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A Note on Frobenius Groups

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1. INTRODUCTION

Two celebrated applications of the character theory of finite groups are Burnside's $p^{\alpha}q^{\beta}$-Theorem and the theorem of Frobenius on the groups that bear his name. The elegant proofs of these theorems were obtained at the beginning of this century. It was then a challenge to find character-free proofs of these results. This was achieved for the $p^{\alpha}q^{\beta}$-Theorem by Bender [1], Goldschmidt [3], and Matsuyama [8] in the early 1970s. Their proofs used ideas developed by Feit and Thompson in their proof of the Odd Order Theorem. There is no known character-free proof of Frobenius' Theorem.

Recently, Corrádi and Horváth [2] have obtained some partial results in this direction. The purpose of this note is to prove some stronger results. We hope that this will stimulate more interest in this problem.

Throughout the remainder of this note, we let G be a finite Frobenius group. That is, G contains a subgroup H such that $1 \neq H \neq G$ and

$$H \cap H^{g} = 1 \qquad \text{for all } g \in G - H.$$

A subgroup with these properties is called a *Frobenius complement* of G. The *Frobenius kernel* of G, with respect to H, is defined by

$$K = \left(G - \bigcup_{g \in G} H^{g} \right) \cup \{1\}.$$

In the language of permutation groups, this corresponds to having a transitive permutation group in which any two-point stabilizer is trivial.

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The Frobenius kernel corresponds to the set of fixed-point-free elements together with the identity. Clearly, K is a normal subset of G. Character theory is required to prove the following.

THEOREM (Frobenius). *K is a subgroup.*

For the proof see [4, Theorem 4.5.1, p. 140] or [7, Theorem 5.9, p. 153].

Next we describe the other major results related to the structure of Frobenius groups that we shall need. If K is a subgroup then it is a normal subgroup and every nonidentity element of H induces a fixed-point-free automorphism of K. The following result is the Frobenius Conjecture which was proved by Thompson in his thesis. See [11–13] and [4, Theorem 10.2.1, p. 337].

THEOREM (Thompson). *A finite group that admits a fixed point free automorphism of prime order is nilpotent.*

The structure of Frobenius complements is also restricted. The following result was obtained by Burnside. See [6, Satz 8.18, p. 506].

THEOREM (Burnside). *Let L be a finite group of fixed point free automorphisms of a finite group V. Then every Sylow subgroup of L is cyclic or generalized quaternion. If L has odd order then L is metacyclic.*

Character theory is not required for the proofs of these results. We mention two further significant achievements.

THEOREM (G. Higman [5]). *Let V be a finite nilpotent group that admits a fixed point free automorphism of prime order p. Then the nilpotency class of V is bounded by a function that depends only on p.*

THEOREM (Zassenhaus [9, Theorem 18.6, p. 204]). *Let L be a finite insoluble group of fixed point free automorphisms of a finite group V. Then L has a normal subgroup L_0 of index at most two such that $L_0 \cong SL_2(5) \times M$ with M metacyclic.*

Corrádi and Horváth show that K is a subgroup provided that G has odd order and $N(D) \nsubseteq K$ for every Sylow subgroup D of G that is contained in K. Without the aid of character theory, we shall prove the following.

THEOREM A. *Suppose that G contains a subgroup $D \neq 1$ such that $D \subseteq K$, D is not a 2-group, and $N(D) \nsubseteq K$. Then K is a subgroup.*

COROLLARY B. *If G is not simple then K is a subgroup.*

COROLLARY C. *If G contains a soluble subgroup L such that $L \cap K \neq 1$, $L \nsubseteq K$, and $L \cap K$ contains an element of odd order then K is a subgroup.*

Burnside, using the fact that any two involutions generate a soluble subgroup, proved Frobenius' Theorem in the case that H has even order. Corrádi and Horváth extend Burnside's argument to prove that K is a subgroup provided that H contains an element h of order p, where $p = \min \pi(H)$, such that $\langle h, h^g \rangle$ is soluble for all $g \in G$. We shall prove the following.

THEOREM D. *Suppose that there exists $h \in H^{\#}$ such that $\langle h, h^g \rangle$ is soluble for all $g \in G$. Then K is a subgroup.*

The theorem of Frobenius, together with an elementary argument [6, Satz 8.17, p. 506], implies that G has exactly one conjugacy class of Frobenius complements. As far as the author is aware, there is no known character-free proof of this fact. However, a counting argument inspired by Corrádi and Horváth [2, Lemma 1.2] enables us to prove the following result, which we shall need.

THEOREM E. *If L is another Frobenius complement in G then there exists $g \in G$ such that $H^g \leq L$ or $L^g \leq H$.*

2. PRELIMINARIES

LEMMA 2.1. (i) *H is a Hall subgroup of G.*

(ii) $|G| = |H||K|$.

(iii) *If $k \in K^{\#}$ then $C(k) \subseteq K$.*

(iv) *If K is a subgroup then it is a normal Hall subgroup of G and $G = HK$.*

(v) *If K is a subgroup and H has odd order then H is soluble.*

Proof. Statements (i)–(iv) are well known and easy to prove. For (v) see [4, Theorem 10.3.1(vi), p. 339]. ∎

LEMMA 2.2. *Any of the following imply that K is a normal subgroup of G.*

(i) *H has even order.*

(ii) *H is soluble.*

(iii) *K contains a nontrivial normal subgroup of G.*

(iv) *K contains a nontrivial subgroup of G that is normalized by H.*

Proof. For (i) see [9, Prop. 8.3, p. 60]. Statement (ii) is the obvious transfer argument, see [6, Satz 2.4, p. 417]. Suppose that (iii) or (iv) holds. Then there exists a subgroup $D \neq 1$ such that $D \subseteq K$ and $H \leq N(D)$. Now H is a Frobenius complement in HD and the kernel of HD has order $|HD : H| = |D|$ by Lemma 2.1(ii). Since $D \subseteq K$ it follows that D is the kernel of HD. If H has odd order then Lemma 2.1(v) implies that H is soluble and then the result follows from (ii). Otherwise, apply (i). ∎

3. THE PROOF OF THEOREM E

Assume Theorem E to be false. If H has nontrivial intersection with some conjugate of L then replace L by that conjugate. Let $D = H \cap L$. By assumption, $H \not\subseteq L$ so $D \neq H$. Moreover, if $D \neq 1$ then D is a Frobenius complement in H. Let

$$\tilde{H} = \Big(H - \bigcup_{h \in H} D^h \Big) \cup \{1\}.$$

Using Lemma 2.1(ii) in the case that $D \neq 1$, we see that $|\tilde{H}| = |H : D|$.

We assert that $\tilde{H}$ has trivial intersection with every conjugate of L. Indeed, suppose that $x \in G$ is such that $H \cap L^x \neq 1$. By assumption, $H \not\subseteq L^x$ so $H \cap L^x$ is a Frobenius complement in H. By induction, $H \cap L^x \cap D^h \neq 1$ for some $h \in H$. Then $L^x \cap L^h \neq 1$ so $L^x = L^h$. Now $h \in H$ whence $H \cap L^x = H \cap L^h = D^h$ and we deduce that $\tilde{H} \cap L^x = 1$. This proves the assertion.

Since $\tilde{H}$ is a normal subset of the Frobenius complement H, we see that $\tilde{H}$ has trivial intersection with its conjugates and that $N(\tilde{H}) = H$. This, together with the previous assertion and the fact that L has trivial intersection with its conjugates, implies that

$$|G| > |G : H|\big(|\tilde{H}| - 1\big) + |G : L|\big(|L| - 1\big).$$

Dividing through by $|G|$ and using $|\tilde{H}| = |H : D|$, we obtain

$$1 > \frac{1}{|D|} - \frac{1}{|H|} + 1 - \frac{1}{|L|},$$

whence

$$\frac{|D|}{|H|} + \frac{|D|}{|L|} > 1.$$

Recall that $D = H \cap L$. It follows that $D = H$ or $D = L$ and then that $H \leq L$ or $L \leq H$, a contradiction. ∎

COROLLARY 3.1. *Let L be a subgroup of G such that $H \cap L \neq 1$ and $L \nleq H$. Then L is a Frobenius group with complement $H \cap L$ and whose kernel with respect to $H \cap L$ is $K \cap L$.*

Proof. Set $H_0 = H \cap L$. Then H_0 is a Frobenius complement in L. Let $K_0 = (L - \bigcup_{l \in L} H_0^l) \cup \{1\}$. Now $K = (G - \bigcup_{g \in G} H^g) \cup \{1\}$ so $K \cap L \subseteq K_0$. Let $x \in G$ be such that $H^x \cap L \neq 1$. Now $H^x \cap L \neq L$ since otherwise $H^x = H$ and then $L \leq H$, a contradiction. Thus $H^x \cap L$ is also a Frobenius complement in L. Theorem E implies that $(H^x \cap L) \cap (H \cap L)^l \neq 1$ for some $l \in L$. Then $H^x \cap H^l \neq 1$ so $H^x = H^l$ and then $H^x \cap L = H_0^l$. In particular, $H^x \cap K_0 = 1$. It follows that $K \cap L = K_0$. ∎

COROLLARY 3.2 (Corrádi and Horváth). *Any $\pi(H)$-subgroup of G is conjugate to a subgroup of H.*

Proof. Let $D \neq 1$ be a $\pi(H)$-subgroup of G. Using Sylow's Theorem and the fact that H is a Hall subgroup of G we may suppose that $D \cap H \neq 1$. Moreover, every element of K has order coprime to $|H|$ so $D \cap K = 1$. Lemma 2.1(ii) implies that Frobenius kernels have cardinality greater than one so using Corollary 3.1 we deduce that $D \leq H$. ∎

COROLLARY 3.3. *Let $P \neq 1$ be a subgroup of G with $P \subseteq K$. Then $N(P) \cap K$ is a normal subgroup of $N(P)$.*

Proof. We may suppose that $N(P) \nsubseteq K$ and then that $N(P) \cap H \neq 1$. Now apply Corollary 3.1 and Lemma 2.2(iii). ∎

Proof of Theorem D. Let $g \in G - H$ and set $L = \langle h, h^g \rangle$. Then by Corollary 3.1, $H \cap L$ is a Frobenius complement in L and the corresponding kernel is $K \cap L$. By hypothesis, L is soluble so Lemma 2.2(ii) implies that $K \cap L$ is a subgroup and then Lemma 2.1(iv) implies that $L = (H \cap L)(K \cap L)$. Also, $H^g \cap L$ is a Frobenius complement in L so by Theorem E there exists $l \in L$ such that $(H \cap L)^l \cap (H^g \cap L) \neq 1$. Since $L = (H \cap L)(K \cap L)$ we may choose l so that $l \in K \cap L$. Now $H^l \cap H^g \neq 1$ whence $gl^{-1} \in H$. We deduce that $G = HK$.

Now $|G : H| = |K|$ by Lemma 2.1(ii) so what we have just proved implies that K is a transversal to H in G. Thus if x and y are distinct elements of K then $xy^{-1} \notin H$. Since K is a normal subset of G, it follows that xy^{-1} is not contained in any conjugate of H and then that $xy^{-1} \in K$. This proves that K is a subgroup. ∎

4. THE PROOF OF THEOREM A

Throughout this section, we assume the hypothesis of Theorem A. Replacing D by a suitable conjugate, we may suppose that $H \cap N(D) \neq 1$. Corollary 3.3 implies that $N(D) \cap K$ is a normal subgroup of $N(D)$. Thus we may choose a subgroup $M \leq G$, maximal subject to

(i) $M \cap K \neq 1$ and $M \cap H \neq 1$,

(ii) $M \cap K$ is a subgroup, and

(iii) $M \cap K$ is not a 2-group.

Let

$$F = M \cap K \qquad \text{and} \qquad \pi = \pi(F).$$

LEMMA 4.1. (i) *F is a nilpotent Hall π-subgroup of G.*

(ii) *If $P \neq 1$ is a characteristic subgroup of F then $M = N(P)$.*

(iii) *$F \trianglelefteq M$ and every π-subgroup of M is contained in F.*

Proof. Corollary 3.1 implies that M is a Frobenius group with kernel F. By construction, F is a subgroup so Lemma 2.1(iv) implies that F is a normal Hall π-subgroup of M. This proves (iii). Suppose that $P \neq 1$ is a characteristic subgroup of F. Then $M \leq N(P)$. Now $N(P) \cap K$ is a subgroup by Corollary 3.3 so the maximal choice of M forces $M = N(P)$. This proves (ii).

Thompson's Theorem implies that F is nilpotent. Let $p \in \pi$. Since F is a nilpotent Hall π-subgroup of M we have $\mathscr{O}_p(F) \in \mathrm{Syl}_p(M)$. Now $N(\mathscr{O}_p(F)) = M$ by (ii) from which it follows that $\mathscr{O}_p(F) \in \mathrm{Syl}_p(G)$. This proves (i). ∎

We recall the following extension of Sylow's theorem due to Wielandt.

THEOREM 4.2. *Let X be a finite group that possesses a nilpotent Hall π-subgroup. Then the following hold.*

(W1) *X has a single conjugacy class of Hall π-subgroups.*

(W2) *Every π-subgroup of X is contained in a Hall π-subgroup of X.*

(W3) *If P is a π-subgroup of X then $N(P)$ possesses a nilpotent Hall π-subgroup.*

(W4) *If $Y \trianglelefteq X$ and T is a Hall π-subgroup of X then $T \cap Y$ is a nilpotent Hall π-subgroup of Y.*

Proof. For W1–W3 see [10, Theorem 3.2, p. 166, and Example 2, p. 186]. W4 follows from the corresponding assertion for Sylow subgroups. ∎

We recall also that a group X is π-nilpotent if $X/\mathscr{O}_{\pi'}(X)$ is a π-group. If, in addition, X has a Hall π-subgroup T then $X = T\mathscr{O}_{\pi'}(X)$.

LEMMA 4.3. *Let L be a subgroup of G with $L \subseteq K$. Then L is π-nilpotent.*

Proof. It suffices to show that L is p-nilpotent for all $p \in \pi$. Choose $p \in \pi$.

Case (i). $|\pi| \geq 2$. Let $P \neq 1$ be a p-subgroup of L. Since $\mathscr{O}_p(F) \in \mathrm{Syl}_p(G)$, we may replace L by a suitable conjugate to suppose that $P \leq \mathscr{O}_p(F)$. Let $N = N(P) \cap K$. Note that N is a subgroup by Corollary 3.3 and also that $C(P) \trianglelefteq N$ by Lemma 2.1(iii). Choose $q \in \pi - \{p\}$. Then $\mathscr{O}_q(F) \in \mathrm{Syl}_q(G)$ by Lemma 4.1(i) and then the Frattini Argument yields

$$N = C(P)\big(N \cap N\big(\mathscr{O}_q(F)\big)\big).$$

Using Lemma 4.1(ii), (i) we obtain $N \cap N(\mathscr{O}_q(F)) \leq F = \mathscr{O}_{p'}(F) \times \mathscr{O}_p(F)$ and hence $N = C(P)N_{\mathscr{O}_p(F)}(P)$. Consequently, $N/C(P)$ is a p-group. Since $L \subseteq K$, we deduce that $N_L(P)/C_L(P)$ is a p-group for all nontrivial p-subgroups $P \leq L$. Frobenius' Normal p-Complement Theorem [4, Theorem 7.4.5, p. 253] implies that L is p-nilpotent.

Case (ii). $|\pi| = 1$. Then $F \in \mathrm{Syl}_p(G)$ and $p \neq 2$ since the choice of M implies that F is not a 2-group. Choose $P \in \mathrm{Syl}_p(L)$. We may assume that $P \neq 1$ and we shall proceed by induction on $|F|/|P|$. Replacing L by a suitable conjugate, we may suppose that $P \leq F$. We use $J(P)$ to denote the Thompson subgroup of P as defined in [13] and note that it is a characteristic subgroup of P. Suppose that $P = F$. Using Lemma 4.1(ii) we have $N_L(J(P)) \leq K \cap M = F$ and also $C_L(Z(P)) \leq F$. Hence $N_L(J(P))$ and $C_L(Z(P))$ are p-nilpotent. Thompson's Normal p-Complement Theorem [13] implies that L is p-nilpotent.

Suppose that $P < F$. Now $P < N_F(P) \leq N_F(J(P)) \leq N(J(P)) \cap K$ so Corollary 3.3 and induction imply that $N(J(P)) \cap K$ is a p-nilpotent subgroup. Thus $N_L(J(P))$ is p-nilpotent. Similarly, $N_L(Z(P))$ and hence $C_L(Z(P))$ are p-nilpotent. Again it follows that L is p-nilpotent. ∎

LEMMA 4.4. *Let P be a nontrivial subgroup of F. Then*

(i) *$N_F(P)$ is a Hall π-subgroup of $N(P)$, and*

(ii) *$N(P) = \mathscr{O}_{\pi'}(N(P))(N(P) \cap M)$.*

Proof. We proceed by induction on $|F : P|$. If $P = F$ then $N(P) = M$ and the result follows. Suppose that $P < F$. Let $Q = N_F(P)$, so that $Q > P$. By induction we have

$$N(Q) = \mathscr{O}_{\pi'}(N(Q))(N(Q) \cap M). \tag{1}$$

Since $P < Q$ and since $C(P) \subseteq K$ by Lemma 2.1(iii), we also have

$$\mathscr{O}_{\pi'}(N(Q)) \leq C(Q) \leq N(P) \cap K. \tag{2}$$

It follows from (1) and (2) that

$$N(P) \cap N(Q) = \mathscr{O}_{\pi'}(N(Q))(N(P) \cap N(Q) \cap M). \tag{3}$$

Using Lemma 4.1(iii) we see that $N(P) \cap N(Q) \cap F$ is a Hall π-subgroup of $N(P) \cap N(Q) \cap M$ and hence of $N(P) \cap N(Q)$. Recall that $Q = N_F(P)$, so $Q = N(P) \cap N(Q) \cap F$ and thus Q is a Hall π-subgroup of $N(P) \cap N(Q)$. Now $N(P)$ possesses a nilpotent Hall π-subgroup by Theorem 4.2(W3), from which it follows that Q is a Hall π-subgroup of $N(P)$. This proves (i).

By Corollary 3.3 we have $N(P) \cap K \trianglelefteq N(P)$ so (i) and the Frattini Argument yield

$$N(P) = (N(P) \cap K)(N(P) \cap N(Q)).$$

Using (3) and (2) we obtain

$$\begin{aligned} N(P) &= (N(P) \cap K)(N(P) \cap N(Q) \cap M) \\ &= (N(P) \cap K)(N(P) \cap M). \end{aligned}$$

Now $N(P) \cap K$ is π-nilpotent by Lemma 4.3 and Q is a Hall π-subgroup of $N(P) \cap K$. Consequently, $N(P) \cap K = \mathscr{O}_{\pi'}(N(P) \cap K)Q$. Since $\mathscr{O}_{\pi'}(N(P) \cap K) \leq \mathscr{O}_{\pi'}(N(P))$ and $Q \leq N(P) \cap M$ we deduce that $N(P) = \mathscr{O}_{\pi'}(N(P))(N(P) \cap M)$, which proves (ii). ∎

LEMMA 4.5. *Any two elements of F that are conjugate in G are already conjugate in M.*

Proof. Let $f \in F$ and $g \in G$ be such that $f^g \in F$. Now $Z(F)^g \leq C(f^g)$ and Lemma 4.3, together with Theorem 4.2(W4), imply that $C_F(f^g)$ is a nilpotent Hall π-subgroup of $C(f^g)$. Thus there exists $c \in C(f^g)$ such that $Z(F)^{gc} \leq F$. It suffices to show that $gc \in M$. Now $N(Z(F)^{gc}) = M^{gc}$ and hence F^{gc} is the unique Hall π-subgroup of $N(Z(F)^{gc})$. Moreover, $N_F(Z(F)^{gc})$ is a Hall π-subgroup of $N(Z(F)^{gc})$ by Lemma 4.4(i). We deduce that $F^{gc} = F$ and then that $gc \in N(F) = M$, as desired. ∎

LEMMA 4.6. $H \leq M$.

Proof. Let $\{f_1, \dots, f_\alpha\}$ be a set of representatives for the conjugacy classes of nontrivial π-elements of G. By Theorem 4.2(W2 and W1) we may suppose that $f_i \in F$ for all i.

For each i let

$$T_i = f_i \mathscr{O}_{\pi'}(C(f_i)),$$

and note that $T_i \subseteq C(f_i) \subseteq K$. Recall that each element of G can be expressed uniquely as a commuting product of a π-element and a π'-element. It follows that for all i, j and all $g \in G$ that

$$T_i \cap T_j^g \neq \varnothing \qquad \text{if and only if } i = j \text{ and } g \in C(f_i).$$

Thus

$$|K| > \sum_{i=1}^{\alpha} \left|G : C(f_i)\right| |T_i|.$$

It follows from Lemma 4.4 that $C(f_i) = \mathcal{O}_{\pi'}(C(f_i))C_F(f_i)$ for each i. Moreover, $|G| = |H||K|$ by Lemma 2.1(ii). Consequently

$$\begin{aligned} 1 &> \sum_{i=1}^{\alpha} |H| / \left|C_F(f_i)\right| \\ &= \frac{|H : H \cap M|}{|F|} \sum_{i=1}^{\alpha} |H \cap M| |F| / \left|C_F(f_i)\right|. \end{aligned}$$

We chose M such that $H \cap M \neq 1$ so Corollary 3.1 and Lemma 2.1(ii) imply that $|M| = |H \cap M| |F|$. By Lemma 2.1(iii) we have $C_F(f_i) = C_M(f_i)$ for all i, whence

$$1 > \frac{|H : H \cap M|}{|F|} \sum_{i=1}^{\alpha} \left|M : C_M(f_i)\right|.$$

Lemma 4.5 implies that $\{f_1, \dots, f_\alpha\}$ is a set of representatives for the conjugacy classes of nontrivial π-elements of M. By Lemma 4.1(iii), the set of π-elements of M is F whence

$$\begin{aligned} 1 &> \frac{|H : H \cap M|}{|F|}(|F| - 1) \\ &= |H : H \cap M|\left(1 - \frac{1}{|F|}\right). \end{aligned}$$

Now $|F| \geq 2$ so $2 > |H : H \cap M|$ and then $H \leq M$ as required. ∎

An application of Lemma 2.2(iv) completes the proof of theorem A.

Proof of Corollary B. Assume false and choose $L \trianglelefteq G$ such that $1 \neq L \neq G$. Then $L \nsubseteq K$ by Lemma 2.2(iv) so as $L \trianglelefteq G$ we deduce that $L \cap H \neq 1$. Now $L \cap H \neq L$ since H cannot contain a nontrivial normal subgroup of G. Thus $L \cap H$ is a Frobenius complement in L so Corollary 3.2 and the Frattini Argument imply that $G = LN(L \cap H)$. Consequently,

$$G = LH. \tag{4}$$

Let p be any prime divisor of $|G : H|$ and choose $P \in \mathrm{Syl}_p(G)$. Since H is a Hall subgroup of G we have $P \subseteq K$ and using (4) we also have $P \leq L$. Thus $G = LN(P)$ and then

$$\left|N(P) : N_L(P)\right| = |G : L| = |H : H \cap L|.$$

Now $|G:L| \neq 1$ and H is a Hall subgroup of G so it follows that $N(P) \not\subseteq K$. Theorem A implies that $p = 2$. We deduce that $|G:H|$ is a power of two and then that $|G| = |H||P|$. Now $P \subseteq K$ and $|G| = |H||K|$ whence $P = K$. But P is a subgroup, a contradiction. ∎

Proof of Corollary C. This follows from Corollary 3.1, Lemma 2.2(ii), and Theorem A. ∎

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