Generating finite groups with conjugates of a subgroup

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1 Introduction

Suppose that H is a subgroup of a finite group G and that G is generated by the conjugates of H. In this paper we consider the question:

how many conjugates of H are needed to generate G?

In order to answer this question we must study chains of subgroups that start with H and end with G. The *chain length of H in G* is defined by

$$\mathrm{cl}_G(H) = \max\{n \in \mathbf{N} \mid \text{ there is a chain } H = M_0 < M_1 < \ldots < M_n = G\}.$$

We are using the notation $A < B$ to mean that A is a proper subgroup of B. It is almost trivial to see that G can be generated by a set consisting of at most $\mathrm{cl}_G(H) + 1$ conjugates of H. Indeed, suppose that every set consisting of at most $\mathrm{cl}_G(H)$ generates a proper subgroup of G. Then using the fact that G is generated by the conjugates of H we can construct a chain

$$H = H_0 < \langle H_0, H_1\rangle < \langle H_0, H_1, H_2\rangle < \ldots < \langle H_0, \ldots, H_{\mathrm{cl}_G(H)}\rangle \leq G$$

where each H_i is a conjugate of H. This chain has length $\mathrm{cl}_G(H) + 1$ so the last inclusion cannot be proper. Thus G can be generated by $\mathrm{cl}_G(H) + 1$ conjugates of H.

If H is a maximal subgroup of G then $\mathrm{cl}_G(H) = 1$ and $\mathrm{cl}_G(H) + 1$ conjugates of H are needed to generate G. However, if H is not maximal in G the situation is different. We will prove that fewer than $\mathrm{cl}_G(H) + 1$ conjugates of H are required to generate G unless the structure of G is very restricted. Our main theorem is:

Theorem A *Let H be a subgroup of a finite group G such that $G = \langle H^G \rangle$ and $\mathrm{cl}_G(H) \geq 2$. Then every set consisting of at most $\mathrm{cl}_G(H)$ conjugates of H generates a proper subgroup of G if and only if G/H_G has the following structure.*

(i) *G/H_G is a Frobenius group with cyclic complement H/H_G;*

(ii) *There is a prime p such that the Frobenius kernel of G/H_G is an elementary abelian p-group;*

(iii) *Considered as a GF(p)H-module, the kernel of G/H_G is the direct sum of $\mathrm{cl}_G(H)$ irreducible, nontrivial isomorphic GF(p)H-modules.*

Theorem A has two immediate corollaries, the latter of which is reminiscent of Baer's Criterion for a group to have a nontrivial normal p-subgroup.

Corollary B *Let H be a subgroup of a finite group G such that $G = \langle H^G \rangle, \mathrm{cl}_G(H) \geq 2$ and G/H_G is insoluble. Then G can be generated by a set consisting of at most $\mathrm{cl}_G(H)$ conjugates of H.*

Corollary C *Let H be a subgroup of a finite group G such that $\mathrm{cl}_G(H) \geq 2$ and G/H_G is insoluble. Suppose that every set consisting of at most $\mathrm{cl}_G(H)$ conjugates of H generates a proper subgroup of G. Then $G \neq \langle H^G \rangle$.*

Corollary B raises the following question for further study:

> *suppose that H is a subgroup of a finite group G such that $G = \langle H^G \rangle, \mathrm{cl}_G(H) \geq 3, G/H_G$ is insoluble and every set consisting of at most $\mathrm{cl}_G(H) - 1$ conjugates of H generates a proper subgroup of G. What can be said about the structure of G?*

This situation can happen if $H \cong \mathbf{Z}_2$ and $\mathrm{cl}_G(H) = 3$. This occurs in A_5.

The proof of Theorem A uses ideas similar to those used by the author in [2] and [3].

2 Notation and Quoted Results

Throughout this paper, group means finite group, $A \leq B$ means A is a subgroup of B, $A < B$ means A is a proper subgroup of B and $A \trianglelefteq B$ means A is a normal subgroup of B. If A and B are subgroups of a group G then

$$A^B = \{A^b \mid b \in B\}.$$

Note that A^B is a set of conjugates of A, not the subgroup generated by those conjugates, which we shall denote by

$$\langle A^B \rangle.$$

If $H < G$ then

$$H_G = \bigcap \{H^g \mid g \in G\} = \text{ the core of } H \text{ in } G.$$

Notice that H_G is the largest subgroup of H that is normal in G. If $H \leq G$ then $H^\sharp = H - \{1\}$ and

$$\mathrm{cl}_G(H) = \max \{n \in \mathbf{N} \mid \text{ there is a chain } H = H_0 < H_1 < \ldots < H_n = G\}.$$

A subgroup H of a group G is a *Frobenius complement* in G if $1 < H < G$ and $H \cap H^g = 1$ for all $g \in G - H$. G is a *Frobenius group* if it has a Frobenius complement. If H is a Frobenius complement in a group G then the set

$$K = G - \cup\{H^{\sharp g} \mid g \in G\}$$

is called the *Frobenius kernel* of G.

Frobenius' Theorem *Let G be a Frobenius group with complement H. Then the Frobenius kernel K of G is a normal subgroup of G with order coprime to the order of H, $G = HK$ and $H \cap K = 1$.*
[1, (35.24) page 191]

Thompson's Theorem *Frobenius kernels are nilpotent.*
[1, (40.8) page 207]

3 Preliminary Lemmas

Lemma 3.1 *Let* $1 < V \trianglelefteq G, H \leq G$ *and suppose that* $V \cap H = 1$. *Set* $G^* = G/V$. *Then.*

(i) $\mathrm{cl}_{G^*}(H^*) \leq \mathrm{cl}_G(H) - 1$;

(ii) *If* H *does not normalize any nontrivial proper subgroups of* V *then* $\mathrm{cl}_{G^*}(H^*) = \mathrm{cl}_G(H) - 1$.

Proof Suppose $H^* = H_0^* < H_1^* < \ldots < H_r^* = G^*$ is a chain of subgroups in G^*. Let H_i be the inverse image of H_i^* in G. Then $H < HV = H_0 < H_1 < \ldots < H_r = G$ is a chain of subgroups in G of length $r + 1$. Thus $r + 1 \leq \mathrm{cl}_G(H)$ and hence $\mathrm{cl}_{G^*}(H^*) \leq \mathrm{cl}_G(H) - 1$.

Now suppose that H does not normalize any nontrivial proper subgroups of V. Let $c = \mathrm{cl}_G(H)$. Then there exists a chain $H = H_0 < H_1 < \ldots < H_c = G$ of length c in G. Thus we have

$$H^* = H_0^* \leq H_1^* \leq \ldots \leq H_c^* = G^* \tag{1}$$

and since $\mathrm{cl}_{G^*}(H^*) \leq c-1$, there exists i such that $H_i^* = H_{i+1}^*$. Choose i minimal with this property. Then $H_iV = H_{i+1}V$ so $H_i < H_{i+1} \leq H_iV$ whence $H_{i+1} = H_i(H_{i+1} \cap V)$. Now $H_i \neq H_{i+1}$ so $H_{i+1} \cap V \neq 1$ and as $H \leq H_{i+1}$, we have that $H_{i+1} \cap V = V$ so $V \leq H_{i+1}$. Since $H_{i+1} < \ldots < H_c$ this implies that $H_{i+1}^* < H_{i+2}^* < \ldots < H_c^*$ so all except one of the containments in (1) is strict. We deduce that $\mathrm{cl}_{G^*}(H^*) = c - 1$.

Lemma 3.2 *Let* $1 < H < G$ *and suppose that* $N_G(P) \leq H$ *whenever* $1 < P \leq H$. *Then* H *is a Frobenius complement in* G.

Proof This is well known.

Lemma 3.3 *Let* H *be a Frobenius complement in a group* G. *Let* $g \in G$. *Then* $\langle H, g\rangle = \langle H, H^g\rangle = H\langle g^H\rangle$.

Proof We may assume that $g \notin H$. Let $M = \langle H, H^g \rangle$. Then $1 < H < M$ so H is a Frobenius complement in M, as is H^g. Let K be the Frobenius kernel of M. Then K is a nilpotent Hall subgroup of M and H and H^g are complements to K in M. The Schur-Zassenhaus Theorem implies that there exists $m \in M$ such that $H^g = H^m$. Then $H^{gm^{-1}} = H$ forcing $gm^{-1} \in H \leq M$ whence $g \in M$. We deduce that $\langle H, g \rangle = \langle H, H^g \rangle$. Clearly $\langle H, g \rangle = H \langle g^H \rangle$.

Lemma 3.4 *Let p be a prime; H a cyclic p'-group; X a faithful irreducible $GF(p)H$-module; n a natural number; V a $GF(p)H$-module that is the direct sum of n submodules each isomorphic to X and let $G = HV$, the semidirect product of V considered as an abelian group with H. Then $G = \langle H^G \rangle, \mathrm{cl}_G(H) = n$ and every set consisting of at most n conjugates of H generates a proper subgroup of G.*

Proof Let $V = U_1 \oplus \ldots \oplus U_n$ where each U_i is a submodule isomorphic to X. Since each U_i is nontrivial and irreducible we have $U_i = [U_i, H]$ and so $V = [V, H] \leq \langle H^G \rangle$. Now $H \leq \langle H^G \rangle$ hence $G = \langle H^G \rangle$.

Since X is irreducible, we know that $\mathrm{End}_H(X)$ is a field. This field contains H since H is abelian. Let F be the subfield of $\mathrm{End}_H(X)$ generated by H. Then X is an F-vectorspace. The irreducibility of X implies that $\dim_F(X) = 1$. Also, V and each U_i are F-vectorspaces, $\dim_F(U_i) = 1$ and $\dim_F(V) = n$.

Suppose that $H = H_0 < H_1 < \ldots < H_r = G$ is a chain of subgroups. Since $G = HV$, for each i we have $H_i = HW_i$ where $W_i = H_i \cap V$. Each W_i is H-invariant and hence an F-vectorspace. Moreover, $0 = W_0 < W_1 < \ldots < W_r = V$ so as $\dim_F(V) = n$ we deduce that $r \leq n$. Thus $\mathrm{cl}_G(H) \leq n$. The chain $H < HU_1 < H(U_1 \oplus U_2) < \ldots < H(U_1 \oplus \ldots \oplus U_n) = G$ has length n so $\mathrm{cl}_G(H) = n$.

Let $g_1, \ldots, g_n \in G$. We will show that $\langle H^{g_1}, \ldots, H^{g_n} \rangle \neq G$. By conjugating by g_1^{-1} there is no loss of generality in supposing that $g_1 = 1$. For each i, choose $h_i \in H$ and $v_i \in V$ such that $g_i = h_i v_i$. Then

$$\langle H^{g_1}, \ldots, H^{g_n} \rangle = \langle H, H^{v_2}, \ldots, H^{v_n} \rangle \leq \langle H, v_2, \ldots, v_n \rangle = HW$$

where

$$W = \langle \langle v_2, \ldots, v_n \rangle^H \rangle.$$

Now W is a subgroup of V that is H-invariant. Hence it is F-invariant and therefore an F-subspace of V. Considered as an F-subspace, we see that every member of W can be written as a linear combination of the v_i with coefficients in H. Thus $\{v_2, \ldots, v_n\}$ is an F-spanning set for W. Hence $\dim_F(W) \leq n-1 < \dim_F(V)$. We deduce that $W \neq V$ and finally that $\langle H^{g_1}, \ldots, H^{g_n} \rangle \neq G$.

4 The Minimal Counterexample

Throughout the remainder of this paper, we assume the following hypothesis, which is satisfied by a minimal counterexample to Theorem A.

Hypothesis

(i) *H is a subgroup of G and $n = \mathrm{cl}_G(H)$;*

(ii) *$G = \langle H^G \rangle$ and $n \geq 2$;*

(iii) *Every set consisting of at most n conjugates of H generates a proper subgroup of G;*

(iv) *$H_G = 1$;*

(v) *If $Y < X$ with $|X| < |G|, X = \langle Y^X \rangle, \mathrm{cl}_X(Y) \geq 2$ and if every set consisting of at most $\mathrm{cl}_X(Y)$ conjugates of Y generates a proper subgroup of X then X has the structure given in the conclusion of Theorem A.*

We will eventually prove that G has the structure given in the conclusion of Theorem A. This, together with Lemma 3.4 will prove Theorem A.

Definition *An r-tuple $(H_1, \ldots, H_r)$ of conjugates of H is* good *if*

$$H_i \not\leq N_G(H_1) \text{ for all } i, 2 \leq i \leq r$$

and

$$H_1 < \langle H_1, H_2 \rangle < \langle H_1, H_2, H_3 \rangle < \ldots < \langle H_1, H_2, H_3, \ldots, H_r \rangle.$$

Notice that the above chain has length $r-1$.

Lemma 4.1 *Let $(H_1, \ldots, H_r)$ be a good r-tuple of conjugates of H, with $r < n+1$. Then there exist $H_{r+1}, \ldots, H_{n+1} \in H^G$ such that $(H_1, \ldots, H_{n+1})$ is good.*

Proof Since $r \leq n$ we have $H_1 \leq \langle H_1, \ldots, H_r \rangle < G = \langle H_1^G \rangle$ so there exists a conjugate D of H_1 that does not normalize $\langle H_1, \ldots, H_r \rangle$. In particular, $D \not\leq \langle H_1, \ldots, H_r \rangle$. In the case that D does not normalize H_1, let $H_{r+1} = D$. In the case that D does normalize H_1 we observe that since D does not normalize $\langle H_1, \ldots, H_r \rangle$, there exists $i, 2 \leq i \leq r$ and $d \in D$ such that $H_i^d \not\leq \langle H_1, \ldots, H_r \rangle$. Since $(H_1, \ldots, H_r)$ is good it follows that H_i does not normalize H_1, but as d does normalize H_1 we see that H_i^d cannot normalize H_1. Let $H_{r+1} = H_i^d$. In both cases $(H_1, \ldots, H_{r+1})$ is good.

Repeated application of the above procedure completes the proof of this lemma.

Lemma 4.2 *Let $(H_1, \ldots, H_{n+1})$ be a good $(n+1)$-tuple of conjugates of H. Then the following hold.*

(i) *$G = \langle H_1, \ldots, H_{n+1} \rangle$;*

(ii) *If σ is any permutation of $\{2, \ldots, n+1\}$ then $(H_1, H_{2\sigma}, \ldots, H_{(n+1)\sigma})$ is good;*

(iii) *$\langle H_1, \ldots, H_i \rangle$ is a maximal subgroup of $\langle H_1, \ldots, H_{i+1} \rangle$ for each i, $1 \leq i \leq n$;*

(iv) *If $Y = \langle H_1, \ldots, H_n \rangle$ then $\mathrm{cl}_Y(H_1) = n - 1$.*

Proof Trivial.

Lemma 4.3 *Let $(H_1, \ldots, H_r)$ be a good r-tuple of conjugates of H, then*

$$\langle H_1, \ldots, H_r \rangle = \langle H_1^{\langle H_1, \ldots, H_r \rangle} \rangle.$$

Proof Let $2 \leq i \leq r$. Then (H_1, H_i) is a good 2-tuple. Using Lemmas 4.1 and 4.2(ii),(iii) we see that H_1 is a maximal subgroup of $\langle H_1, H_i \rangle$.

Since H_i does not normalize H_1, we have $\langle H_1, H_i \rangle = \langle H_1^{\langle H_1, H_i \rangle} \rangle$ hence $H_i \leq \langle H_1^{\langle H_1, \ldots, H_r \rangle} \rangle$ and the result follows.

For the remainder of this paper we fix the following notation:

> Let $H_1 = H$ and let $H_2, \ldots, H_{n+1}$ be chosen in accordance with Lemma 4.1, so that $(H_1, \ldots, H_{n+1})$ is good.

Let

$$\begin{aligned} M &= \langle H_1, \ldots, H_{n-1}, H_n \rangle, \\ L &= \langle H_1, \ldots, H_{n-1}, H_{n+1} \rangle, \\ D &= \langle H_1, \ldots, H_{n-1} \rangle, \text{ and} \\ N &= N_G(H). \end{aligned}$$

Lemma 4.4 (i) *$(H_1, \ldots, H_{n-1}, H_n)$ and $(H_1, \ldots, H_{n-1}, H_{n+1})$ are good;*

(ii) *M and L are distinct maximal subgroups of G;*

(iii) *$D = M \cap L$ and D is a maximal subgroup of M and of L.*

Proof Trivial.

5 The Case $n = 2$

In this section we assume that $n = 2$. In particular, we have $D = H$.

Lemma 5.1 *Let $k \in G$ be such that $H^k \neq H$. Let $K = \langle H, H^k \rangle$. Let $g \in G$. Then $H^g \leq K$ or $H^g \cap K \leq H_K$.*

Proof Since G cannot be generated by two conjugates of H we have $H < K < G$. Since $n = 2$ it follows that H is maximal in K and that K is maximal in G.

Let $x \in G$ be such that $H^x \not\leq K$. Let $E = H^x \cap K$. If $E \not\leq H$ then $K = \langle H, E \rangle$ whence $K \leq \langle H, H^x \rangle$. Since G cannot be generated by two conjugates of H, and as K is a maximal subgroup of G, we see that $K = \langle H, H^x \rangle$ hence $H^x \leq K$, a contradiction. Thus $H^x \cap K \leq H$.

Now suppose that $H^g \not\leq K$. Then $H^{gk} \not\leq K$ for all $k \in K$ so the previous paragraph implies $H^{gk} \cap K \leq H$ for all $k \in K$. Thus $H^g \cap K \leq \bigcap \{ H^{k^{-1}} \mid k \in K \} = H_K$.

Lemma 5.2 $H_M = H_L = 1$.

Proof Since $M = \langle H, H_2 \rangle \not\leq N$, we may choose $m \in M$ such that $H^m \neq H$. By Lemma 4.4(iii) we have $M \cap L = H$ so $H^m \not\leq L$. Now $L = \langle H, H_3 \rangle$ so Lemma 5.1 implies that $H^m \cap L \leq H_L$. Then $H_M = H_M^m \leq H^m \cap H \leq H^m \cap L \leq H_L$ so $H_M \leq H_L$.

The preceding argument with L in place of M implies that $H_L \leq H_M$, hence $H_M = H_L \trianglelefteq \langle M, L \rangle$. Lemma 4.4(ii) implies that $\langle M, L \rangle = G$ so H_M is a normal subgroup of G contained in H. Thus $H_M \leq H_G = 1$. Hence result.

Lemma 5.3 *Let $1 < P \leq H$. Then $N_G(P) \leq N$.*

Proof Let $g \in N_G(P)$. Then $1 < P \leq H^g \cap M$ and as $M = \langle H, H_2 \rangle$, Lemmas 5.1 and 5.2 imply that $H^g \leq M$. Similarly $H^g \leq L$ whence $H^g \leq M \cap L = H$. We deduce that $N_G(P) \leq N$.

Lemma 5.4 *Let $g \in G$ and suppose that $H^g \leq N$. Then $g \in N$.*

Proof Assume false. Then $H^g \neq H$ so $H < N < G$ and since $\mathrm{cl}_G(H) = 2$ we see that H is a maximal subgroup of N and that N is a maximal subgroup of G. Since $H \trianglelefteq N$ it follows that $N = HH^g$ and as $g \notin N$ we also have $G = \langle N, g \rangle$.

Next we consider the subgroup $N^{g^{-1}}$. The factorization $N^{g^{-1}} = H^{g^{-1}}H$ implies that H is not a Hall subgroup of $N^{g^{-1}}$. Thus there exists a Sylow subgroup P of H such that $N_{N^{g^{-1}}}(P) \not\leq H$. Using the previous lemma, we see that $H < \langle H, N_{N^{g^{-1}}}(P) \rangle \leq N \cap N^{g^{-1}} \leq N$. The maximality of H in N forces $N = N^{g^{-1}}$. Since $G = \langle N, g \rangle$ we have that $H \trianglelefteq N \trianglelefteq G$ whence $G = \langle H^G \rangle \leq N < G$, a contradiction. We deduce that $g \in N$.

Lemma 5.5 *G is a Frobenius group with complement H*

Proof First we prove that N is a Frobenius complement in G. Suppose not. Then by Lemma 3.2 there exists P such that $1 < P \leq N$ and $N_G(P) \not\leq N$. Choose $g \in N_G(P) - N$ and set $T = \langle H, H^g \rangle$. Then $H < T < G$ so as $\mathrm{cl}_G(H) = 2$, we see that H is maximal in T and that T is maximal in G.

Observe that P normalizes both H and H^g so $P \leq N_G(T)$. Since H is not contained in any proper normal subgroup of G, we have that $N_G(T) = T$, whence $P \leq T$. We have $H \leq T \cap N \leq T$ so either $H = T \cap N$ or $T \cap N = T$. Lemma 5.4 and the fact that $g \notin N$ imply that $H^g \not\leq N$ so as $H^g \leq T$, we see that $T \cap N \neq T$. Thus $H = T \cap N$, and in particular, $P \leq H$. Lemma 5.3 implies that $N_G(P) \leq N$, a contradiction. We deduce that N is a Frobenius complement in G.

Frobenius' Theorem implies that G contains a normal subgroup K such that $G = NK$ and $N \cap K = 1$. But $H \trianglelefteq N$ whence $HK \trianglelefteq G$. So as $G = \langle H^G \rangle$, we have $NK = HK$. Since $N \cap K = 1$, we see that $N = H$, hence result.

The previous lemmas together with Frobenius' Theorem imply that G contains a normal subgroup K, the Frobenius kernel of G, such that

$$G = HK \text{ and } H \cap K = 1.$$

Lemma 5.6 *There is a prime p such that K is an elementary abelian p-group. Considered as a $GF(p)H$-module, K is the direct sum of two irreducible, nontrivial isomorphic $GF(p)H$-modules. Moreover, H is cyclic.*

Proof Let $a \in K^\sharp$ and let $U = \langle a^H \rangle$. Lemma 3.3 implies that $\langle H, H^a \rangle = HU$ so using the hypothesis that G cannot be generated by two conjugates of H we see that $H < HU < G$. Since $\mathrm{cl}_G(H) = 2$ it follows that H is maximal in HU and that HU is maximal in G. In particular, U is both a maximal and a minimal H-invariant subgroup of K.

Thompson's Theorem implies that K is nilpotent, hence $N_K(U) > U$ and it follows that $U \trianglelefteq K$. Now U is nilpotent and is a minimal H-invariant subgroup of K so we see that U is an elementary abelian p-group for some prime p on which H acts irreducibly.

Now choose $b \in K - U$ and let $W = \langle b^H \rangle$. Then again, $W \trianglelefteq K$ and W is an elementary abelian q-group for some prime q on which H acts irreducibly.

If $p \neq q$ then as K is nilpotent we have $[a, b] = 1$ so ab is an element with order pq. However, the previous argument shows that every element of $K^\sharp$ has prime order, a contradiction. Thus $p = q$. Since U and W are minimal H-invariant subgroups of K we have $U \cap W = 1$ and as they are maximal H-invariant subgroups of K we have $\langle U, W \rangle = K$. It follows that $K = U \times W$. Thus K is an elementary abelian p-group.

We are left with the task of proving that U is isomorphic to W as an H-module and that H is cyclic. Choose $u \in U^\sharp$ and $w \in W^\sharp$. Set $t = uw$ and $T = \langle t^H \rangle$. The projection maps

$$\pi_U : T \longrightarrow U \quad \text{and} \quad \pi_W : T \longrightarrow W$$

are H-module homomorphisms. Since $t\pi_U = u$ and $t\pi_W = w$ it follows that π_U and π_W are nontrivial. As previously we have that T is an irreducible H-module, as are U and W. It follows that π_U and π_W are H-module isomorphisms. Also, $\pi_U^{-1}\pi_W$ is an H-module isomorphism $U \to W$ that maps u to w.

By keeping w fixed and letting u range over $U^\sharp$ we see that $\mathrm{End}_H(U)$ acts transitively on $U^\sharp$. Let $E = \mathrm{End}_H(U)$. Then U is an irreducible E-module and so $\mathrm{End}_E(U)$ is a field. Now $H \subseteq \mathrm{End}_E(U)$ and as the multiplicative group of a finite field is cyclic, we deduce that H is cyclic.

6 The Case $n \geq 3$

In this section we assume that $n \geq 3$.

Lemma 6.1 (i) *M and L are Frobenius groups with cyclic complement H.*

(ii) *There is a prime p such that the Frobenius kernels of M and L are elementary abelian p-groups.*

(iii) *Considered as a $GF(p)H$-module, the kernel of M is the direct sum of $n-1$ nontrivial, irreducible isomorphic $GF(p)H$-modules.*

(iv) *The kernel of L satisfies (iii) also.*

Proof Lemma 4.3 implies that $M = \langle H^M \rangle$ and Lemma 4.2(iv) implies that $\mathrm{cl}_H(M) = n - 1$. Since $G = \langle M, H_{n+1} \rangle$ and since G cannot be generated by n conjugates of H, we see that M cannot be generated by $n - 1$ conjugates of H. Then M/H_M is a Frobenius group with complement H/H_M. Let $\overline{M} = M/H_M$. Since $n \geq 3$ we have that $D > H$ so $\overline{D}$ is a Frobenius group with complement $\overline{H}$. In particular, the only subgroup of $\overline{H}$ that is normal in $\overline{D}$ is 1. Now $\overline{H_L} \trianglelefteq \overline{D}$, whence $\overline{H_L} = 1$ forcing $H_L \leq H_M$. A similar argument with the roles of M and L interchanged proves $H_M \leq H_L$. Thus $H_L = H_M \trianglelefteq \langle M, L \rangle = G$ and as $H_G = 1$, we deduce that $H_L = H_M = 1$.

We have that M is a Frobenius group with cyclic complement H and whose kernel is an elementary abelian p-group, which considered as a $\mathrm{GF}(p)H$-module is the direct sum of $n-1$ irreducible, nontrivial isomorphic $\mathrm{GF}(p)H$-modules. Similarly, there is a prime q such that L is a Frobenius group with complement H whose kernel is an elementary abelian q-group etc. Since $n \geq 3$, we have $M \cap L = D > H$ whence $p = q$.

Lemma 6.2 *G is a Frobenius group with complement H and kernel $O_p(G)$. Moreover, $O_p(G) = \langle O_p(M), O_p(L) \rangle$.*

Proof Lemma 6.1 implies that $M = HO_p(M)$. Since $H \leq D$ we have $D = H(D \cap O_p(M))$ and as H has order coprime to p we see that $D \cap O_p(M) = O_p(D)$. By Lemma 6.1, $O_p(M)$ is abelian so $O_p(D) \trianglelefteq O_p(M)$ and hence $O_p(D) \trianglelefteq M$. Similarly, $O_p(D) \trianglelefteq L$ and it follows that $O_p(D) \trianglelefteq G$. As $n \geq 3$, we have $H < D$, but as $D = HO_p(D)$ it follows that $O_p(D) \neq 1$.

Let V be a minimal normal subgroup of G that is contained in $O_p(D)$ and set $G^* = G/V$. Since $O_p(M)$ and $O_p(L)$ centralize $O_p(D)$, we have the factorization $G = HC_G(V)$ so H normalizes no nontrivial proper subgroup of V. Moreover, $H \cap V = 1$ so Lemma 3.1 implies that $\mathrm{cl}_{G^*}(H^*) = n - 1$. Since $G = \langle H^G \rangle$ we also have $G^* = \langle H^{*G^*} \rangle$.

Suppose that G^* can be generated by $n - 1$ conjugates of H^*. Then there exist $A_1, \ldots, A_{n-1} \in H^G$ such that $G = V\langle A_1, \ldots, A_{n-1} \rangle$. Since V is an abelian minimal normal subgroup of G we see that $\langle A_1, \ldots, A_{n-1} \rangle$ is a maximal subgroup of G. Since $G = \langle H^G \rangle$, there exists $A_n \in H^G$ such that $A_n \not\leq \langle A_1, \ldots, A_{n-1} \rangle$. Then $G = \langle A_1, \ldots, A_n \rangle$, contrary to hypothesis. Thus G^* cannot be generated by $n - 1$ conjugates of H^*.

Next we show that $H^*_{G^*} = 1$. Let E be the inverse image of $H^*_{G^*}$ in G. Then $V \leq E = V(H \cap E) \trianglelefteq G$ and $E \trianglelefteq M$. Using a Frattini Argument and the Schur-Zassenhaus Theorem, we have that $M = EN_M(H \cap E)$ and then $M = VN_M(H \cap E)$. But $VH \leq D < M$ so as H is a Frobenius complement in M we must have $H \cap E = 1$. Then $E = V$ and $H^*_{G^*} = 1$.

The previous two paragraphs together with the hypothesis on G imply that G^* is a Frobenius group with complement H^* and kernel $O_q(G^*)$ for some prime q. Now $M = HO_p(M)$ and as $O_p(M) > V$, we see that $|M^* : H^*|$ is divisible by p, so as $G^* = H^*O_q(G^*)$ we have $p = q$. Thus $G = HO_p(G)$ and as H has order coprime to p, it follows that $\langle O_p(M), O_p(L)\rangle \leq O_p(G)$. Since M is a maximal subgroup of G, we see that $O_p(M)$ is a maximal H-invariant subgroup of $O_p(G)$, whence $\langle O_p(M), O_p(L)\rangle = O_p(G)$.

Let $g \in G$ and suppose that $H \cap H^g \neq 1$. Now $H \cap V = 1$ hence $H^* \cap H^{*g^*} \neq 1$ so the fact that H^* is a Frobenius complement in G^* forces $g^* \in H^*$. Then $g \in HV \leq M$ and since H is a Frobenius complement in M, we have $g \in H$. We deduce that G is a Frobenius group with complement H. Since $H \cap O_p(G) = 1$ and as $G = HO_p(G)$, it follows that $O_p(G)$ is the Frobenius kernel of G.

Lemma 6.3 *$O_p(G)$ is an elementary abelian p-group that, considered as a $GF(p)H$-module is, the direct sum of n nontrivial, irreducible isomorphic $GF(p)H$-submodules.*

Proof Let $m \in O_p(M) - O_p(D)$ and $l \in O_p(L) - O_p(D)$. Then $H \neq H^m, m \in \langle H, H^m\rangle \leq M, H \neq H^l$ and $l \in \langle H, H^l\rangle \leq L$. Now $O_p(D) = O_p(M) \cap O_p(L)$ so as $l \notin M$ and $l \in \langle H, H^l\rangle$, it follows that $H^l \not\leq \langle H, H^m\rangle$. Since $N_G(H) = H$, the 3-tuple (H, H^m, H^l) is good. By Lemma 4.1, there exist $B_4, \ldots, B_{n+1} \in H^G$ such that $(H, H^m, H^l, B_4, \ldots, B_{n+1})$ is a good $(n+1)$-tuple. All the preceding arguments can be carried out using this $(n+1)$-tuple in place of $(H_1, \ldots, H_{n+1})$. Thus $T = \langle H, H^m, H^l, B_4, \ldots, B_n\rangle$ is a Frobenius group with complement H and abelian kernel $O_p(T) = T \cap O_p(G)$. Since $m, l \in T \cap O_p(G)$, it follows that m and l commute. Since $O_p(M)$ and $O_p(L)$ are elementary abelian, we deduce that $O_p(G)$ is elementary abelian.

Now we regard $O_p(G)$ as a GF$(p)H$-module. Maschke's Theorem, Lemma 6.1(iii) and the fact that D is maximal in M imply that there exist nontrivial isomorphic irreducible GF$(p)H$-modules $V_1, \ldots, V_{n-1}$ such that

$O_p(M) = V_1 \oplus \ldots \oplus V_{n-1}$ and $O_p(D) = V_1 \oplus \ldots \oplus V_{n-2}$. Since $O_p(M) \nleq O_p(L)$, there exists an irreducible GF$(p)H$-submodule W of $O_p(L)$ such that $W \nleq O_p(M)$. Since $O_p(M)$ is a maximal GF$(p)H$-submodule of $O_p(G)$, we have $O_p(G) = O_p(M) \oplus W$. Lemma 6.1(iv) and the fact that $n \geq 3$ imply $W \cong V_1$. This completes the proof of Lemma 6.3.

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