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A characterisation of $F_2(G)$

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In [1] we obtained a short proof of the theorem of Thompson that a finite group is soluble if and only if every two of its elements generate a soluble subgroup. A natural next question to ask is *what happens if we keep one of the generators fixed?* For a finite group we define

$$\mathrm{sol}(G),$$

the soluble radical of G, to be the largest normal soluble subgroup of G, and

$$\mathrm{SOL}(G) = \bigl\{x \in G \bigm| \langle x, g\rangle \text{ is soluble for all } g \in G\bigr\}.$$

Conjecture. *For any finite group G,*

$$\mathrm{SOL}(G) = \mathrm{sol}(G).$$

Some progress towards a proof of this conjecture is reported in [2] and the author's work in this area has led to a number of results of independent interest, notably [3,4]. The interested reader is referred to the survey article [5].

It is the purpose of this article to record further progress towards a proof of the above conjecture. Recall that for a group G,

$$F_2(G)$$

is defined to be the inverse image of $F(G/F(G))$ in G. Equivalently, $F_2(G)$ is the largest normal soluble subgroup of G with Fitting height at most two. Define

$$\mathcal{F}_2(G) = \bigl\{x \in G \bigm| x \in F_2\bigl(\langle x, g\rangle\bigr) \text{ for all } g \in G\bigr\}.$$

We shall prove the following result.

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Theorem A. *Let G be a finite group. Then*

$$\mathcal{F}_2(G) = F_2(G).$$

We remark that Theorem A may be restated as follows:

Let x be an element of the finite group G. Then $\langle x^G \rangle$ is soluble with Fitting height at most two if and only if $\langle x^{\langle g \rangle} \rangle$ has this property for all $g \in G$.

The corresponding assertion with 'Fitting height at most two' replaced by 'Fitting height at most one' is an immediate consequence of the Baer–Suzuki Theorem.

It should be pointed out that if $x \in G$ satisfies the following:

$\langle x, x^g \rangle$ is soluble with Fitting height at most two for all $g \in G$,

then one cannot conclude that $x \in F_2(G)$. Indeed, a counterexample may be found in any non-abelian finite simple group.

We hope that the proof of Theorem A presented here provides a model for an eventual proof of the conjectured characterisation of the soluble radical of a finite group.

1. Preliminaries

Henceforth the word group will mean finite group. We shall need a number of results concerning the action of a p-group on a p'-group.

Lemma 1.1. *Let the p-group A act on the p'-group G. Then*:

(i) $[G, A] \trianglelefteq G$,
(ii) $G = C_G(A)[G, A]$, *and*
(iii) $[G, A, A] = [G, A]$.

Proof. This is well known. (i) follows from commutator identities and (ii) follows from a Frattini Argument. □

The following is a special case of a result of Goldschmidt [6, (11.12), p. 589].

Lemma 1.2. *Let the abelian p-group A act on the soluble p'-group G. Then*

$$C_{[G,A]}(A) = \left\langle C_{[C_G(B),A]}(A) \;\middle|\; B \leqslant A \text{ and } A/B \text{ is cyclic} \right\rangle.$$

Definition. Let A be an elementary abelian p-group that acts on the q-group Q, $q \neq p$. Then Q is A-minimal provided:

(i) $A/C_A(Q)$ is cyclic, and
(ii) if $p=2$ then Q is cyclic.

Lemma 1.3. *Let the elementary abelian p-group A act on the q-group Q, $q \neq p$. If 1 and Q are the only A-invariant subgroups of Q then Q is A-minimal.*

Proof. Note that Q is elementary abelian and that an abelian group with a faithful irreducible representation must be cyclic. This proves (i). Recall that if an involution t acts on a $2'$-group X then every element of X can be written in the form xy where t centralizes x and inverts y. This proves (ii). □

Lemma 1.4. *Let the elementary abelian p-group A act on the q-group Q, $q \neq p$. Let $P \leqslant A$. Then*

$$[Q,P] = \big\langle [T,P] \bigm| T \leqslant Q \text{ is } A\text{-invariant and } A\text{-minimal} \big\rangle.$$

Proof. By a result of Goldschmidt [6, (7.13), p. 484], we have

$$[Q,P] = \big\langle \big[C_Q(B),P\big] \bigm| B \leqslant A \text{ and } A/B \text{ is cyclic} \big\rangle.$$

Thus if $p>2$ the lemma is proved. If $p=2$ note that $[C_Q(B),P]$ is generated by elements that are inverted by a generator for A modulo B. □

The preceding results of Goldschmidt are particularly effective when a non-cyclic abelian p-group is available. In the contrary case, the following result can act as a substitute.

Theorem 1.5. *Let p be an odd prime and suppose that the p-group P acts on the p'-group G. Then*

$$C_{[G,P]}(P) = \big\langle C_{[x,P]}(P) \bigm| x \in G \text{ and } x \in [x,P] \big\rangle.$$

Note that the subgroups $[x,P]$ are P-invariant, that $\langle P, P^x\rangle = P[x,P]$ and that any subgroup of the form $[y,P]$ may be written in the form $[x,P]$ with $x \in [x,P]$.

Theorem 1.5 is proved in [3] using the Glauberman Character Correspondence. We shall only need this result in the case that G is a q-group and an elementary proof in this case may be found in [7]. We note that Theorem 1.5 is simply not true if $|P| = 2$, the right-hand side being trivial. This is the cause of some difficulty.

The proof of Theorem A proceeds by constructing collections of subgroups Ω with the property that any member of Ω is contained in a unique maximal member of Ω. The following hypothesis and lemma describe the basic idea.

Hypothesis $(\mathcal{U})$.

(i) G is a group with $F(G)=1$, A is a subgroup of G and Ω is a collection of A-invariant nilpotent subgroups of G.
(ii) If $R \leqslant Q \in \Omega$ and R is A-invariant then $R \in \Omega$.
(iii) If $Q, R \in \Omega$ and $A\langle Q, R\rangle \neq G$ then $\langle Q, R\rangle \in \Omega$.

Note that Ω is partially ordered by inclusion so we may consider the maximal and minimal members of Ω.

Lemma $(\mathcal{U})$. *Assume Hypothesis* $(\mathcal{U})$.

(i) *If Q and R are maximal members of Ω then $Q=R$ or $Q \cap R = 1$.*
(ii) *If $A\langle Q, R\rangle \neq G$ whenever Q and R are minimal members of Ω then Ω possesses a unique maximal member.*

Proof. Assume (i) to be false and choose a counterexample with $Q \cap R$ maximal. Let $T = Q \cap R \neq 1$ and $H = N_G(T)$. Since $F(G)=1$, we have $H \neq G$. Let $Q_0 = N_Q(T)$ and $R_0 = N_R(T)$. Since Q and R are nilpotent and $Q \neq R$, we have $Q_0 > T$ and $R_0 > T$. Now $Q_0, R_0 \in \Omega$ and $A\langle Q_0, R_0\rangle \leqslant H \neq G$; so $\langle Q_0, R_0\rangle \in \Omega$. Let S be a maximal member of Ω that contains $\langle Q_0, R_0\rangle$. We have $T < Q_0 < Q \cap S$; so the maximal choice of $Q \cap R$ forces $Q = S$. Similarly $R = S$; so $Q = R$, a contradiction.

To prove (ii), let M and N be maximal members of Ω. Let Q and R be minimal members of Ω that are contained in M and N, respectively. By assumption, $A\langle Q, R\rangle \neq G$; so $\langle Q, R\rangle \in \Omega$. Let L be a maximal member of Ω containing $\langle Q, R\rangle$. Then $Q \leqslant M \cap L$; so $M = L$. Similarly $L = N$; so $M = N$. $\square$

We shall refer to this lemma simply as $(\mathcal{U})$.

We shall need a number of nonsimplicity criteria. The first is a generalization of Wielandt's characterization of subnormal subgroups.

Theorem 1.6 (D. Bartels [8]). *Let P be a subgroup of the group G. Then*

$$\left\langle P^x \;\middle|\; x \in G \text{ and } x \in \left\langle P, P^x\right\rangle\right\rangle$$

is the smallest subnormal subgroup of G that contains P.

We shall also use the following result.

The Baer–Suzuki Theorem [6, Theorem 4.8, p. 195]. *Let x be a q-element of the group G. If $\langle x, x^g\rangle$ is a q-group for all $g \in G$ then $x \in O_q(G)$.*

Unfortunately, the Baer–Suzuki Theorem cannot be generalized from a single prime to a set of primes. However, the following observation, when combined with the Goldschmidt Lemma can sometimes be effective.

Lemma 1.7. *Let t be a p'-element of the group G. If tg is a p'-element for every p'-element $g \in G$ then $t \in O_{p'}(G)$.*

Proof. Use induction on the length of a word to show that every member of $\langle t^G \rangle$ is a p'-element. □

The Goldschmidt Lemma [6, (5.18), p. 112]. *Let u be a p-element of the soluble group H. Then*

$$O_{p'}\big(C_H(u)\big) \leqslant O_{p'}(H).$$

The following result is used to eliminate the final configuration in the proof of Theorem A.

Lemma 1.8. *Let $M_1, \dots, M_n$ and $H \neq 1$ be subgroups of the group G. Suppose that*

$$G = M_1 \cup \cdots \cup M_n, \qquad G > M_i > H, \qquad M_i \cap M_j = H \quad \text{for all } i \neq j,$$

that $n \geqslant 2$, and that H does not contain a nontrivial normal subgroup of G. Then H is a Frobenius complement in G and $F(G) \neq 1$.

Proof. Let $i \neq j$, choose $m_i \in M_i - H$ and choose $m_j \in M_j - H$. Then $m_i m_j^{-1} \in M_k$ for some $k \neq i, j$. Now

$$\big(H \cap H^{m_i}\big)^{m_j^{-1}} \leqslant M_j \cap M_k = H;$$

so $H \cap H^{m_i} \leqslant H^{m_j}$. We deduce that $H \cap H^{m_i} = H \cap H^{m_j}$, that $H \cap H^{m_i} = H_{M_j}$, that $H \cap H^{m_j} = H_{M_i}$, and then that $H_{M_i} = H_{M_j}$. Note that $1 = H_G = H_{M_1} \cap \cdots \cap H_{M_n}$; whence $H \cap H^{m_i} = 1$.

It follows that $H \cap H^g = 1$ for all $g \in G - H$; so H is a Frobenius complement in G. Frobenius' Theorem implies that G possesses a normal complement K to H. Thompson's Theorem on fixed-point free automorphisms implies that K is nilpotent. Hence $F(G) \neq 1$ as claimed. □

2. The soluble case

We need to establish Theorem A in the case that G is soluble. For these groups, a more general result is provable, which we shall now describe.

For a group G the characteristic subgroups $F_k(G)$, $k \geqslant 0$ are defined by

$$F_0(G) = 1 \quad \text{and} \quad F_{k+1}(G) = \text{ the inverse image of } F\big(G/F_k(G)\big) \text{ in } G.$$

If G is soluble then $f(G)$, the Fitting height of G, is the smallest integer k such that $G = F_k(G)$. For any k we note that $F_k(G)$ has Fitting height at most k and contains every normal subgroup of G with this property. Define

$$\mathcal{F}_k(G) = \big\{x \in G \bigm| x \in F_k\big(\langle x, g\rangle\big) \text{ for all } g \in G\big\}$$

or equivalently

$$\mathcal{F}_k(G) = \big\{x \in G \bigm| f\big(\langle x^{\langle g\rangle}\rangle\big) \leqslant k \text{ for all } g \in G\big\}.$$

It is trivial that $F_k(G) \subseteq \mathcal{F}_k(G)$.

Theorem 2.1. *Let G be a soluble group. Then*

$$\mathcal{F}_k(G) = F_k(G) \quad \textit{for all } k \geqslant 0.$$

Proof. Assume false and let G be a minimal counterexample. Then $k \geqslant 1$ since $\mathcal{F}_0(G) = 1 = F_0(G)$. Choose $x \in \mathcal{F}_k(G) - F_k(G)$. Then $G = \langle x^G\rangle$, $f(G) > k$, and every proper quotient of G has Fitting height at most k. Consequently, G has a unique minimal normal subgroup V, since otherwise G would embed into a direct product of groups each of which has Fitting height at most k. Note that V is an elementary abelian q-group for some prime q.

It is a general fact that $F(G/\Phi(G)) = F(G)/\Phi(G)$, where $\Phi(G)$ is the Frattini subgroup of G. Thus $f(G/\Phi(G)) = f(G)$ and we deduce that $\Phi(G) = 1$. Since V is the unique minimal normal subgroup of G, it follows that $C_G(V) = V = O_q(G)$. The solubility of G implies that there is a unique conjugacy class of complements to V in G.

Set $\overline{G} = G/V$. Then $f(\overline{G}) = k$. Since $\overline{G} = \langle \bar{x}^{\overline{G}}\rangle$, the minimality of G implies that there exists $g \in G$ such that $f(\langle \bar{x}^{\langle \bar{g}\rangle}\rangle) = k$. Set $M = \langle x^{\langle g\rangle}\rangle$ and $H = \langle M^V\rangle$. Note that $\overline{H} = \overline{M} = \langle \bar{x}^{\langle \bar{g}\rangle}\rangle$ and that $f(H) \geqslant f(M) = f(\overline{M}) = k$.

We claim that $f(H) > k$. Indeed, let

$$T = \bigcap \big\{N \trianglelefteq H \bigm| f(H/N) < f(H)\big\}.$$

Notice that H/T embeds into a direct product of groups each of which has Fitting height less than $f(H)$. Thus $1 \neq T \leqslant F(H)$. Now V normalizes H; whence $[O_{q'}(F(H)), V] \leqslant O_{q'}(F(H)) \cap V = 1$, since V is a q-group. Recall that $C_G(V) = V$; whence $O_{q'}(F(H)) = 1$ and then T is a q-group. Now

$$f\big(\overline{H}/\overline{H} \cap F\big(\overline{G}\big)\big) \leqslant f\big(\overline{G}/F\big(\overline{G}\big)\big) < k;$$

whence $T \leqslant \ker(H \to \overline{H}/\overline{H} \cap F(\overline{G}))$ and then $\overline{T} \leqslant F(\overline{G})$. Since $V = O_q(G)$, it follows that $F(\overline{G})$ is a q'-group. But T is a q-group; so $\overline{T} = 1$. Thus $T \leqslant V$ and so $f(\overline{H}) < f(H)$. Since $f(\overline{H}) = k$, this proves the claim.

By its definition, H is generated by conjugates of x; so the minimality of G forces $G = H$. Thus $G = MV$. Now $x \in \mathcal{F}_k(G)$ and $M = \langle x^{\langle g\rangle}\rangle$; so $f(M) \leqslant k$. Then $M \neq G$ and, since V is abelian and a minimal normal subgroup of G, we have $M \cap V = 1$. Thus M is a complement to V in G. Using the fact that V is the unique minimal normal subgroup of G, it follows that $N_G(M) = M$. But $g \in N_G(M)$; whence $M = \langle x, g\rangle$.

Choose $v \in V$. Then $\langle \bar{x}^{\langle \overline{gv}\rangle}\rangle = \langle \bar{x}^{\langle \bar{g}\rangle}\rangle$; so the previous argument, with gv in place of g, implies that $\langle x, gv\rangle$ is also a complement to V. Recall that there is a unique conjugacy class of complements to V in G. Thus

$$\langle x, gv\rangle^u = \langle x, g\rangle$$

for some $u \in V$. We have

$$x^{-1}x^u = [x, u] \in V \cap \langle x, g\rangle = 1$$

and

$$g^{-1}(gv)^u = g^{-1}g^u v = [g, u]v \in V \cap \langle x, g\rangle = 1.$$

Thus $u \in C_V(x)$ and $v = [u, g]$. Since v is an arbitrary element of V, it follows that $|C_V(x)| \geqslant |V|$. Consequently,

$$x \in C_G(V) = V \leqslant F_k(G).$$

This final contradiction completes the proof. $\square$

Corollary 2.2. *Let G be a soluble group, $k \geqslant 0$, and $x \in G$. Then $\langle x^G\rangle$ has Fitting height at most k if and only if $\langle x^{\langle g\rangle}\rangle$ has Fitting height at most k for all $g \in G$.*

3. The minimal counterexample

Henceforth we let G be a minimal counterexample to Theorem A. Then $G = \langle \mathcal{F}_2(G)\rangle$. Theorem 2.1 implies that $F(G) = 1$. Fix a prime p such that $\mathcal{F}_2(G)$ contains p-elements. Set

$$\mathcal{F} = \bigl\{P \leqslant G \bigm| P \text{ has order } p \text{ and is generated by a member of } \mathcal{F}_2(G)\bigr\}$$

and

$$\mathcal{A} = \left\{A \leqslant G \;\middle|\; \begin{array}{l} A \text{ is an elementary abelian } p\text{-group that is generated} \\ \text{by members of } \mathcal{F}_2(G) \text{ and one of the following holds:} \\ \text{(i) } m(A) \geqslant 3 \text{ and } p > 2, \text{ or} \\ \text{(ii) } m(A) = 2 \text{ and } A \text{ contains at most one subgroup of} \\ \qquad \text{order } p \text{ that is not a member of } \mathcal{F} \end{array}\right\}.$$

Here, $m(A)$ is the minimal number of generators for A. Note that $\langle P, g\rangle \neq G$ for all $P \in \mathcal{F}$ and $g \in G$.

If $\mathcal{A} \neq \emptyset$ then we use $(\mathcal{U})$ and Bartels's Theorem to construct a nilpotent normal subgroup of G. If $\mathcal{A} = \emptyset$ then the p-local structure of G is severely restricted, and we obtain a contradiction using Lemma 1.8.

Fundamental to the whole argument is the construction of collections of subgroups satisfying Hypothesis $(\mathcal{U})$. If A is a p-subgroup of G we define

$$\Omega_q(A) = \{Q \in \mathsf{N}(A, q) \mid C_Q(A) \leqslant O_q(C_G(A))\}.$$

Recall that $\mathsf{N}(A, q)$ is the set of q-subgroups of G that are normalized by A.

We remark that the even prime is particularly pleasant to deal with in the case $\mathcal{A} \neq \emptyset$. In the case $\mathcal{A} = \emptyset$, Theorem 1.5 plays a crucial role. However, this result is only available for odd primes and thus the even prime creates severe difficulties in the case $\mathcal{A} = \emptyset$.

Lemma 3.1. *Let A be a p-subgroup of G that is generated by members of $\mathcal{F}_2(G)$ and suppose that $A \leqslant H < G$. Then $A \leqslant F_2(H)$. If Q is a p'-subgroup of H that is normalized by A then $[Q, A]$ is nilpotent and $[Q, A] \leqslant O_{p'}(F(H))$. Moreover, if Q is a q-group then $[Q, A] \leqslant O_q(H)$.*

Proof. The minimality of G forces $A \leqslant F_2(H)$; so by setting $\overline{H} = H/F(H)$ we have $\overline{A} \leqslant O_p(\overline{H})$. Thus $[\overline{Q}, \overline{A}] \leqslant O_p(\overline{H}) \cap \overline{Q} = 1$, hence the result. □

Lemma 3.2. *Let $P \in \mathcal{F}$ and suppose that $x \in G$ satisfies $x \in [x, P]$. Let $n \in N_G(P)$ and set $H = \langle P, n, x\rangle$. Then*:

(i) $H \neq G$.
(ii) $[x, P] \leqslant O_{p'}(F(H))$.
(iii) $\langle [x, P], [x, P]^n\rangle$ *is nilpotent.*

Proof. We have

$$x \in [x, P] \leqslant \langle P, P^x\rangle = \langle P, P^{nx}\rangle \leqslant \langle P, nx\rangle$$

and it follows that $H = \langle P, nx\rangle$. Now $P \in \mathcal{F}$; so $P \leqslant F_2(H)$ and then $H \neq G$.

Set $\overline{H} = H/O_{p'}(F(H))$. Then $\overline{P} \leqslant O_p(\overline{H})$ and so $\bar{x} \in [\bar{x}, \overline{P}] \leqslant O_p(\overline{H})$. But $[\bar{x}, \overline{P}, \overline{P}] = [\bar{x}, \overline{P}]$ and $O_p(\overline{H})$ is nilpotent; so we have $[\bar{x}, \overline{P}] = 1$. This proves (ii), and (iii) follows immediately. □

4. Uniqueness results

Lemma 4.1. *Let $P \in \mathcal{F}$ and let D be a nilpotent p'-subgroup of G that is normalized by P. Suppose that $D = [D, P]$ and that $|D|$ is divisible by at least two primes. Then $\langle D^{N(P)}\rangle$ is nilpotent and, in particular, $\langle N(P), D\rangle \neq G$.*

Proof. It suffices to assume that D is maximal subject to the above conditions and prove that $N(P)$ normalizes D. Let q be a prime divisor of $|D|$, set $Q = O_q(D)$ and $R = O_{q'}(D)$. Since D is nilpotent, we have $D = Q \times R$ and, by hypothesis, $R \neq 1$. Moreover, as $D = [D, P]$ we have $Q = [Q, P]$ and $R = [R, P]$.

Choose $y \in Q$ such that $y \in [y, P]$ and $z \in R$ such that $z \in [z, P]$. Set $x = yz$ and note that $x \in [x, P]$ since $[Q, R] = 1$. Let $n \in N(P)$ and set $H = \langle P, x, n\rangle$. Lemma 3.2 implies that $x \in F(H)$. Consequently, $y \in O_q(H)$, $z \in O_{q'}(H)$, and we deduce that $[y^n, z] = 1$. Now Q is generated by such elements y and R is generated by such elements z; so it follows that $Q^n \leqslant C_G(R)$. Let $M = N_G(R)$. Now $PD \leqslant M$; so using Lemmas 3.1 and 1.1 we have $D \leqslant [O_{p'}(F(M)), P]$ and then the maximal choice of D forces $D = [O_{p'}(F(M)), P]$. Since $n \in N(P)$ and $Q = [Q, P]$, we have $Q^n = [Q^n, P]$; whence $Q^n \leqslant D$. Then $Q^n = Q$ and we deduce that $N_G(P) \leqslant N_G(Q)$. Now q was an arbitrary prime divisor of $|D|$; so $N_G(P) \leqslant N_G(D)$ as desired. □

Lemma 4.2. *Let $P \in \mathcal{F}$, let $q \neq p$ and suppose that $p \neq 2$. Then the following hold*:

(i) *If $Q \in \mathcal{V}(P, q)$ then $[Q, P] \in \Omega_q(P)$.*
(ii) *$\Omega_q(P)$ satisfies Hypothesis ($\mathcal{U}$).*
(iii) *$N_G(P)$ normalizes the maximal members of $\Omega_q(P)$.*

Proof. (i) By Theorem 1.5 we have

$$C_{[Q,P]}(P) = \big\langle C_{[x,P]}(P) \bigm| x \in Q \text{ and } x \in [x, P]\big\rangle;$$

hence we may suppose that $Q = [x, P]$ for some $x \in Q$. Let $c \in C_G(P)$. Lemma 3.2 implies that $\langle Q, Q^c\rangle$ is nilpotent and hence a q-group. Thus $\langle C_Q(P), C_Q(P)^c\rangle$ is a q-group and, since c was arbitrary, the Baer–Suzuki Theorem forces $C_Q(P) \leqslant O_q(C_G(P))$. This proves (i).

(ii) By the definition $\Omega_q(P)$, it suffices to verify (iii) of Hypothesis ($\mathcal{U}$). Let $Q, R \in \Omega_q(P)$, set $H = P\langle Q, R\rangle$, and suppose that $H \neq G$. Set $S = (H \cap O_q(C_G(P)))[O_q(H), P]$. Using (i) we have $S \in \Omega_q(P)$. Note that $Q = C_Q(P)[Q, P]$. Now $C_Q(P) \leqslant O_q(C_G(P))$ since $Q \in \Omega_q(P)$ and $[Q, P] \leqslant [O_q(H), P]$ by Lemma 3.1. Thus $Q \leqslant S$. Similarly, $R \leqslant S$; so $\langle Q, R\rangle \in \Omega_q(P)$ as desired.

(iii) Let Q be a maximal member of $\Omega_q(P)$ and let $n \in N_G(P)$. Note that Q^n is also a maximal member of $\Omega_q(P)$. If $[Q, P] = 1$ then $Q = O_q(C_G(P))$ and the result is clear. Hence, we may suppose that $[Q, P] \neq 1$. Choose $x \in Q$ such that $1 \neq x \in [x, P]$. Set $T = \langle [x, P], [x, P]^n\rangle$. Lemma 3.2 implies that T is a q-group. Since $n \in N_G(P)$, we have $T = [T, P]$; so $T \in \Omega_q(P)$ by (i). Now $1 \neq [x, P] \leqslant Q \cap T$; so ($\mathcal{U}$) implies that $T \leqslant Q$. Then $1 \neq [x, P]^n \leqslant Q \cap Q^n$ and another application of ($\mathcal{U}$) yields $Q = Q^n$. Thus $N_G(P)$ normalizes Q. □

Lemma 4.3. *Let $A \in \mathcal{A}$, let $q \neq p$ and $r \neq p$ be primes and suppose that $Q \in \mathsf{V}\!\mathsf{I}(A, q)$ and $R \in \mathsf{V}\!\mathsf{I}(A, r)$ are A-minimal. Then $A\langle Q, R\rangle \neq G$.*

Proof. Let $B = C_A(Q)$ and $C = C_A(R)$, so that A/B and A/C are cyclic. Then $A\langle Q, R\rangle \leqslant C_G(B \cap C)$; so we may assume that $B \cap C = 1$. The members of $\mathcal{A}$ are noncyclic and elementary abelian; so it follows that $A = B \times C$ and that B and C are cyclic of order p. We also have $[Q, C] \neq 1$ and $[R, B] \neq 1$. Since $m(A) = 2$, the definition of $\mathcal{A}$ implies that at least one of B or C is a member of $\mathcal{F}$. Without loss of generality we may suppose that $C \in \mathcal{F}$.

Suppose that $p > 2$. Lemma 4.2 implies that $[Q, C] \in \Omega_q(C)$. Let Q^* be a maximal member of $\Omega_q(C)$ that contains $[Q, C]$. Then $C_G(C) \leqslant N_G(Q^*)$ by Lemma 4.2. Note that $Q = C_Q(C)[Q, C]$ and that $AR \leqslant C_G(C)$. Thus $A\langle Q, R\rangle \leqslant N_G(Q^*)$ and the result is proved in this case.

Suppose that $p = 2$. Then the definition of A-minimality implies that Q and R are cyclic. Let x be a generator for Q. Since $[Q, C] \neq 1$ it follows that $x \in [x, C]$. Set $H = \langle C, x, R\rangle$. Lemma 3.2 implies that $x \in O_q(H)$. Since $Q = \langle x\rangle$, it follows that $F(\langle Q, R\rangle) \neq 1$. Now A normalizes $\langle Q, R\rangle$ and $F(G) = 1$; so $A\langle Q, R\rangle \neq G$ in this case also. □

Lemma 4.4. *Let $A \in \mathcal{A}$ and suppose that $q \neq p$. Then*:

(i) *If $Q \in \mathsf{V}\!\mathsf{I}(A, q)$ then $[Q, A] \in \Omega_q(A)$.*
(ii) *$\Omega_q(A)$ has a unique maximal member.*
(iii) *Let $P \in \mathcal{F}$ with $P < A$. If $Q \in \mathsf{V}\!\mathsf{I}(P, q)$ then $[Q, P]$ is contained in the unique maximal member of $\Omega_q(A)$.*

Proof. (i) We must show that $C_{[Q,A]}(A) \leqslant O_q(C_G(A))$. By Lemma 1.2, we have

$$C_{[Q,A]}(A) = \bigl\langle C_{[C_Q(B),A]}(Q) \bigm| B \leqslant A \text{ and } A/B \text{ is cyclic}\bigr\rangle.$$

Hence we may suppose that $Q = [C_Q(B), A]$ for some $B \leqslant A$ with A/B cyclic. The members of $\mathcal{A}$ are noncyclic and abelian; so $A \leqslant C_G(B) < G$. Then $[C_Q(B), A] \leqslant O_q(C_G(B))$ by Lemma 3.1. Since $C_G(A) \leqslant C_G(B)$, we have $[C_Q(B), A] \cap C_G(A) \leqslant O_q(C_G(A))$. This proves (i).

(ii) An argument identical to the one employed in the proof of Lemma 4.2(ii) shows that $\Omega_q(A)$ satisfies Hypothesis $(\mathcal{U})$. Then $(\mathcal{U})$ together with Lemmas 1.3 and 4.3 imply that $\Omega_q(A)$ has a unique maximal member.

(iii) We may suppose that $Q = [x, P]$ for some $x \in Q$. Set $S = \langle [x, P]^A\rangle$ and note that $S = [S, A]$. By (i) it suffices to prove that S is a q-group.

If $p > 2$ then $[x, P] \in \Omega_q(P)$ by Lemma 4.2(i), and then the desired conclusion follows from Lemma 4.2(iii). Hence, we may assume that $p = 2$. The definition of $\mathcal{A}$ implies that $m(A) = 2$ so $A = \langle P, n\rangle$ for some $n \in A$. Set $H = \langle P, x, n\rangle$. By Lemma 3.2 we have $[x, P] \leqslant O_q(H)$ so S is a q-group in this case also. □

The following result terminates our study of the set $\mathcal{A}$.

Theorem 4.5. $\mathcal{A} = \emptyset$.

Proof. Assume false, let $A \in \mathcal{A}$, and choose $P \in \mathcal{F}$ with $P < A$. The minimality of G implies that G is the smallest subnormal subgroup of G that contains P and then Bartels's Theorem yields $G = \langle P^x \mid x \in G \text{ and } x \in \langle P, P^x \rangle\rangle$. Note that $\langle P, P^x \rangle = P[x, P]$ for any x. Thus

$$G = PK \quad \text{where} \quad K = \bigl\langle [x, P] \bigm| x \in G \text{ and } x \in [x, P] \bigr\rangle.$$

We shall prove that K is nilpotent. Note that P normalizes K. For each prime $q \neq p$, set

$$K_q = \bigl\langle [x, P] \bigm| x \in G,\ x \in [x, P] \text{ and } [x, P] \text{ is a } q\text{-group} \bigr\rangle.$$

If $x \in G$ satisfies $x \in [x, P]$ then by Lemma 3.2 we have $x \in O_{p'}(F(\langle P, x \rangle))$ and it follows that

$$K = \langle K_q \mid q \neq p \rangle.$$

Lemma 4.4(iii) implies that each K_q is a q-group. Thus it suffices to show that $[K_q, K_r] = 1$ whenever $q \neq p$ and $r \neq p$ are distinct primes. Note that K_q and K_r are A-invariant since A centralizes P.

Let Q and R be A-invariant A-minimal subgroups of K_q and K_r, respectively. Set $H = A\langle Q, R \rangle$. Lemma 4.3 implies that $H \neq G$ and then Lemma 3.1 forces $[Q, P] \leqslant O_q(H)$ and $[R, P] \leqslant O_r(H)$. Thus $[Q, P]$ commutes with $[R, P]$. Using Lemma 1.4 we deduce that $[K_q, K_r] = 1$ and then that K is nilpotent.

Since $G = PK$ and P normalizes K, we have $K \leqslant F(G) = 1$; so $G = P \leqslant F_2(G)$, a contradiction. □

5. Reduction to the isolated case

Throughout the remainder of this paper, we fix $P \in \mathcal{F}$. We say that P is *isolated in* G if the only conjugate of P that commutes with P is P itself.

Lemma 5.1. *If $p = 2$ then P is isolated in G.*

Proof. Assume false and let $P_1 \neq P$ be a conjugate of P that commutes with P. Set $A = \langle P, P_1 \rangle$. Then $m(A) = 2$ and A has only one subgroup of order 2 that is not equal to P or P_1. Thus $A \in \mathcal{A}$, contrary to Theorem 4.5. □

The goal of this section is to extend the above result to odd primes.

Lemma 5.2. *If $P \leqslant S$ with S a p-group then $P \leqslant Z(S)$.*

Proof. Assume false and let S be a minimal counterexample. Set $A = \langle P^S \rangle$. Then A is noncyclic and elementary abelian. The previous lemma forces $p > 2$; so as $\mathcal{A} = \emptyset$ it follows that $m(A) = 2$. If S fails to centralize A then it has two orbits on the set of subgroups of A with order p, one of size 1, the other of size p. But A contains at least two members of $\mathcal{F}$; so it follows that $A \in \mathcal{A}$, a contradiction. Thus $A \leqslant Z(S)$ as claimed. $\square$

Lemma 5.3. *Let $P < A < G$ with A abelian and generated by conjugates of P. Then $A \leqslant O_p(C_G(P))$.*

Proof. Since $A \leqslant F_2(C_G(P))$, it suffices to show that $[O_{p'}(F(C_G(P))), A] = 1$. Note that $p > 2$ by Lemma 5.1.

Let $q \neq p$, choose $y \in [O_q(C_G(P)), A]$ and $g \in G$; let u be a generator for P and set $H = \langle P, (uy)^g \rangle$. Observe that $u^g, y^g \in H$ since u and y commute and have coprime orders. Now $P \leqslant F_2(H)$; so H is soluble. Since $y^g \in O_{p'}(C_G(u^g))$, we may apply the Goldschmidt Lemma to conclude that $y^g \in O_{p'}(H)$.

We have $O_{p'}(H) = C_{O_{p'}(H)}(P)[O_{p'}(H), P]$ and Lemma 3.1 implies that $[O_{p'}(H), P]$ is nilpotent. Set $M = \langle N_G(P), [O_{p'}(H), P] \rangle$. Lemmas 4.1 and 4.2 imply that $M \neq G$. Note that $y^g \in O_{p'}(H) \leqslant M$. Lemma 3.1 yields $[O_q(C_G(P)), A] \leqslant O_q(M)$ and we deduce that $\langle y, y^g \rangle$ is a q-group. Since g was arbitrary, the Baer–Suzuki Theorem forces $y \in O_q(G)$. Consequently, $[O_q(C_G(P)), A] = 1$, which completes the proof of this lemma. $\square$

Theorem 5.4.
(i) *P is isolated in G.*
(ii) *If $P \leqslant M < G$ then $M = N_M(P)[O_{p'}(F(M)), P]$.*

Proof. Suppose that $P \leqslant M < G$ and that P is isolated in M. Since $P \leqslant F_2(M)$ we may choose $S \in \mathrm{Syl}_p(F_2(M))$ with $P \leqslant S$. Then $SO_{p'}(F(M)) \trianglelefteq M$ and the Frattini Argument yields $M = N_M(S)O_{p'}(F(M))$. Now $P \leqslant Z(S)$ by Lemma 5.2 and since we are assuming that P is isolated in M we obtain $P \trianglelefteq N_M(S)$. Consequently, $M = N_M(P)O_{p'}(F(M))$ and applying Lemma 1.1(ii) we deduce that $M = N_M(P)[O_{p'}(F(M)), P]$. In particular, (ii) follows from (i).

Assume (i) to be false. Then there exists a conjugate P_1 of P such that $[P, P_1] = 1$ and $P \neq P_1$. Set $A = \langle P, P_1 \rangle \cong C_p \times C_p$. Note that $p > 2$ by Lemma 5.1. We shall derive a contradiction by showing that $A \in \mathcal{A}$. Let $g \in G$ and set $M = \langle P, g \rangle$.

Suppose that P is isolated in M. Then $M = N_M(P)[O_{p'}(F(M)), P]$. Lemmas 4.1 and 4.2 imply that $\langle N_G(P), [O_{p'}(F(M)), P] \rangle \neq G$ and we deduce that $\langle A, g \rangle \neq G$.

Suppose that P is not isolated in M. Then we can find $B \leqslant M$ such that $P < B \cong C_p \times C_p$ and B is generated by conjugates of P. By Lemma 5.3 we have $\langle A, B \rangle \leqslant O_p(C_G(P))$ and then Lemma 5.2 implies that $\langle A, B \rangle$ is elementary abelian. Since $\mathcal{A} = \emptyset$ we have $m(\langle A, B\rangle) < 3$; whence $A = B$ and we deduce that $\langle A, g\rangle \neq G$ in this case also.

What we have just done implies that $A \leqslant F_2(\langle A, g\rangle)$ for all $g \in G$. Thus $A \subseteq \mathcal{F}_2(G)$ and so $A \in \mathcal{A}$. This contradicts Theorem 4.5 and completes the proof. □

6. The even prime

The purpose of this section is to establish the following result.

Theorem 6.1. *Suppose that Q is a nilpotent p'-subgroup of G that is normalized by P and that $Q = [Q, P]$. Then $\langle Q^{N_G(P)}\rangle$ is nilpotent and, in particular, $\langle Q^{N_G(P)}\rangle \neq G$.*

If p is odd, the result follows from Lemmas 4.1 and 4.2. One way of dispensing with the case $p = 2$ is as follows: Theorem 4.5 and Glauberman's Z^*-Theorem imply that $P \leqslant Z(G \bmod O(G))$, consequently $G = PO(G)$. The Odd Order Theorem implies that $O(G)$ is soluble, a contradiction.

However, we prefer a more elementary approach since this may shed more light on more general problems than the one considered in this paper. Throughout the remainder of this section we assume that $p = 2$.

Lemma 6.2. $G = PO(G)$.

Proof. Recall that any element g of a group can be expressed uniquely as a commuting product of a 2-element, the 2-part of g, and a $2'$-element. Let u be a generator for P. Set

$$\Delta_0 = \{g \in G \mid g \text{ is a } 2'\text{-element}\} \quad \text{and}$$
$$\Delta_1 = \{g \in G \mid \text{ the 2-part of } g \text{ is conjugate to } u\}.$$

We claim that $\Delta_0 u \subseteq \Delta_1$. Indeed, let $g \in \Delta_0$ and set $M = \langle P, g\rangle$. Theorem 5.4 implies that $M = C_M(u)O(M)$. Let $\overline{M} = M/O(M)$ so then $O(\overline{M}) = 1$ and $\bar{u} \in Z(\overline{M})$. Since $\overline{M} = \langle \bar{u}, \bar{g}\rangle$, it follows that $\overline{M}$ is an abelian 2-group. Then $\bar{g} = 1$ and $\langle u\rangle \in \mathrm{Syl}_2(M)$. The 2-part of gu is nontrivial since $\overline{gu} = \bar{u}$ and is conjugate to u since $\langle u\rangle \in \mathrm{Syl}_2(M)$. Thus $\Delta_0 u \subseteq \Delta_1$, as claimed.

We claim also that $\Delta_1 u \subseteq \Delta_0$. Let $g \in \Delta_1$ and let M and $\overline{M}$ be as in the previous paragraph. Again, $\overline{M}$ is an abelian 2-group. Let S be a Sylow 2-subgroup of M that contains u. The 2-part of g is conjugate in M to a member of S. Using Theorem 5.4, we see that the 2-part of g is conjugate in M to u. Since $\overline{M}$ is abelian and $\overline{M} = \langle \bar{u}, \bar{g}\rangle$, we deduce that $\bar{u} = \bar{g}$. Thus $\overline{gu} = 1$; so $gu \in O(M)$ and then $gu \in \Delta_0$.

What we have just done implies that u permutes transitively the set $\Gamma = \{\Delta_0, \Delta_1\}$ by right multiplication. Observe that Δ_0 and Δ_1 are unions of conjugacy classes so any conjugate of u also has this property. The minimality of G implies that $G = \langle u^G \rangle$; so G acts transitively by right multiplication on Γ. Let

$$K = \ker\bigl(G \to \mathrm{Sym}(\Gamma)\bigr).$$

Then $K \trianglelefteq G$ and $G = PK$. Now $1 \in \Delta_0$, whence $K \subseteq \Delta_0$. Thus $K = O(G)$ and the lemma is proved. □

Lemma 6.3. *Let r be an odd prime and let $z \neq 1$ be an r-element of $C_G(P)$. Then $[O(F(C_G(z))), P]$ has order divisible by r.*

Proof. Assume false and let R be a Sylow r-subgroup of $C_G(P)$ that contains z. Set $H = N_G(R)$. Theorem 5.4 implies that $H = C_H(P)[O(F(H)), P]$. Now $P \leqslant C_H(R)$; whence $[O(F(H)), P] \leqslant C_H(R)$. Since $[O(F(H)), P] \leqslant [O(F(C_G(z))), P]$, we deduce that $[O(F(H)), P]$ is an r'-group and then that $R \in \mathrm{Syl}_r(N_G(R))$. Consequently, $R \in \mathrm{Syl}_r(G)$. Thus P centralizes a Sylow r-subgroup of G and it follows readily from Sylow's Theorem and from $G = PO(G)$ that P centralizes every P-invariant r-subgroup of G.

Let $g \in G$ be an r'-element and $M = \langle P, g \rangle$; so $M = C_M(P)[O(F(M)), P]$. Now $[O(F(M)), P]$ is nilpotent; so $[O_r(M), P]$ is a Sylow r-subgroup of $[O(F(M)), P]$. The previous paragraph and Lemma 1.1 imply that $[O(F(M)), P]$ is an r'-group. Considering the abelian group $M/[O(F(M)), P]$, we see that M is an r'-group. However, g was an arbitrary r'-element; so Lemma 1.7 forces $P \leqslant O_{r'}(G)$. But $G = \langle P^G \rangle$; so G is an r'-group, contrary to $z \neq 1$ and completing the proof of this lemma. □

We observe that since $G = PO(G)$ we have available the extension of Sylow's Theorems to groups with operators. Thus for $q \neq 2$, G possesses P-invariant Sylow q-subgroups, $C_G(P)$ acts transitively on these subgroups, and any P-invariant q-subgroup is contained in a P-invariant Sylow q-subgroup. It follows that G possesses a unique maximal P-invariant q-subgroup that is normalized by $C_G(P)$.

Lemma 6.4. *Let q be an odd prime and let $z \neq 1$ be a q-element that is inverted by P. Then $\langle z^{C_G(P)} \rangle$ is a q-group.*

Proof. Assume false and set $H = C_G(z)$. Note that H has odd order, that H is normalized by P, that $H = C_H(P)[H, P]$ and that $[H, P]$ is nilpotent. Now $z \in [z, P] \leqslant [H, P]$; so using Lemma 4.1 we conclude that $[H, P]$ is a q-group. Suppose that $1 \neq h \in C_H(P)$ has prime order $r \neq q$. Set $L = C_G(h)$. Then $z \in L$, whence $z \in [z, P] \leqslant [O(F(L)), P]$. Applying the previous lemma it follows that the nilpotent group $[O(F(L)), P]$ has order divisible by both q and r. Again

Lemma 4.1 supplies a contradiction. We deduce that $C_H(P)$ is a q-group and then that $C_G(z)$ is a q-group.

We claim that if Q and R are P-invariant Sylow q-subgroups of G that contain z then $[Q, P] = [R, P]$. We proceed by reverse induction on $|T|$ where $T = Q \cap R$. The result is vacuously true if $Q = R$; so assume that $Q \neq R$. Then $T < N_Q(T)$ and $T < N_R(T)$.

Let M be a maximal subgroup of G that contains $N_G(T)$. Let Q^* be a P-invariant Sylow q-subgroup of G chosen so that $Q \cap M \leqslant Q^* \cap M \in \mathrm{Syl}_q(M)$. We have $T < Q \cap M \leqslant Q \cap Q^*$; so the inductive hypothesis yields $[Q, P] = [Q^*, P]$. Similarly there exists a P-invariant Sylow q-subgroup R^* of G such that $R^* \cap M \in \mathrm{Syl}_q(M)$ and $[R, P] = [R^*, P]$.

Now $z \in [z, P] \leqslant [O_q(M), P]$ and $C_G(z)$ is a q-group. Hence, $C_M(O_q(M)) \leqslant O_q(M)$. Also, $G = PO(G)$; so G cannot involve $\mathrm{SL}_2(q)$. Glauberman's ZJ-Theorem implies that $ZJ(Q^* \cap M) \trianglelefteq M$. Consequently, $N_{Q^*}(Q^* \cap M) \leqslant N_{Q^*}(ZJ(Q^* \cap M)) \leqslant N_G(ZJ(Q^* \cap M)) = M$ and we deduce that $Q^* \leqslant M$. Since $M = C_M(P)[O(F(M)), P]$ and $[O(F(M)), P]$ is nilpotent, it follows that $[Q^*, P] = [O_q(M), P]$. Similarly $[R^*, P] = [O_q(M), P]$. Thus $[Q, P] = [R, P]$ as claimed.

Let Q be a P-invariant Sylow q-subgroup of G that contains z and choose $c \in C_G(P)$. Since P inverts z, we may invoke Lemma 3.2 to conclude that $\langle z, z^c\rangle$ is a P-invariant q-group. Let R be a P-invariant Sylow q-subgroup of G that contains $\langle z, z^c\rangle$. Then $[Q, P] = [R, P]$. Now P inverts z^c; whence $z^c \in [R, P]$. We deduce that $z^c \in Q$ for all $c \in C_G(P)$ and then that $\langle z^{C_G(P)}\rangle$ is a q-group. □

The proof of Theorem 6.1. By Lemma 4.1, we may suppose that Q is a q-group for some odd prime q. As we have already remarked, G possesses a unique maximal P-invariant q-subgroup V that is normalized by $C_G(P)$. If z is an element of Q that is inverted by P then the previous lemma implies that $z \in V$. But $Q = [Q, P]$; so Q is generated by such elements. Thus $Q \leqslant V$ and the theorem is proved. □

7. The final contradiction

Lemma 7.1. *If M is a maximal subgroup of G that contains P then $N_G(P) \leqslant M$ and $M = N_G(P)[O_{p'}(F(M)), P]$.*

Proof. This follows from Theorems 5.4 and 6.1. □

Lemma 7.2. $O_{p'}(F(C_G(P))) = 1$.

Proof. Set $T = O_{p'}(F(C_G(P)))$. Let $g \in G$ and let M be a maximal subgroup of G that contains $\langle P, g\rangle$. The previous lemma implies that $M =$

$N_G(P)O_{p'}(F(M))$. Consequently, $T \leqslant F_2(M)$ and also $T \leqslant O_{p'}(M)$. Since g was arbitrary, we deduce that $T \subseteq \mathcal{F}_2(G)$ and that $\langle T, g\rangle$ is a p'-group whenever g is a p'-element of G. Lemma 1.7 forces $T \leqslant O_{p'}(G)$. Then $T \subseteq \mathcal{F}_2(O_{p'}(G))$ and using the minimality of G we have $T \leqslant F_2(O_{p'}(G)) \leqslant F_2(G) = 1$. The result follows. $\square$

We are now in a position to derive a contradiction. We shall use $(\mathcal{U})$ to show that G satisfies the hypotheses of Lemma 1.8. Set

$$\Omega = \bigl\{Q \leqslant G \bigm| Q \text{ is a nilpotent } p'\text{-subgroup of } G \text{ that is normalized by } P \text{ and } Q = [Q, P]\bigr\}.$$

Lemma 3.1 shows that Ω satisfies (iii) of Hypothesis $(\mathcal{U})$; so it remains to verify (ii). Let $Q \in \Omega$ and let M be a maximal subgroup of G that contains PQ. Then $Q = [Q, P] \leqslant O_{p'}(F(M))$ and by Lemma 7.1 we have $N_G(P) \leqslant M$. The previous lemma forces $C_Q(P) = 1$. Consequently, if R is a P-invariant subgroup of Q then $C_R(P) = 1$; so $R = [R, P] \in \Omega$. We deduce that Ω satisfies Hypothesis $(\mathcal{U})$ and then $(\mathcal{U})$ implies that distinct maximal members of Ω have trivial intersection.

Let $M_1, \dots, M_n$ be the distinct maximal subgroups of G that contain P. Since $\langle P, g\rangle \neq G$, for all $g \in G$ we have

$$G = M_1 \cup \cdots \cup M_n \quad \text{and} \quad n \geqslant 2.$$

Set $H = N_G(P)$. Lemma 7.1 implies that $H \leqslant M_i$ for all i. Also, $H < M_i$ since otherwise H would be a maximal subgroup of G, contrary to $n \geqslant 2$. It is an easy consequence of the minimality of G that $H_G = 1$.

For each i, set $Q_i = [O_{p'}(F(M_i)), P]$. Lemma 7.1 implies that $M_i = HQ_i$ and so $1 \neq Q_i \in \Omega$. Now H permutes the maximal members of Ω that contain Q_i; so $(\mathcal{U})$ and the maximality of M_i imply that Q_i is a maximal member of Ω. Let $i \neq j$. Then

$$M_i \cap M_j = H(Q_i \cap M_j).$$

Set $T = Q_i \cap M_j$. Now $C_T(P) \leqslant C_{Q_i}(P) \leqslant O_{p'}(F(C_G(P))) = 1$; so $T = [T, P] \leqslant Q_i \cap [O_{p'}(F(M_j)), P] = Q_i \cap Q_j$. By $(\mathcal{U})$ we have $Q_i \cap Q_j = 1$; so we deduce that $M_i \cap M_j = H$ for all $i \neq j$. Lemma 1.8 implies that $F(G) \neq 1$, a contradiction. This completes the proof of Theorem A.

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