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Fixed Points of Coprime Automorphisms and Generalizations of Glauberman's Z^*-Theorem

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One of the most widely used applications of block theory to the structure theory of finite groups is Glauberman's Z^*-theorem [2], which asserts that if t is an involution of a finite group G which is not conjugate in G to any other involution of a Sylow 2-subgroup containing t, then $G = O_{2'}(G)C_G(t)$. Once this case is established, the corresponding result for arbitrary 2-elements follows without difficulty. Notice that an involution t satisfies the hypotheses above if and only if $\langle t, t^g \rangle$ is a dihedral group of twice odd order whenever $g \in G \setminus C_G(t)$. No general analogue of this result for odd primes has been proved without making use of the classification of finite simple groups, as far as we are aware, though several analogues which make use of the classification have appeared.

We here give a character-theoretic proof of a somewhat analogous result for general elements. The hypotheses of (ii) below are somewhat less natural than for the case $p = 2$, but they are in one sense analogous to the Z^*-situation.

The second author's interest in odd analogues of the Z^*-theorem was re-awoken after reading some work of the first author on deducing properties of the group generated by all conjugates of a given element from properties of the group generated by any pair of its conjugates. it turns out that the character-theoretic methods used below are also (somewhat unexpectedly) relevant to proving a conjecture of the first author from [1]. This conjecture was proved by the first author in the special case of solvable groups, and may be viewed as a "lemma of generation" useful in the construction of signalizer functors. Our two main results here are:

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THEOREM 1. *Let π be a set of primes and let x be a π-element of the finite group G. Then the following are equivalent*:

(i) $G = O_{\pi'}(G)C_G(x)$.

(ii) *For each $g \in G \setminus C_G(x)$, the subgroup $\langle x, x^g \rangle$ has a non-trivial normal π-complement, and furthermore $[O_{\pi'}(\langle x, x^g \rangle), x] \cap C_G(x) \leq O_{\pi'}(C_G(x))$.*

(iii) *For each $g \in G \setminus C_G(x)$, the subgroup $\langle x, x^g \rangle$ has a non-trivial normal π-complement, and furthermore $\langle [O_{\pi'}(\langle x, x^g \rangle), x] \cap C_G(x) : g \in G \rangle$ is a π'-group.*

Remarks. The equivalence of conditions (ii) and (iii) of Theorem 1 is almost immediate, but we feel that the formulation of (iii) is worth explicit mention. Notice that when $\pi = \{2\}$ and x is an involution, the theorem reduces to a re-statement of Glauberman's Z^*-theorem (the group exhibited in (iii) is trivial in this case). The general method of proof of the theorem breaks down in this case, so we do not offer an alternative proof of that result, and we assume it as a starting point.

THEOREM 2. *Let G be a finite group and let α be a coprime automorphism of odd order of G. Then*

$$C_{[G, \langle \alpha \rangle]}(\alpha) = \langle C_{[g, \langle \alpha \rangle]}(\alpha) : g \in [G, \langle \alpha \rangle] \rangle.$$

Remarks. By definition, $[g, \langle \alpha \rangle]$ is $\langle [g, \beta] : \beta \in \langle \alpha \rangle \rangle$. It is an α-invariant subgroup of G, and in the semi-direct product $G\langle \alpha \rangle$, we have $\langle \alpha, \alpha^g \rangle = \langle \alpha \rangle [g, \langle \alpha \rangle]$. Another useful alternative expression for $[g, \langle \alpha \rangle]$ is $[\langle \alpha, \alpha^g \rangle, \alpha]$. We note for later convenience that $[H, \alpha] = [H, \langle \alpha \rangle]$ whenever H is an α-invariant subgroup of G.

Proof of Theorem 1. We have already noted that the equivalence of (ii) and (iii) is immediate. Notice that condition (i) implies that $\langle x^g : g \in G \rangle \leq O_{\pi'}(G)\langle x \rangle$, and it follows easily that (i) implies (ii). Suppose that the hypotheses of (ii) hold for G and x, but that (i) fails, and G, x have been chosen so that first $|\langle x \rangle|$, then $|G|$, are minimized subject to this failure. It is then immediate that there is no prime p such that x_p is non-trivial and central in G, for if $\tau = \pi \setminus \{p\}$ and x_p is central in G, then x_τ inherits the hypotheses, while x_τ and x_π have the same centralizer in this situation.

We note that the hypotheses of (ii) exclude the possibility that x commutes with a distinct conjugate. In particular, $N_G(\langle x \rangle) = C_G(x)$. Notice that, by a Frattini-type argument (under the current hypotheses), the conclusion of (i) is therefore equivalent to the assertion that the subgroup of G generated by all conjugates of x has a normal π-complement and a Hall π-subgroup $\langle x \rangle$. This is, in turn, equivalent to the assertion that $[G, x]$ is a π'-group.

We next claim that if x lies in a subgroup M of G, then any other G-conjugate of x in M is already conjugate to x within M. Notice that the hypotheses imply that x is central in any π-subgroup of G containing it. To prove the claim, it suffices to consider the case that $M = \langle x, x^g \rangle$ for some $g \in G \setminus C_G(x)$. In that case, M has a Hall π-subgroup, H say, containing x. Then we have $x^{gm} \in H$ for some $m \in M$, so the hypotheses imply that $x^{gm} = x$.

Now it follows (using the hypotheses and the arguments above) that, for each $g \in G \setminus C_G(x)$, $T = \langle x, x^g \rangle$ satisfies $T = O_{\pi'}(T) C_T(x)$ (T has a normal π-complement, after all, and x is central in some Hall π-subgroup of T). Since $x^g = x^t$ for some $t \in T$, we conclude that we may assume that $t \in O_{\pi'}(T)$, so that $\langle x, x^g \rangle \leq \langle x \rangle [O_{\pi'}(T), x]$.

Letting N denote the subgroup of G generated by all conjugates of x, we conclude from the discussions above that $[N, x] = [G, x]$. The minimal choice of G forces $G = N$. Choose a prime divisor p of the order of x, and let x_p denote the p-part of x. We first claim that x_p commutes with none of its other conjugates. For choose g outside $C_G(x_p)$. Then $\langle x_p, x_p^g \rangle$ is contained in $\langle x, x^g \rangle$, so certainly has a normal p-complement by hypothesis. Also, $\langle x_p \rangle$ is a Sylow p-subgroup of $\langle x, x^g \rangle$. Suppose that x_p and x_p^g commute. Then we are forced to conclude that $g \in N_G(\langle x_p \rangle) = L$.

Now $L < G$, for otherwise $x \in C_G(x_p) \lhd G$, which forces $G = C_G(x_p)$, as G is generated by conjugates of x. This is a contradiction, as we have already seen that x_p is not central in G. Hence we have $L = O_{\pi'}(L) C_L(x)$ by the minimal choice of G. Then $[L, x_p] \leq O_{\pi'}(L) \cap \langle x_p \rangle = 1$. This contradicts the choice of g. Hence x_p and x_p^g do not commute.

We have established that x_p does not commute with any of its distinct conjugates, so that x_p is central in any p-subgroup of G containing it. Let $H = C_G(x_p)$. Then $H < G$ and $C_G(x) = C_H(x)$. By the minimality of G, we have $H = O_{\pi'}(H) C_H(x)$. Hence we have $O_{\pi'}(C_H(x)) \leq O_{\pi'}(H) \leq O_{p'}(H)$. Notice also that p does not divide $[G : C_G(x)]$.

We are now in a position to complete the proof in the case that $p = 2$. For, in that case, the fact that x_p does not commute with any of its other conjugates is already sufficient to ensure that $G = O_{p'}(G) C_G(x_p)$ by Glauberman's Z^*-theorem. Then $O_{p'}(H) \leq O_{p'}(G)$, so the argument above shows that $G = O_{p'}(G) C_G(x)$. Now G is generated by conjugates of x, so that G is solvable in this case. Let X be any Hall π-subgroup of G containing x. Then x lies in $Z(X)$. Thus x lies in $O_{\pi', \pi}(G)$. Then G has a normal π-complement, as G is generated by conjugates of x. But then G is not a counterexample after all.

From now on, then, we may assume that p is odd and that x has odd order. We wish to prove next that $G = O_{p'}(G) C_G(x)$. We will prove that, for each $g \in G$ and each irreducible character χ in the principal p-block of G, we have $[G, x] \leq \ker \chi$.

Since p does not divide $[G : C_G(x)]$, we deduce that $\chi(x) \neq 0$ by the standard congruence for the principal p-block. We will next prove that $\chi(xx^g) = \chi(x^2)$ for each $g \in G$. It suffices to consider $g \in G \backslash C_G(x)$. Fix such a choice of g and define $T = \langle x, x^g \rangle$. Then for some $t \in T$, $xx^g = xx^t = x^2(x^{-1}x^t) \in x^2[O_{\pi'}(T), x]$. Thus, within $\langle x \rangle [O_{\pi'}(T), x]$, the element xx^t is in the π-section of x^2 and has π'-part in $[O_{\pi'}(T), x]$. Hence (within the same group), it is conjugate to an element of the form x^2c, where c is an element of $[O_{\pi'}(T), x] \cap C_G(x^2)$. Since x has odd order, we see that $c \in [O_{\pi'}(T), x] \cap C_G(x)$. Thus $c \in O_{\pi'}(C_G(x))$ by hypothesis. We have seen above that $O_{\pi'}(C_G(x)) \leq O_{p'}(C_G(x_p))$. By (a well-known corollary of) Brauer's second main theorem, it follows that $\chi(x^2c) = \chi(x^2)$, as claimed. We conclude that $\chi(xx^g) = \chi(x^2)$ for our chosen character χ.

Now consideration of a representation affording χ above shows that

$$\sum_{g \in C_G(x) \backslash G} \chi(xx^g) = \frac{[G : C_G(x)]\, \chi(x)^2}{\chi(1)}.$$

From the above, we conclude that $\chi(1)\chi(x^2) = \chi(x)^2$. We may repeat this argument for each generator of $\langle x \rangle$. Notice that x^2 runs through such generators as x does, since x has odd order. Let n $(= \phi(|\langle x \rangle|))$ denote the number of such generators. Taking the product across all such generators yields $\chi(1)^n = \prod_{\{j=1,\ldots,|\langle x \rangle| : (|\langle x \rangle|, j)=1\}} \chi(x^j)$. We conclude that $|\chi(x)| = \chi(1)$ and that $[G, x] \leq \ker \chi$, as claimed. Since χ was arbitrary in the principal p-block, we conclude that $[G, x] \leq O_{p'}(G)$, as claimed.

Thus we have $G = O_{\tau'}(G)C_G(x)$, where τ is the set of all prime divisors of the order of x (for we have established that $[G, x]$ is a τ'-group). Since G is generated by conjugates of x, we see that $G = O_{\tau'}(G)\langle x \rangle$. We claim that a proof of Theorem 2 now more than suffices to complete this proof (with x in the role of α and $O_{\tau'}(G)$ in the role of G). For choose a prime p in $\pi \backslash \tau$. Then x normalizes some Sylow p-subgroup of $O_{\tau'}(G)$, say P, and our hypotheses ensure that $[P, x] = 1$. However, Theorem 2 yields

$$C_{O_{\tau'}(G)}(x) = \langle [\langle x, x^g \rangle, x] \cap C_G(x) : g \in G \rangle \leq O_{\pi'}(C_G(x))$$

under our hypotheses. Hence $P = 1$, so that $G = O_{\pi'}(G)\langle x \rangle$, as required to complete the proof of Theorem 1. We now proceed with the proof of Theorem 2.

Proof of Theorem 2. We may immediately reduce to the case that $G = [G, \alpha]$, and we do so. We will employ Glauberman correspondence in conjunction with some of the ideas above. Let $H = C_G(\alpha)$ and set $H_0 =$

$\langle C_{[g,\langle\alpha\rangle]}(\alpha): g \in G\rangle$. Notice that $H_0 \lhd H$. Let μ be an irreducible character of H with H_0 in its kernel. We need to show that μ is trivial.

By Glauberman correspondence, there is a unique irreducible character χ of $G\langle\alpha\rangle$ and a unique sign ϵ such that $\chi(\alpha^j h) = \epsilon\mu(h)$ for each $h \in H$ and each generator α^j of $\langle\alpha\rangle$. Notice, in particular, that $\chi(\alpha^j h) = \chi(\alpha^j) \neq 0$ for each $h \in H_0$ and each generator α^j of $\langle\alpha\rangle$, by the choice of μ. We will prove that $\chi(\alpha\alpha^g) = \chi(\alpha^2)$ for each $g \in G$. As in Theorem 1, this yields $\chi(1)\chi(\alpha^2) = \chi(\alpha)^2$. We could continue to argue as in Theorem 1 again, but with the more precise information above, we see immediately that $\epsilon = 1$, $\mu(1) = \chi(1)$. Hence $\alpha \in \ker\chi$, so that χ is trivial, as $G\langle\alpha\rangle$ is generated by conjugates of α. Hence μ is also trivial, as required.

We have $\alpha\alpha^g = \alpha^2[\alpha, g]$. Hence, within $\langle\alpha, \alpha^g\rangle$, $\alpha\alpha^g$ is conjugate to $\alpha^2 c$, where c is an element of $C_{[g,\langle\alpha\rangle]}(\alpha)$. But then $c \in H_0$, so we do have $\chi(\alpha\alpha^g) = \chi(\alpha^2 c) = \chi(\alpha^2)$, as required to complete the proof.

Examples. As is already noted in [1], Theorem 2 is false in general if α has even order. For example, let G be extra-special of order 27 and exponent 3, and let α be an automorphism of order 2 of G which acts without non-trivial fixed points on $G/Z(G)$ and (necessarily) centralizes $Z(G)$. Then α acts without non-trivial fixed points on $[g, \langle\alpha\rangle]$ for each $g \in G$, but α has non-trivial fixed points on $G = [G, \langle\alpha\rangle]$.

To obtain the factorization of (i) of Theorem 1, the hypotheses of (ii) cannot be weakened to the assumption that x commutes with none of its distinct conjugates. For example, let $G = C \times D$, where D is a Frobenius group of order 21 and C is cyclic of order 7. Let $\pi = \{3, 7\}$, $x \in G$ have order 21. Then x does not commute with any of its other conjugates, $x \notin Z(G)$, and $O_{\pi'}(G) = 1$.

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