$ we see that H is cyclic.

Proof of Theorem B. Let $N = N_G(H_M)$. First we will show that (N, M, H) is a γ-triple. It suffices to prove that $N > M$. If $H_M \trianglelefteq G$ this is clear. If H_M is not normal in G then, since it is subnormal by Theorem A, there exists $g \in G$ such that $H_M \neq H_M^g \leq N$. Now $H_M \trianglelefteq M$ so $g \in G - M$. Lemma 2.1(iii) implies that (G, M, H_M) is a γ-triple and thus $H_M^g \cap M \leq \langle H_M, g \rangle \cap M = H_M$. Hence H_M^g is a subgroup of N not contained in M and so $N > M$.

Set $N^* = N/H_M$, $M^* = M/H_M$, and $H^* = H/H_M$. Then (N^*, M^*, H^*) is a γ-triple and $H^*_{M^*} = 1$. If $H^* = 1$ then H/H_M is cyclic. If $H^* \neq 1$ then Lemma 3.2 implies H^* is cyclic and thus H/H_M is cyclic as claimed.

Proof of Theorem C. Assume the theorem false and let G be a minimal counterexample. Lemma 3.1 proves (i). Note that (iv) follows from (ii) and (iii). Thus (ii) or (iii) is false. Lemma 3.2 implies that $H_M \neq 1$. Wielandt's Theorem and (i) imply the existence of a subgroup K of G with the properties

$$G = HK, \qquad H \cap K = H_M, \qquad \text{and} \qquad K \trianglelefteq G.$$

Let $L = \langle H_M^G \rangle$ and set $X = ML$. Since $X \leq N_G(H_M)\langle H_M^G \rangle$ and as H_M is not normal in G, Theorem A and Lemma 2.6 imply $X < G$. Next we claim that (X, M, H) is a γ-triple. This will be immediate once we have shown $X > M$. Since H_M is subnormal but not normal in G, there exists $g \in G$ such that $H_M \neq H_M^g$. Lemma 2.1 implies that (G, M, H_M) is a γ-triple and as $g \notin M$ we obtain $H_M^g \cap M \leq H_M$. Thus H_M^g is a subgroup of X not contained in M.

Let $X^* = X/H_X$, $M^* = M/H_X$, and $H^* = H/H_X$. Then (X^*, M^*, H^*) is an irreducible γ-triple and since $H > H_M \geq H_X$ we see that $H^* > H^*_{M^*}$. Minimality of G implies there is a prime p such that $X^* = H^*O_p(X^*)$, $H^* \cap O_p(X^*) = H^*_{M^*}$, and $H^*/H^*_{M^*}$ is a cyclic p'-group. Since $H/H_M \cong H^*/H^*_{M^*}$ and as (ii) or (iii) is false, we see that K is not a p-group.

Let N be the inverse image of $O_p(X^*)$ in X. Then $X = HN$ and $H \cap N = H_M$. Also, $X = H(K \cap X)$ and $H \cap K \cap X = H_M$. Since (X, H, H_M) is a W-triple, Lemma 2.5(ii) implies that $K \cap X = N$. In particular, $(K \cap X)/H_X$ is a p-group. Let h be a p'-element of H_X. Then for each $g \in G$ we see that h^g must have a trivial image in $(K \cap X)/H_X$ and hence $h \in (H_X)_G = 1$. We deduce that $K \cap X$ is a p-group, as are H_M and L.

Since H/H_M is a cyclic p'-group and H_M is a p-group, the Schur–Zassenhaus Theorem implies that there exists $R \leq H$ such that R is a cyclic p'-group, $H = RH_M$, and $R \cap H_M = 1$. Let $x \in R^\#$ and $c \in C_K(x)$. Then $x \in H \cap H^c - H_M$ and as (G, H, H_M) is a W-triple it follows that $c \in H \cap K = H_M$. Thus $C_K(x) \leq H_M$ for all $x \in R^\#$. Since H_M is a p-group and R is a p'-group, we may use Sylow's Theorem to see that K and R have coprime orders.

Let Z be a subgroup of R with prime order r. Let q be a prime divisor of $|K|$ not equal to p. Since K is an r'-group, Lemma 1.4 implies that Z normalizes a nontrivial q-subgroup of K. Let Q be a minimal such q-subgroup. Then Q is elementary abelian and Z acts irreducibly on Q. Since $C_K(Z) \leq H_M$ we see that Z is faithful on Q. Moreover, $C_K(Z) \leq L \trianglelefteq G$ so Lemma 1.2(ii) implies $C_{K/L}(Z) = 1$, hence K/L is nilpotent by Thompson's Theorem. Then $[K \cap X, Q] \leq L \leq K \cap X$ so Q normalizes $K \cap X$. Since $Z \leq X$ we see that QZ normalizes $K \cap X$ also. Lemma 2.7 yields

$$[K \cap X, Q] \leq \langle C_{K \cap X}(Z)^Q \rangle = \langle H_M^Q \rangle.$$

and using Lemma 1.2(i) we obtain

$$K \cap X = \langle H_M^Q \rangle C_{K \cap X}(Q). \qquad (*)$$

Now $M = H(M \cap K \cap X) > H$ so we may select $m \in M \cap K \cap X - H$. By $(*)$ there exists $d \in \langle H_M^Q \rangle$ and $c \in C_{K \cap X}(Q)$ such that $m = dc$. Choose $e \in Q^\#$ and set $g = ce$. Since c and e are commuting elements of coprime orders we have $e \in \langle g \rangle$. Now $M \leq X = H(K \cap X)$ so M has order divisible by only the prime p and the primes dividing $|R|$. Thus $Q \cap M = 1$ and hence $g \notin M$. We deduce that

$$\langle H, g \rangle \cap M = H.$$

Now $e \in \langle g \rangle \cap Q$ and choice of Q implies that $Q = \langle e^Z \rangle \leq \langle H, g \rangle$. Thus $\langle H_M^Q \rangle \leq \langle H, g \rangle$, in particular, $d \in \langle H, g \rangle$. Then as $c \in \langle g \rangle$ we obtain $m \in \langle H, g \rangle \cap M = H$, contradicting $m \notin H$ and completing the proof of Theorem C.

Proof of Corollary D. Assume the corollary false. Then G is noncyclic and hence insoluble. Since M is a maximal subgroup of G and G is noncyclic we see that $M \neq 1$. Burnside's transfer lemma implies that M is not cyclic and it follows that $H \neq 1$. We have shown that $G > M > H > 1$.

Let $g \in G - M$. Then $\langle H, g \rangle \neq G$ so as M is maximal in G we see that $M \nleq \langle H, g \rangle$, and as H is maximal in M this forces $M \cap \langle H, g \rangle = H$. Thus (G, M, H) is a γ-triple.

Theorem A and the simplicity of G imply that $H_M = 1$. Thus $H > H_M$ and Theorem C implies that G is soluble, a contradiction.

4. EXAMPLES

First we have the trivial γ-triples.

EXAMPLE 1. Let p be a prime, G be an elementary abelian p-group, and let $H < M < G$. Using elementary linear algebra it follows that (G, M, H) is a γ-triple.

Next we construct some more complex examples.

EXAMPLE 2. Let p be a prime, H a cyclic p'-group, and X a faithful irreducible GF(p)H-module. Let U and W be nontrivial GF(p)H-modules all of whose composition factors are isomorphic to X. Let $V = U \oplus W$ and $G = HV$, the semidirect product of V considered as an abelian group and H considered as a group of automorphisms of V. Finally, let $M = HU$. Then (G, M, H) is a γ-triple.

Proof. Let $g \in G - M$. Since $G = HV$, there exist $h \in H$ and $v \in V - U$ such that $g = hv$. Then

$$\langle H, g\rangle \cap M = \langle H, v\rangle \cap M = H\langle v^H\rangle \cap M = H(\langle v^H\rangle \cap U).$$

Thus all we must do is prove that $\langle v^H\rangle \cap U = 0$. Choose $u \in U$ and $w \in W$ such that $v = u + w$.

Let $x \in X^{\#}$. Since H is cyclic, we see that $\mathrm{End}_H(X)$ is transitive on $X^{\#}$ and then that there are H-homomorphisms $\theta : X \to U$ and $\psi : X \to W$ such that $x\theta = u$ and $x\psi = w$. Then $\langle v^H\rangle = \{y\theta + y\psi : y \in X\}$. Since $v \notin U$ we have $w \neq 0$ and hence ψ is a monomorphism. Hence

$$\langle v^H\rangle \cap U = \{y\theta + y\psi : y \in X, y\psi = 0\} = 0$$

as required.

The next example is similar to the previous one except that $O_p(G)$ is not abelian.

EXAMPLE 3. Let p be an odd prime such that 3 does not divide $p - 1$. Let

$$P = \langle x_1, y_1, x_2, y_2, z_1, z_2 : x_i^p = y_i^p = z_i^p = [x_i, y_i] = [x_i, z_j] = 1,$$
$$[y_i, z_j] = [z_1, z_2] = 1, [x_1, x_2] = z_1,$$
$$[y_1, y_2] = z_2, [x_1, y_2] = z_1^{-1}z_2^{-1}, [x_2, y_1] = z_1 z_2\rangle.$$

Then P is a p-group of exponent p, class two, order p^6, and $[P, P] = \Phi(P) = Z(P) = \langle z_1, z_2\rangle$.

It is possible to define an automorphism α of P by

$$x_i^\alpha = y_i, \quad y_i^\alpha = x_i^{-1}y_i^{-1}, \quad z_1^\alpha = z_2, \quad \text{and} \quad z_2^\alpha = z_1^{-1}z_2^{-1}.$$

Set $H = \langle\alpha\rangle$, $G = HP$, $M = H\langle x_1, y_1\rangle$. Then (G, M, H) is a γ-triple and $H_M = 1$.

Proof. First observe that $\langle x_1, y_1\rangle$, $\langle x_2, y_2\rangle$, and $\langle z_1, z_2\rangle$ are isomorphic two-dimensional GF(p)H-modules, and as 3 does not divide $p - 1$, they are irreducible. Let $v \in P$ and suppose that $\langle x_1, y_1\rangle \leq \langle v^H\rangle$. Let $P^* = P/\langle z_1, z_2\rangle$. Then P^* is a H-module that is the direct sum of two isomorphic irreducible two-dimensional H-submodules. Since $\langle x_1, y_1\rangle \cap \langle z_1, z_2\rangle = 1$ we have $v^* \neq 0$, and a similar argument to that used in the previous example reveals that $\dim\langle v^{*H}\rangle = 2$. Thus $\langle v^H\rangle \leq \langle z_1, z_2\rangle\langle x_1, y_1\rangle$. Repeating the above argument forces $|\langle v^H\rangle| = p^2$ and thus $\langle v^H\rangle = \langle x_1, y_1\rangle$. We deduce that if $v \in P$ and $\langle x_1, y_1\rangle \leq \langle v^H\rangle$ then $v \in \langle x_1, y_1\rangle$. It now follows readily that (G, M, H) is a γ-triple. Clearly $H_M = 1$.

Note that the above construction can be carried out even if 3 divides $p - 1$; however, in this case (G, M, H) is not a γ-triple.

The following example shows that the conclusion of Theorem A, that H_M is subnormal in G, cannot be strengthened to H_M is normal in G. None of the previous examples does this.

EXAMPLE 4. Let p, P, x_i, y_i, z_i, and α be the same as in the previous example. Let P' be an isomorphic copy of P and let x_i', y_i', z_i' denote the images of x_i, y_i, z_i respectively. Let $Q = P \times P'$ and extend the action of α to Q by letting it act on P' the same as it acts on P. Let $G = \langle\alpha\rangle Q$, $A = \langle x_1, y_1\rangle$, $Z' = \langle z_1', z_2'\rangle$, $H = \langle\alpha\rangle A$, and $M = \langle\alpha\rangle AZ'$.

A similar argument to the one used in the previous example proves that (G, M, H) is a γ-triple. Also, $H_M = A \neq 1$ but $A_G = 1$. Hence (G, M, H) is irreducible, $H > H_M > 1$, and H_M is not normal in G.

Next we give an example of an irreducible γ-triple in which $H > H_M$ but in which G is not a Frobenius group. This shows that the conclusion (i) of Theorem C cannot be strengthened to G is a Frobenius group. None of the previous examples do this.

EXAMPLE 5. Let p be an odd prime and let

$$\begin{aligned} Q &= \langle x, y, z : x^p = y^p = z^p = [x, z] = [y, z] = 1, [x, y] = z\rangle \\ &= \text{the extraspecial group of order } p^3 \text{ and exponent } p; \\ V &= \langle v : v^p = 1\rangle; \\ P &= Q \times V; \end{aligned}$$

define an automorphism α of P with order two by

$$x^\alpha = x, \qquad y^\alpha = y^{-1}, \qquad z^\alpha = z^{-1}, \qquad v^\alpha = v^{-1};$$

set $G = \langle\alpha\rangle P$, $H = \langle\alpha, x\rangle$, and $M = \langle\alpha, x, v\rangle$.

It is left as an exercise to show that (G, M, H) is an irreducible γ-triple.

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