On the Fitting height of a soluble group that is generated by a conjugacy class

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Let G be a finite group and suppose that P is a soluble $\{2,3\}'$-subgroup of G. The reader will lose only a little by assuming that P is a subgroup of prime order $p > 3$. Define

$$\Sigma_G(P) \;=\; \{A \leq G \mid A \text{ is soluble and } A = \langle P, P^a\rangle \text{ for some } a \in A\}.$$

This set is partially ordered by inclusion and we let

$$\Sigma^*_G(P)$$

denote the set of maximal members of $\Sigma_G(P)$.

This article grew out of the following discovery:

Theorem A. *Let G be a finite group, let P be a soluble $\{2,3\}'$-subgroup of G and choose $A \in \Sigma^*_G(P)$. Then*

$$F(A)V$$

is nilpotent for every nilpotent subgroup V of G that is normalized by A.

The article of Bender [1] indicates the usefulness of such results in the study of finite groups. An immediate consequence is the following:

Corollary B. *Assume the hypotheses of Theorem A and that G is soluble. Then*

$$\pi(F(A)) \;\subseteq\; \pi(F(G)).$$

Thus at least when G is soluble, the members of $\Sigma_G^*(P)$ reflect the global properties of G. This is a little surprising since, in some senses, the members of $\Sigma_G^*(P)$ can be small compared to G. Indeed if P is cyclic then every member of $\Sigma_G^*(P)$ has cyclic abelianization.

It is tempting to conjecture that the conclusion of Corollary B can be replaced by the stronger assertion that

$$F(A) \;\leq\; F(G).$$

This is false. However, by considering the notion of Fitting height, it is possible to prove a result that is just as good.

The following consequence of Theorem A is the starting point.

Theorem C. *Let G be a finite soluble group and let $\mathcal{C}$ be a conjugacy class of $\{2,3\}'$-subgroups of G. If G is generated by $\mathcal{C}$ then there exist two members of $\mathcal{C}$ that generate a subgroup with the same Fitting height as G.*

Moreover, the two members of $\mathcal{C}$ may be chosen to be conjugate in the subgroup that they generate.

This leads us to define:

$$\Sigma_G^f(P)$$

to be the set consisting of those members of $\Sigma_G(P)$ with maximal Fitting height. Moreover, if $G \neq 1$ is soluble we let

$$\psi(G) \;=\; \bigcap\{\; K \trianglelefteq G \mid f(G/K) < f(G) \;\},$$

where $f(G)$ denotes the Fitting height of G. Then $\psi(G)$ is the unique smallest normal subgroup of G such that $f(G/\psi(G)) < f(G)$ and it is also the case that $1 \neq \psi(G) \leq F(G)$. If $G = 1$ then we let $\psi(G) = 1$. Using Theorem C we obtain:

Theorem D. *Let P be a $\{2,3\}'$-subgroup of the finite soluble group G and choose $A \in \Sigma_G^f(P)$. Then*
$$\psi(A) \;\leq\; F(G).$$

Combining this with the Baer-Suzuki Theorem we obtain:

Corollary E. *Let $\mathcal{C}$ be a conjugacy class of $\{2,3\}'$-subgroups of the finite group G. Then $\mathcal{C}$ generates a soluble subgroup if and only if every four members of $\mathcal{C}$ generate a soluble subgroup.*

Assume now the hypotheses of Theorem D. The conclusion of that theorem says that we can write down a subnormal nilpotent subgroup of G just by examining the subgroups that are generated by two conjugates of P. If $G = \langle P^G \rangle$ then we can go a little further:

Theorem F. *Let P be a $\{2,3\}'$-subgroup of the finite soluble group G and suppose that $G = \langle P^G \rangle$. Then*

$$\psi(G) \;=\; \left\langle \psi(A) \mid A \in \Sigma_G^f(P) \right\rangle.$$

In other words, we can write down a characteristic nilpotent subgroup of G in terms of subgroups that are 'localized' at P.

1 Preliminaries

Henceforth the word 'group' shall mean 'finite group'.

Lemma 1.1. *Let P be a soluble $\{2,3\}'$-subgroup of the group G, let V be a soluble normal subgroup of G, set $\overline{G} = G/V$ and suppose that $\overline{A} \in \Sigma_{\overline{G}}(\overline{P})$. Then $\overline{A}$ has an inverse image which is a member of $\Sigma_G(P)$.*

Proof. Choose $a \in G$ with $\overline{a} \in \langle \overline{P}, \overline{P}^{\overline{a}} \rangle = \overline{A}$ and $\langle P, P^a \rangle$ minimal. Then $\langle P, P^a \rangle \in \Sigma_G(P)$. □

For a group G we let $F(G)$ be the Fitting subgroup of G, that is, the largest normal nilpotent subgroup of G. For a prime q we let $\mathcal{O}_q(G)$ be the largest normal q-subgroup of G.

If G is soluble we let $f(G)$ denote the Fitting height of G. This is the smallest integer n such that G possesses a series

$$1 = F_0 \trianglelefteq F_1 \trianglelefteq \cdots \trianglelefteq F_n = G$$

with F_{i+1}/F_i nilpotent for all i. If $G \neq 1$ then $f(G/F(G)) = f(G) - 1$.

The following two results are elementary.

Lemma 1.2. *Let $H \leq G$ with G soluble and $f(H) = f(G)$. Then*

$$\psi(H) \leq \psi(G) \leq F(G).$$

Lemma 1.3. *Let G be soluble, let $N \trianglelefteq G$, set $\overline{G} = G/N$ and suppose that $\overline{G} \neq 1$. Then the following are equivalent:*

(i) $\overline{\psi(G)} \neq 1$.

(ii) $f(\overline{G}) = f(G)$.

(iii) $\psi(\overline{G}) = \overline{\psi(G)}$.

2 Modules for soluble groups

Lemma 2.1. *Let G be a group, let $V \neq 1$ be an irreducible G-module over an algebraically closed field and let $K \trianglelefteq G$ be such that G/K is cyclic. If V is homogeneous as a K-module then it is irreducible as a K-module.*

Proof. This follows from [2, Theorems 11.20, p.278 and 11.46, p.303]. $\square$

Lemma 2.2. *Let Q be an extraspecial q-group of exponent q and order q^{1+2t}. Suppose that A is a noncyclic abelian normal subgroup of Q with order q^{1+k}. Then the following hold:*

(a) *$Z(Q) \leq A$ and A possesses exactly q^k hyperplanes $A_1, \ldots, A_{q^k}$ that do not contain $Z(Q)$. These hyperplanes are permuted transitively by Q.*

(b) *Let V be a faithful homogeneous Q-module over a field of characteristic prime to q. For each i set $V_i = C_V(A_i)$ and set $\Omega = \{V_1, \ldots, V_{q^k}\}$. Then Ω is permuted transitively by Q,*

$$V = V_1 \oplus \cdots \oplus V_{q^k}$$

and $C_A(V_i) = A_i$ for all i.

(c) *Let the q-group P act as a group of automorphisms of Q. Then Q possesses a P-invariant abelian normal subgroup with order q^{1+t}.*

Proof. Let $\overline{Q} = Q/Z(Q)$. Since Q is extraspecial we have $Z(Q) = Q' = \Phi(Q) \cong \mathbf{Z}_q$ so $\overline{Q}$ may be regarded as a $GF(q)$-vectorspace and the map $(\ ,\ ) : \overline{Q} \times \overline{Q} \longrightarrow Z(Q)$ defined by

$$(Z(Q)x, Z(Q)y) \;=\; [x, y]$$

is a nondegenerate symplectic form on $\overline{Q}$.

(a). The first assertion is true since $Z(Q) = Q' \cong \mathbf{Z}_q$ and $1 \neq A \trianglelefteq Q$. The second assertion follows from a counting argument and the fact that A is elementary abelian. Let $B = A_1$. The group $\overline{Q}$ acts by conjugation on the set of hyperplanes of A. Since $Z(Q) \cap B = 1$ it follows that

$$N_{\overline{Q}}(B) \;=\; \overline{B}^{\perp},$$

so B has $|\overline{Q} : \overline{B}^{\perp}|$ conjugates. Now $\operatorname{codim} \overline{B}^{\perp} = \dim \overline{B}$ so $|\overline{Q} : \overline{B}^{\perp}| = |\overline{B}|$. As $Z(Q) \cap B = 1$ we have $|\overline{B}| = q^k$. Consequently, B has q^k conjugates. This proves (a).

(b). The first assertion follows from (a). We may suppose that Q acts irreducibly on V. In particular, $C_V(Z(Q)) = 0$. Let $U \leq V$ be an irreducible A-module. Since A is elementary abelian it follows that $C_A(U)$ is a hyperplane of A. As $C_V(Z(Q)) = 0$ we have $C_A(U) = A_i$ for some i. Then $\langle \Omega \rangle \neq 0$ and then the irreducibility of Q on V forces $V = \langle \Omega \rangle$.

If $i \neq j$ then $V_i \cap V_j \leq C_V(A) = 0$. Let $i < q^k$ be such that $V_1 + \cdots + V_i = V_1 \oplus \cdots \oplus V_i$. Now A is abelian so it normalizes each V_j and as $V_{i+1} = C_V(A_{i+1})$ we have

$$V_{i+1} \cap (V_1 \oplus \cdots \oplus V_i) = (V_{i+1} \cap V_1) \oplus \cdots \oplus (V_{i+1} \cap V_i) = 0.$$

We deduce that $V = V_1 \oplus \cdots \oplus V_{q^k}$. The final assertion in (b) follows from $C_V(Z(Q)) = 0$.

(c). Since $Z(Q) \cong \mathbf{Z}_q$ we have $[Z(Q), P] = 1$ so P acts as a group of isometries on $\overline{Q}$. Let U be a maximal P-invariant isotropic subspace of $\overline{Q}$. Suppose that $U < U^{\perp}$. Then P acts on the nontrivial $GF(q)$-vectorspace $U^{\perp}/U$. Since P is a q-group, it fixes a nonzero vector $v + U \in U^{\perp}/U$. Then $U \oplus \langle v \rangle$ is a P-invariant isotropic subspace, contrary to the maximal choice of U. Thus $U = U^{\perp}$ and then $\dim U = t$. The inverse image of U in Q has the desired properties. $\square$

Lemma 2.3. *Let G be a soluble nonnilpotent primitive linear group over an algebraically closed field. Then there exists $Q \trianglelefteq G$ such that Q is an extraspecial q-group and G acts nontrivially and irreducibly on $Q/\Phi(Q)$. Moreover, if $q \neq 2$ then Q has exponent q.*

Proof. Let $\overline{G} = G/F(G)$ and choose a prime r such that $\mathcal{O}_r(\overline{G}) \neq 1$, let K be the inverse image of $\mathcal{O}_r(\overline{G})$ in G and choose $R \in \mathrm{Syl}_r(K)$. Then $K = R\mathcal{O}_{r'}(F(G))$ and since $R \nleq \mathcal{O}_r(G)$ we have $[\mathcal{O}_{r'}(F(G)), R] \neq 1$. Choose a prime $q \neq r$ such that $[\mathcal{O}_q(G), R] \neq 1$ and set $S = [\mathcal{O}_q(G), R]$. Note that $S = [S, R]$. The Frattini Argument implies that

$$G \;=\; N_G(R)\mathcal{O}_{r'}(F(G))$$

and it follows that $S \trianglelefteq G$.

Let Q be a subgroup of S that is minimal subject to $Q \trianglelefteq G$ and $[Q, R] \neq 1$. The hypotheses imply that every abelian normal subgroup of G is cyclic and contained in $Z(G)$. In particular, Q is nonabelian. Suppose that $T < Q$ with $T \trianglelefteq G$. The minimality of Q implies that R acts trivially on T and as $S = [S, R]$ it follows that S acts trivially on T, whence $T \leq Z(S)$. We deduce that

$$1 \neq Q' \leq \Phi(Q) \leq Z(Q) \leq Z(S) \leq Z(G). \tag{1}$$

Since S is nilpotent we have $[Q, S] < Q$ whence

$$[Q, S] \leq Z(S) \leq Z(G)$$

also.

We claim that $Q' = [Q, S] \cong \mathbf{Z}_q$. Let $x \in Q$ and $y \in S$. Since $[Q, S] \leq Z(S)$ we have $[x, y]^q = [x^q, y]$. But $x^q \in \Phi(Q) \leq Z(S)$ whence $[x, y]^q = 1$. Since $Z(S)$ is cyclic and since $Q' \neq 1$ the claim follows.

Now $[C_Q(R), S] \leq [Q, S] \leq C_Q(R)$ so $C_Q(R) \trianglelefteq S$. As $S = [S, R]$ it follows that $C_Q(R) \leq Z(S)$. Since $Q = C_Q(R)[Q, R]$ we have $Q = Z(Q)[Q, R]$ and then $Q' = [Q, R]'$. By the previous paragraph we have $Q' = [Q, S]$ whence $[Q, S] \leq [Q, R]$ and in particular $[Q, R] \trianglelefteq S$. Recall that $G = N_G(R)\mathcal{O}_{r'}(F(G))$ and that $Q \leq S = [\mathcal{O}_q(G), R]$. It follows that $[Q, R] \trianglelefteq G$ and as $1 \neq [Q, R] = [Q, R, R]$, the minimal choice of Q yields

$$Q = [Q, R].$$

Let $Q^* = Q/Q'$ and consider the action of G on Q^*. Now $Q^* = [Q^*, R]$ and since Q^* is abelian we have $C_{Q^*}(R) = 1$ by [3, Theorem 5.2.3, p.177]. Suppose that T^* is a proper G-invariant subgroup of Q^*. Let T be the inverse image of T^* in G. The minimal choice of Q implies that $[T, R] = 1$ whence $T^* \leq C_{Q^*}(R) = 1$. This implies that Q^* is elementary abelian and then that the action of G on Q^* is irreducible. This action is nontrivial since $Q^* = [Q^*, R]$. From (1) we have $1 \neq Q' \leq \Phi(Q) \leq Z(Q) < Q$ whence $Q' = \Phi(Q) = Z(Q)$. We have seen that $Q' \cong \mathbf{Z}_q$ so Q is extraspecial. If $q \neq 2$ then by [3, Theorem 5.3.10, p.184] and the minimal choice of Q we have $Q = \Omega_1(Q)$ and then the fact that $Q' \leq Z(Q)$ implies that Q has exponent q. □

Theorem 2.4. *Let G be a soluble group, suppose that P is a $\{2,3\}'$-subgroup of G such that $G = \langle P^G \rangle$, suppose that $V \neq 0$ is a G-module that does not involve the trivial G-module and let $p = \min \pi(P)$. Then*

$$\dim C_V(P) \;\leq\; \frac{2}{p} \dim V.$$

Proof. Assume false and consider a counterexample in which $|G| + |P|$ is minimized and then $\dim V$ is minimized. We may suppose that F, the underlying field for V, is algebraically closed and that G acts irreducibly on V. We have

$$\dim C_V(P) \;>\; \frac{2}{p} \dim V.$$

Step 1 *Let $\Omega = \{V_1, \dots, V_n\}$ be a P-invariant collection of subspaces of V, all of the same dimension, such that*

$$V \;=\; V_1 \oplus \cdots \oplus V_n.$$

Let $m = |\mathrm{Fix}_\Omega(P)|$. Then

$$n \;<\; pm.$$

Proof. Let $\Omega_1, \dots, \Omega_k$ be the orbits of P on Ω. Then

$$\dim C_V(P) \;=\; \sum_{i=1}^{k} \dim C_{\langle \Omega_i \rangle}(P) \;\leq\; \sum_{i=1}^{k} \frac{1}{|\Omega_i|} \dim \langle \Omega_i \rangle.$$

We may suppose that $\Omega_1, \dots, \Omega_m$ are the orbits of size 1, so the other orbits all have size at least p. Then

$$\begin{aligned}
\dim C_V(P) \;&\leq\; m \dim V_1 \;+\; \frac{1}{p} \sum_{i=m+1}^{k} \dim \langle \Omega_i \rangle \\
&\leq\; m \dim V_1 \;+\; \frac{1}{p} \left(\dim V - m \dim V_1 \right).
\end{aligned}$$

Now $\dim C_V(P) > \frac{2}{p} \dim V$ whence

$$\frac{1}{p} \dim V \;\leq\; m \left(1 - \frac{1}{p} \right) \dim V_1.$$

Since $\dim V = n \dim V_1$, the result follows. □

Step 2 *G is primitive on V. In particular, every normal subgroup of G is homogeneous on V and $Z(G)$ is the unique maximal abelian normal subgroup of G.*

Proof. Assume false. Then there exists a collection $\Omega = \{V_1, \ldots, V_n\}$ of subspaces of V that are permuted transitively by G such that

$$V \;=\; V_1 \oplus \cdots \oplus V_n \quad \text{and} \quad n \geq 2.$$

Choose such a collection with n minimal. Let

$$K = \ker(G \longrightarrow S_\Omega) \quad \text{and} \quad \overline{G} = G/K.$$

Then $\overline{G}$ is a faithful primitive permutation group on Ω. Let $\overline{L}$ be a minimal normal subgroup of $\overline{G}$. Since $\overline{G}$ is soluble and primitive on Ω it follows that $\overline{L}$ is an elementary abelian l-group for some prime l, that $\overline{L}$ is regular on Ω and that

$$\overline{G} = \mathrm{Stab}_{\overline{G}}(U)\overline{L} \quad \text{and} \quad \mathrm{Stab}_{\overline{G}}(U) \cap \overline{L} = 1 \quad \text{for all} \quad U \in \Omega.$$

Step 1 implies that $\mathrm{Fix}_\Omega(\overline{P}) \neq \emptyset$. We claim that $C_{\overline{L}}(\overline{P})$ acts regularly on $\mathrm{Fix}_\Omega(\overline{P})$. Let $U, W \in \mathrm{Fix}_\Omega(\overline{P})$. There exists $\overline{g} \in \overline{L}$ such that $W\overline{g} = U$. Then $\overline{P}, \overline{P}^{\overline{g}} \leq \mathrm{Stab}_{\overline{G}}(U)$ so

$$[\overline{g}, \overline{P}] \;\leq\; \mathrm{Stab}_{\overline{G}}(U) \cap \overline{L} = 1$$

and hence $\overline{g} \in C_{\overline{L}}(\overline{P})$. Since $\overline{L}$ is regular on Ω we have proved the claim.

Now $|\Omega| = |\overline{L}|$ and $|\mathrm{Fix}_\Omega(\overline{P})| = |C_{\overline{L}}(\overline{P})|$ so Step 1 implies that

$$|\overline{L} : C_{\overline{L}}(\overline{P})| \;<\; p.$$

Using the fact that every nonidentity element of $\overline{P}$ has order at least p, it follows that $\overline{L} = C_{\overline{L}}(\overline{P})$ and then that $\overline{P}$ acts trivially on Ω. Since $\overline{G} = \langle \overline{P}^{\overline{G}} \rangle$ and since $\overline{G}$ acts transitively on Ω, we have obtained a contradiction. Thus every normal subgroup of G acts homogeneously on V.

The final two assertions follow from the first. □

Now G is irreducible on V so $C_V(G) = 0$ whence $P < G$. Since $G = \langle P^G \rangle$, it follows that G is not nilpotent. Step 2 and Lemma 2.3 imply that there exists a prime q and $Q \trianglelefteq G$ such that Q is an extraspecial q-group and that G acts irreducibly and nontrivially on $Q/\Phi(Q)$. Also, $C_V(Q) = 0$ so $q \neq \mathrm{char}(F)$.

Step 3 *$P/C_P(Q)$ is a q-group.*

Proof. Assume false. Then there exists a prime $p_0 \neq q$ and a cyclic p_0-group $P_0 \leq P$ such that $[Q, P_0] \neq 1$. Set $G_0 = P_0[Q, P_0]$. Since $p_0 \neq q$ we have $[Q, P_0] = [Q, P_0, P_0]$ whence $G_0 = \langle P_0^{G_0} \rangle$. Now Q acts homogeneously and faithfully on V and as $[Q, P_0] \trianglelefteq Q$ it follows from Maschke's Theorem that V does not involve the trivial $[Q, P_0]$-module. In particular, V does not involve the trivial G_0-module and then the minimality of $|G| + |P|$ forces $G = G_0, P = P_0, p = p_0$ and $Q = [Q, P]$. Then $G = PQ$ and $C_P(Q) \trianglelefteq G$. Since $C_V(P) \neq 0$ and since G is irreducible on V we have $C_P(Q) = 1$. Also, Lemma 2.1 implies that Q acts irreducibly on V.

By [3, Theorem 5.5.5, p.208] there exists an integer t such that

$$|Q| \;=\; q^{1+2t} \quad \text{and} \quad \dim V \;=\; q^t.$$

We have $|P| = p^n$ for some integer $n \geq 1$. Since P is a $\{2, 3\}'$-group we have $p > 3$ so the first paragraph of the proof of [3, Lemma 11.2.5, p.368] implies that

$$p^n \;\text{ divides }\; q^t + 1.$$

The argument now splits into two cases depending on whether F has characteristic p or not.

Case $p = \mathrm{char}(F)$. The Hall-Higman Theorem [3, Theorem 11.2.1, p.364] implies that the Jordan canonical form for a generator of P consists of $(q^t + 1)/p^n$ Jordan blocks. Recalling that $\dim V = q^t$ we have

$$\begin{aligned}\dim C_V(P) &= \frac{\dim V + 1}{p^n} \\ &\leq \frac{2}{p} \dim V,\end{aligned}$$

a contradiction.

Case $p \neq \mathrm{char}(F)$. Let χ be the character of G afforded by V. Using the Coprime Hall-Higman Theorem [4, Satz V.17.13, p.574] together with the fact that $p^n | q^t + 1$ we have

$$\chi_P \;=\; \frac{q^t + 1}{p^n} \rho - \mu$$

where ρ is the regular character of P and μ is a linear character of P. Then

$$\begin{aligned} \dim C_V(P) &\leq \frac{\dim V + 1}{p^n} \\ &\leq \frac{2}{p} \dim V. \end{aligned}$$

This contradiction completes the proof of Step 3. $\square$

Recall that G acts nontrivially and irreducibly on $Q/\Phi(Q)$ and that $p = \min \pi(P)$. Since $G = \langle P^G \rangle$ and P is a $\{2,3\}'$-group it follows from the previous step that

$$q > 3 \text{ and that } p \leq q.$$

Then Lemma 2.3 implies that Q has exponent q.

Step 4 *Let A be an abelian normal subgroup of Q that is normalized by P. Then A is centralized by P.*

Proof. Let $|A| = q^{1+k}$ and note that A is elementary abelian. Since $Z(Q) \leq Z(G)$ we have $[Z(Q), P] = 1$. If $k = 0$ then $A = Z(Q)$, hence we may suppose that $k \geq 1$. We assume the notation of Lemma 2.2. Now P normalizes A so it permutes Ω. Let $m = |\mathrm{Fix}_\Omega(P)|$. Step 1 together with $p \leq q$ implies that

$$q^k < qm.$$

Hence $m > 1$. Lemma 2.2(b) implies that if $V_i \in \mathrm{Fix}_\Omega(P)$ then P normalizes A_i. Note that $|A_i| = q^k$ for all i.

Since $m > 1$ we may suppose that $V_1, V_2 \in \mathrm{Fix}_\Omega(P)$. If $k = 1$ then $|A_1| = |A_2| = q$ and then Step 3 implies that P centralizes A_1 and A_2. But $A = \langle A_1, A_2 \rangle$ so P centralizes A. Hence we may assume that $k \geq 2$. Now $q^k < qm$ so $q < m$.

Now $|A : A_1 \cap A_2| = q^2$ and $Z(Q) \cap A_1 \cap A_2 = 1$ so it follows that A contains exactly q hyperplanes which contain $A_1 \cap A_2$ but not $Z(Q)$. Since $q < m$ we may suppose that $V_3 \in \mathrm{Fix}_\Omega(P)$ and that $A_1 \cap A_2 \not\leq A_3$. Then $A_1 \cap A_2$ and $A_1 \cap A_3$ are distinct hyperplanes of A_1 whence $A = Z(Q)(A_1 \cap A_2)(A_1 \cap A_3)$.

Recall that P normalizes A_1 and A_2 so P normalizes $Z(Q)(A_1 \cap A_2)$. This subgroup has index q in A and it is normal in Q since $Q' = Z(Q)$. By induction, P centralizes $Z(Q)(A_1 \cap A_2)$. Similarly P centralizes $Z(Q)(A_1 \cap A_3)$ and we deduce that P centralizes A. $\square$

We are now in a position to obtain a final contradiction. Let $\overline{Q} = Q/Z(Q)$ and regard $\overline{Q}$ as a $GF(q)$G-module. Since G acts irreducibly and nontrivially on $\overline{Q}$ we have that $\overline{Q}$ does not involve the trivial G-module. Since $Z(Q)$ is in the kernel of the action of G on $\overline{Q}$, we may invoke the minimality of G to obtain

$$\dim C_{\overline{Q}}(P) \;\leq\; \frac{2}{p}\dim\overline{Q}.$$

Choose t such that $|Q| = q^{1+2t}$, so that $\dim\overline{Q} = 2t$. Using Step 3, Lemma 2.2(c) and Step 4 we see that

$$\dim C_{\overline{Q}}(P) \;\geq\; \frac{1}{2}\dim\overline{Q}.$$

But $p \geq 5$ so this contradicts the previous inequality and completes the proof of this theorem. $\square$

Remark By modifying the conclusion, it ought to be possible to remove the hypothesis that P is a $\{2,3\}'$-group.

Corollary 2.5. *Assume the hypotheses of Theorem 2.3. Then*

$$\dim C_V(P) \;<\; \frac{1}{2}\dim V.$$

Remark It is in fact Corollary 2.5 that we shall use rather than the stronger Theorem 2.4. If it is desired to prove only Corollary 2.5 then a simpler proof is possible. In particular, the appeal to Hall-Higman theory in Step 3 may be replaced by a more elementary argument.

Indeed, in Step 3 we have

$$G \;=\; PQ$$

where Q is an extraspecial q-group, P is a cyclic p-group that acts faithfully and irreducibly on $Q/\Phi(Q)$ and V is a faithful G-module on which Q acts irreducibly.

Let E be the enveloping algebra of Q on V. By Weddurburn's Theorem [3, Theorem 3.6.3, p.86] we have $E = \mathrm{End}(V)$, so then $\dim E = (\dim V)^2$. Choose $x \in P$ with prime order p.

The linear transformations $y : V \longrightarrow V$ with $[V, x] \leq \ker y$ and $\operatorname{Im} y \leq C_V(x)$ constitute a subspace of $C_E(x)$ with dimension $(\dim C_V(x))^2$. Considering the scalar transformations, it follows that

$$\dim C_E(x) \;\geq\; (\dim C_V(x))^2 + 1.$$

Either by considering the action of $\langle x\rangle Q/\Phi(Q)$ on E or by the argument of [3, Lemma 11.2.4, p.367] we have

$$\dim C_E(x) \;=\; \frac{\dim E - 1}{p} + 1.$$

But $\dim E = (\dim V)^2$ and $p \geq 5$ so these inequalities yield

$$\dim C_V(x) \;<\; \frac{1}{2}\dim V.$$

Also, another proof of Corollary 2.5 is possible by using a result of Robinson [5, Corollary 1.2].

3 The proofs of Theorems A–E

The proof of Theorem A. Assume false and consider a counterexample with $|G| + |V|$ minimal. Then $G = AV$ and there exist distinct primes r and q such that V is an r-group and $\mathcal{O}_q(A)$ does not centralize V. Since $[V, \mathcal{O}_q(A)]$ is normalized by A, the minimality of V forces $V = [V, \mathcal{O}_q(A)]$. Note that $G \notin \Sigma_G(P)$.

Let $\overline{G} = G/\Phi(V)$. Then $\overline{G} = \overline{A}\,\overline{V}$, $\overline{V}$ is elementary abelian, $\overline{V} = [\overline{V}, \overline{\mathcal{O}_q(A)}] \neq 1$ and then $C_{\overline{V}}(\overline{\mathcal{O}_q(A)}) = 1$. Let $\overline{U} \leq \overline{V}$ be a minimal normal subgroup of $\overline{G}$ and suppose that $\overline{U} < \overline{V}$. Let U be the inverse image of $\overline{U}$ in G. The minimality of $|V|$ implies that $[\mathcal{O}_q(A), U] = 1$ whence $\overline{U} \leq C_{\overline{V}}(\overline{\mathcal{O}_q(A)}) = 1$, a contradiction. Thus $\overline{V}$ is a minimal normal subgroup of $\overline{G}$. Since $\overline{V}$ is abelian, this implies that $\overline{A}$ is a maximal subgroup of $\overline{G}$ and that $\overline{A} \cap \overline{V} = 1$.

Clearly $\overline{A} \in \Sigma_{\overline{G}}(\overline{P})$. Suppose that $\overline{A} \notin \Sigma^*_{\overline{G}}(\overline{P})$. Then since $\overline{A}$ is a maximal subgroup of $\overline{G}$ we have $\overline{G} = \langle\, \overline{P}, \overline{P}^{\overline{g}}\, \rangle$ for some $g \in G$. Then $G = \Phi(V)\langle P, P^g\rangle$ whence $V = \Phi(V)(V \cap \langle P, P^g\rangle)$ so $V \leq \langle P, P^g\rangle$ and then $G = \langle P, P^g\rangle \in \Sigma_G(P)$, a contradiction. We deduce that $\overline{A} \in \Sigma^*_{\overline{G}}(\overline{P})$ and then the minimality of $|G|$ forces $\Phi(V) = 1$. In particular, A is a complement to V.

Set $N = \mathcal{O}_q(A)V \trianglelefteq G$ and note that $\mathcal{O}_q(A) \in \mathrm{Syl}_q(N)$. Since $C_V(\mathcal{O}_q(A)) = 1$ we have $V \cap N_G(\mathcal{O}_q(A)) = 1$ and it follows that the complements to V in G are the normalizers of the Sylow q-subgroups of N. In particular, V acts transitively by conjugation on its set of complements.

Choose $a \in A$ such that $A = \langle P, P^a\rangle$. Let $v \in V$ and set $B = \langle P, P^{av}\rangle$. Since $G = AV$ we have $G = BV$. Now $G \notin \Sigma_G(P)$ and V is a minimal normal subgroup of G so it follows that B is a complement to V. By the previous paragraph there exists $u \in V$ such that $B^u = A$. Then

$$\langle P^u, P^{avu}\rangle \;=\; A \;=\; \langle P, P^a\rangle.$$

In particular,

$$[u, P] \;\leq\; \langle P, P^u\rangle \cap V \;\leq A \cap V \;=\; 1,$$

and

$$[vu, P^a] \;\leq\; \langle P^a, P^{avu}\rangle \cap V \;\leq A \cap V \;=\; 1.$$

Thus $u \in C_V(P)$ and $vu \in C_V(P^a)$. Since v was arbitrary, we deduce that

$$V \;=\; C_V(P)C_V(P^a).$$

Regarding V as a $GF(r)A$-module, this implies that

$$\dim C_V(P) \;\geq\; \frac{1}{2}\dim V.$$

But A acts irreducibly and nontrivially on V and $A = \langle P^A \rangle$, so Corollary 2.5 supplies a contradiction. □

The proof of Corollary B. This follows from Theorem A and the fact that $C_G(F(G)) \leq F(G)$. □

The proof of Theorem C. Choose $P \in \mathcal{C}$. It suffices to show that there exists $A \in \Sigma_G(P)$ with $f(A) = f(G)$. Assume this to be false and let G be a minimal counterexample. Choose $q \in \pi(F(G))$ and set

$$\overline{G} \;=\; G/\mathcal{O}_q(G).$$

Using Lemma 1.1 we see that $f(\overline{G}) = f(G) - 1$. Then $F(G) = \mathcal{O}_q(G)$ since otherwise G would embed into a direct product of two groups, both with Fitting height $f(G) - 1$.

The minimality of G implies that there exists $\overline{A} \in \Sigma_{\overline{G}}(\overline{P})$ such that $f(\overline{A}) = f(\overline{G})$. By Lemma 1.1 there exists $A \in \Sigma_G(P)$ such that A maps onto $\overline{A}$. Choose A^* such that

$$A \;\leq\; A^* \;\in\; \overset{*}{\Sigma}_G(P).$$

Now $f(G) - 1 = f(\overline{A}) \leq f(A) \leq f(A^*)$ so as G is a counterexample, we deduce that $f(\overline{A}) = f(A) = f(A^*)$. By Lemma 1.2 we have $\psi(A) \leq F(A^*)$ so Theorem A implies that $\psi(A)\mathcal{O}_q(G)$ is nilpotent. Now $F(G) = \mathcal{O}_q(G)$ and G is soluble so $C_G(\mathcal{O}_q(G)) \leq \mathcal{O}_q(G)$. We deduce that $\psi(A)$ is a q-group. Since $f(\overline{A}) = f(A)$ it follows from Lemma 1.3 that $\psi(\overline{A})$ is a q-group.

Recall that $f(\overline{A}) = f(\overline{G})$ so Lemma 1.2 implies that $\psi(\overline{A}) \leq F(\overline{G})$. However, $\overline{G} = G/\mathcal{O}_q(G)$ so $F(\overline{G})$ is a q'-group and then $\psi(\overline{A}) = 1$. This implies that $\overline{G} = 1$ and then that $G = \mathcal{O}_q(G)$. Since $G = \langle P^G \rangle$, this forces $G = P$ and then P is a member of $\Sigma_G(P)$ with Fitting height $f(G)$. This contradiction completes the proof. □

The proof of Theorem D. Set $H = \langle P^G \rangle$ and note that $A \leq H$. Then $A \in \Sigma_H^f(P)$. If $H < G$ then by induction we have $\psi(A) \leq F(H)$. But $H \trianglelefteq G$ so $F(H) \leq F(G)$. Hence we may suppose that $H = G$. Then by Theorem C we have $f(A) = f(G)$ and then Lemma 1.2 forces $\psi(A) \leq F(G)$. $\square$

The proof of Corollary E. Choose $P \in \mathcal{C}$ and $A \in \Sigma_G^f(P)$. Let $g \in G$ and set $H = \langle A, A^g \rangle$. Then H is soluble since it is generated by four members of $\mathcal{C}$. By Theorem D we have $\langle \psi(A), \psi(A)^g \rangle \leq F(H)$. In particular, $\langle \psi(A), \psi(A)^g \rangle$ is nilpotent for all $g \in G$ so the Baer-Suzuki Theorem forces $\psi(A) \leq F(G)$. Now apply induction to $G/F(G)$. $\square$

4 Generators for $\psi(G)$

Lemma 4.1. *Let G be a soluble group. Suppose that $f(G) \geq 2$ and that $\psi(G)$ is a q-group. Set $\overline{G} = G/\psi(G)$ and let K be the inverse image of $\mathcal{O}_{q'}(\psi(\overline{G}))$ in G. Then*

$$\psi(G) \;=\; [\psi(G), K].$$

Proof. Let L be the inverse image of $\psi(\overline{G})$ in G and choose $Q \in \mathrm{Syl}_q(L)$. Since $\psi(G)$ is a q-group and since $\psi(\overline{G})$ is nilpotent we have

$$L \;=\; KQ, \quad K \trianglelefteq L \;\text{ and }\; Q \trianglelefteq L. \tag{2}$$

Set

$$G^* \;=\; G/[\psi(G), K].$$

Now $K^*/\psi(G)^* \cong K/\psi(G)$, which is nilpotent. Since $\psi(G)^* \leq Z(K^*)$ we deduce that K^* is nilpotent. Then using (2) we see that L^* is nilpotent. We have

$$G^*/L^* \;\cong\; G/L \cong \overline{G}/\psi(\overline{G}).$$

Since $f(G) \geq 2$ we have $f(\overline{G}/\psi(\overline{G})) = f(G) - 2$. Now L^* is nilpotent so $f(G^*/L^*) \geq f(G^*) - 1$ whence $f(G) - 2 \geq f(G^*) - 1$ so $f(G) > f(G^*)$. But then $\psi(G) \leq [\psi(G), K]$. □

The proof of Theorem F. Assume false and let G be a minimal counterexample. Set

$$T = \left\langle \psi(A) \mid A \in \Sigma_G^f(P) \right\rangle.$$

Using Theorem D we have

$$T \leq \psi(G) \text{ but } T \neq \psi(G).$$

Step 1 *Suppose that $V \neq 1$ is a normal subgroup of G such that $f(G/V) = f(G)$. Then*

$$\psi(G) = T(\psi(G) \cap V) \quad \textit{and} \quad \psi(G) \cap V \not\leq T.$$

Proof. Set $\overline{G} = G/V$. Since $f(\overline{G}) = f(G)$ we have $\psi(\overline{G}) = \overline{\psi(G)}$ by Lemma 1.3. The minimality of G implies that $\psi(\overline{G}) = \langle \psi(\overline{A}) \mid \overline{A} \in \Sigma_{\overline{G}}^f(\overline{P}) \rangle$. Let $\overline{A} \in \Sigma_{\overline{G}}^f(\overline{P})$. Theorem C implies that $f(\overline{A}) = f(\overline{G})$ and Lemma 1.1 implies that $\overline{A}$ has an inverse image $A \in \Sigma_G(P)$. Since $f(G) = f(\overline{G})$ it follows that $A \in \Sigma_G^f(P)$ and then Lemma 1.3 yields $\psi(\overline{A}) = \overline{\psi(A)}$. Consequently $\psi(G) \leq \langle \psi(A) \mid A \in \Sigma_G^f(P) \rangle V = TV$. Since $T \leq \psi(G)$ we have $\psi(G) = T(\psi(G) \cap V)$ and since $T \neq \psi(G)$ we have $\psi(G) \cap V \not\leq T$. □

Step 2 *$\psi(G)$ is an elementary abelian q-group for some prime q.*

Proof. Suppose that q and r are distinct prime divisors of $|\psi(G)|$. Then $\psi(G) \not\leq \mathcal{O}_q(\psi(G))$ so $f(G/\mathcal{O}_q(\psi(G))) = f(G)$ and then Step 1 implies that $|\psi(G) : T|$ is a power of q. Similarly, $|\psi(G) : T|$ is a power of r whence $\psi(G) = T$, a contradiction. Thus $\psi(G)$ is a q-group for some prime q. Suppose that $\Phi(\psi(G)) \neq 1$. Since $\Phi(\psi(G)) \neq \psi(G)$ we may apply Step 1 to conclude that $\psi(G) = T\Phi(\psi(G))$. But then $\psi(G) = T$, a contradiction. We deduce that $\Phi(\psi(G)) = 1$ and then that $\psi(G)$ is elementary abelian. □

Let
$$\overline{G} \ = \ G/\psi(G) \ \text{ and } \ \overline{K} = \mathcal{O}_{q'}(\psi(\overline{G})).$$
Let K be the inverse image of $\overline{K}$ in G. The minimality of G implies that
$$\overline{K} \ = \ \langle \ \mathcal{O}_{q'}(\psi(\overline{A})) \mid \overline{A} \in \Sigma^f_{\overline{G}}(\overline{P}) \ \rangle.$$
By Lemma 1.1, each member of $\Sigma^f_{\overline{G}}(\overline{P})$ has an inverse image in $\Sigma_G(P)$ so we let
$$\Sigma \ = \ \{ \ A \in \Sigma_G(P) \mid \overline{A} \in \Sigma^f_{\overline{G}}(\overline{P}) \ \}$$
and for each $A \in \Sigma$ we let
$$\Pi(A)$$
denote the inverse image of $\mathcal{O}_{q'}(\psi(\overline{A}))$ in A. Then
$$K \ = \ \psi(G)\langle \ \Pi(A) \mid A \in \Sigma \ \rangle.$$

Step 3 $\psi(G) = \langle \ [\psi(G), \Pi(A)] \mid A \in \Sigma \ \rangle$.

Proof. We will apply Lemma 4.1. If $f(G) < 2$ then G is nilpotent so as $G = \langle P^G \rangle$ we have $G = P$ and then $G \in \Sigma^f_G(P)$, a contradiction. Thus $f(G) \geq 2$ and Lemma 4.1 implies that
$$\psi(G) \ = \ [\psi(G), K].$$
Now $K = \psi(G)\langle \ \Pi(A) \mid A \in \Sigma \ \rangle$ and $\psi(G)$ is abelian. Then
$$\psi(G) \ = \ \langle \ [\psi(G), \Pi(A)] \mid A \in \Sigma \ \rangle$$
because K centralizes the quotient of the left hand side by the right hand side. $\square$

In what follows, we fix $A \in \Sigma$ such that

$$[\psi(G), \Pi(A)] \not\leq T.$$

Such an A exists by Step 3 and the fact that $\psi(G) \not\leq T$. Set

$$H = A[\psi(G), \Pi(A)].$$

Choose B such that

$$A \leq B \in \overset{*}{\Sigma}_H(P).$$

Step 4 $[\psi(G), \Pi(A)] \ = \ [\psi(G), \Pi(A), \Pi(A)]$.

Proof. This is because $\Pi(A)/\psi(G) \cap \Pi(A)$ is a q'-group and $\psi(G)$ is abelian. □

Step 5 $f(\overline{A}) = f(G) - 1,\ B \in \Sigma_G^f(P)$ *and* $f(H) = f(G)$.

Proof. Since $\overline{A} \in \Sigma_{\overline{G}}^f(\overline{P})$ and $\overline{G} = G/\psi(G) = \langle \overline{P}^{\overline{G}} \rangle$, Theorem C implies that $f(\overline{A}) = f(G) - 1$. We claim that $f(B) = f(G)$. Assume false. Then

$$f(G) - 1 \geq f(B) \geq f(A) \geq f(\overline{A}) = f(G) - 1$$

whence

$$f(B) = f(A) = f(\overline{A}).$$

Lemma 1.3 implies that $\Pi(A) \leq \psi(A)(A \cap \psi(G))$ and then using Lemma 1.2 we have $\Pi(A) \leq F(B)$. Now $B \in \overset{*}{\Sigma}_H(P)$ so Theorem A implies that $\Pi(A)F(H)$ is nilpotent. But $[\psi(G), \Pi(A)] \leq F(H)$ so it follows from Step 4 that $[\psi(G), \Pi(A)] = 1$, contrary to the choice of A. We deduce that $f(B) = f(G)$ so $B \in \Sigma_G^f(P)$ and then also $f(H) = f(G)$. □

Step 6 $[\psi(G), \Pi(A)] \leq \psi(H)$.

Proof. Set $H^* = H/\psi(H)$. By Step 5 we have $f(H) = f(G)$ so $\psi(H) \leq \psi(G)$. In particular, $\overline{A}$ is a homomorphic image of A^*. Then

$$f(G) - 1 = f(H^*) \geq f(A^*) \geq f(\overline{A}) = f(G) - 1$$

so $f(A^*) = f(\overline{A}) = f(H^*)$. Lemma 1.3 yields $\Pi(A)^* \leq \psi(A^*)(A \cap \psi(G))^*$ and then Lemma 1.2 forces $\Pi(A)^* \leq F(H^*)$. From Step 4 we have

$$[\psi(G), \Pi(A)]^* = [[\psi(G), \Pi(A)]^*, \Pi(A)^*].$$

Now $[\psi(G), \Pi(A)]^* \leq F(H^*)$ so as $\Pi(A)^* \leq F(H^*)$ and $F(H^*)$ is nilpotent it follows that $[\psi(G), \Pi(A)]^* = 1$. Hence $[\psi(G), \Pi(A)] \leq \psi(H)$. □

We are now in a position to obtain a final contradiction. Since $A = \langle P^A \rangle$ and $H = A[\psi(G), \Pi(A)]$, it follows from Step 4 that $H = \langle P^H \rangle$. Also, $\Sigma_H^f(P) \subseteq \Sigma_G^f(P)$ since $f(H) = f(G)$. Now $[\psi(G), \Pi(A)] \not\leq T$ so Step 6 and the minimality of G force $G = H$. Since $f(B) = f(G)$ we have $\psi(B) \leq \psi(G)$. Moreover, $A \leq B$, $\psi(G)$ is elementary abelian and $G = A[\psi(G), \Pi(A)]$ so $1 \neq \psi(B) \trianglelefteq G$. By Step 5 and the definition of T we have $\psi(B) \leq T$ so applying Step 1 with $V = \psi(B)$ it follows that $\psi(G) = \psi(B)$. This is a contradiction since $B \in \Sigma_G^f(P)$. □

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