Journal of Algebra **232**, 578–616 (2000)
doi:10.1006/jabr.2000.8407, available online at http://www.idealibrary.com on IDEAL®

Generating Finite Groups with Conjugates of a Subgroup, II

Paul Flavell

The School of Mathematics and Statistics, The University of Birmingham, Birmingham B15 2TT, United Kingdom
E-mail: p.j.flavell@bham.ac.uk

Communicated by G. Glauberman

Received May 6, 1997

1. INTRODUCTION

Suppose that H is a subgroup of a finite group G and that G is generated by the conjugates of H. In this paper, we consider the following question:

when can G be generated by two conjugates of H?

We began the study of this question in [2]. In order to discuss the results proved in [2] and in this paper, we use the following notation. The *chain length* of H in G is defined by

$$\mathrm{cl}_G(H) = \max\{n \in \mathbf{N} \mid \text{there is a chain } H = K_0 < K_1 < \cdots < K_n = G\}.$$

If $\mathrm{cl}_G(H) = n$ then we write $H <_n G$ and say that H is *strongly nth maximal in G*.

If $\mathrm{cl}_G(H) = 1$ then H is maximal in G and the answer to the question is yes. In [2] we considered the case $\mathrm{cl}_G(H) = 2$. We proved that the answer is yes, unless G has a very restricted structure. Note that there is no loss in factoring out by H_G, the largest normal subgroup of G contained in H and hence assuming $H_G = 1$. In [2] we proved:

THEOREM A. *Let H be a subgroup of a finite group G such that $G = \langle H^G \rangle$ and $H_G = 1$. Suppose that $\mathrm{cl}_G(H) = 2$ and that G cannot be generated by two conjugates of H. Then there exists a prime p and a faithful irreducible $GF(p)H$-module U such that*:

- *$G \cong H(U \oplus U)$, the semidirect product of H with $U \oplus U$.*

0021-8693/00 $35.00

• *If* $F = \mathrm{End}_H(U)$ *then* $\dim_F(U) = 1$ *and* H *is isomorphic to a subgroup of* $GL_1(F)$.

In particular, G *is not simple and* G *is a Frobenius group with cyclic complement* H *and elementary abelian kernel.*

In fact, this is a re-statement of [2, Theorem A] in the case $\mathrm{cl}_G(H) = 2$. This paper considers the case $\mathrm{cl}_G(H) = 3$. First we make some general comments.

In considering our question in the case $\mathrm{cl}_G(H) \geq 3$, some obvious examples come to mind. For instance, if G is the alternating group on n letters and H is a subgroup that fixes more than $n/2$ letters then G cannot be generated by two conjugates of H. Another example is if G is a simple group and H has order 2, then any two conjugates of H generate a dihedral group, and hence a proper subgroup of G. In order to avoid examples like these, which we can never hope to classify, some extra hypothesis is needed. A suitable choice seems to be $N(H) = H$, in other words, that H is *self normalizing*. Non nilpotent groups contain a rich supply of self normalizing subgroups. For example, if H contains the normalizer of a Sylow subgroup then H is self normalizing.

We shall prove:

THEOREM B. *Let* H *be a self normalizing subgroup of a finite group* G *such that* $G = \langle H^G \rangle$ *and* $H_G = 1$. *Suppose that* $\mathrm{cl}_G(H) = 3$ *and that* G *cannot be generated by two conjugates of* H. *Then* G *has one of the following structures*:

Type 1. *G has a proper quotient* $\overline{G}$ *in which* $\mathrm{cl}_{\overline{G}}(\overline{H}) = 2$ *such that* $\overline{G}$ *cannot be generated by two conjugates of* $\overline{H}$.

Type 2. *There exists a prime* p *and a faithful irreducible* $GF(p)$ H*-module* U *such that*:

• $G \cong H(U \oplus U \oplus U)$, *the semidirect product of* H *with* $U \oplus U \oplus U$.

• *If* $F = \mathrm{End}_H(U)$ *then* $\dim_F(U) = 2$ *and* H *is isomorphic to a subgroup of* $GL_2(F)$.

Type 3. *There exists* $K <_2 G$ *and a faithful irreducible* $GF(2)K$*-module* U *such that*:

• $G \cong K(U \oplus U)$.

• H *is a maximal subgroup of* K.

• *If* $F = \mathrm{End}_K(U)$ *then* $\dim_F(U) = 2$ *and* K *induces the full special linear group* $SL(U/F)$ *on* U.

• *Let* $q = |F|$. *Then* $q = 2^n$ *for some* $n \geq 2$, $K \cong SL_2(q)$ *and* $H \cong D_{2(q+1)}$ *or* $D_{2(q-1)}$.

In particular, G *is not simple.*

Remark. Groups of type 1 are the obvious "degenerate" examples since if G has a proper quotient $\overline{G}$ which cannot be generated by two conjugates of $\overline{H}$ then G cannot be generated by two conjugates of H. The structure of the quotient $\overline{G}$ is of course given by Theorem A. Groups of type 2 and 3 are clearly similar to the groups in Theorem A.

Theorem B suggests the following problems for further study:

(i) What happens if we remove the condition $\mathrm{cl}_G(H) = 3$? Are the examples that arise similar to those in Theorem B?

(ii) If H is a self normalizing subgroup of a finite simple group G, can G be generated by two conjugates of H?

(iii) What happens if we remove the condition $N(H) = H$ and instead seek to generate G with H and an element of G?

Next we outline the strategy used to prove Theorem B. We assume that G satisfies the hypothesis of Theorem B and impose the "non-degeneracy" condition that G is not of type 1. The crucial move in the proof of Theorem A was to show that the hypothesis of Theorem A implies that H is a TI-subgroup; that is, H has trivial intersection with its conjugates. Groups of types 2 and 3 show that we cannot do this for Theorem B. Instead we consider the set

$$\Delta = \{A \cap B \mid 1 \neq A \cap B \text{ and } A, B \text{ are distinct conjugates of } H\}.$$

We show that any two distinct members of Δ have trivial intersection. In particular, we have constructed TI-subgroups. Here, the crucial move is to show that if $g \in G, H^g \neq H$ and $H \cap H^g \neq 1$ then $H <_1 \langle H, H^g\rangle <_2 G$.

We then consider subgroups K such that $H <_1 K <_2 G$. We show that either $K <_1 \langle K, K^g\rangle \neq G$ whenever $K \cap K^g \neq 1$ or that K contains a non-trivial normal subgroup of G. Again, the first case leads to the construction of TI-subgroups. Also, we define a G-invariant equivalence relation on Δ with the property that the stabilizer of an equivalence class contains the normalizers of many of the TI-subgroups that we have constructed. A detailed analysis of this stabilizer, together with a counting argument, shows that there are two or three distinct K that contain a proper normal subgroup of G. It is then not too difficult to show that G is of type 2 or 3.

It is left as an exercise for the reader to show that groups of types 2 and 3 do indeed satisfy the hypothesis of Theorem B. For groups of type 2 this is elementary linear algebra. For groups of type 3 one needs to use the fact that $|H^1(SL_2(q), V)| = q$ if $q = 2^n, n \geq 2$ and V is the natural module for $SL_2(q)$.

2. PRELIMINARIES

Throughout this paper, group means finite group. If A and B are groups then $A \leq B$ means A is a subgroup of G, $A < B$ means $A \leq B$ and $A \neq B$, and $A \trianglelefteq B$ means A is a normal subgroup of B. The notations $\mathrm{cl}_G(H)$ and $<_n$ were defined in the Introduction. If G is a group then $G^\sharp = G - \{1\}$ and if V is a vectorspace then $V^\sharp = V - \{0\}$. We let $\pi(G)$ be the set of prime divisors of $|G|$ and if p is a prime then $\mathrm{Syl}_p(G)$ is the set of Sylow p-subgroups of G.

If A and B are subgroups of a group G, we define

$$A^B = \left\{A^b \mid b \in B\right\}.$$

This should not be confused with the subgroup generated by the members of A^B, which we shall denote by

$$\langle A^B \rangle.$$

Also, we define

$$A_B = \bigcap \left\{A^b \mid b \in B\right\}$$

and note that A_B is the largest subgroup of A that is normalized by B.

If H is a group then $F(H)$, the *Fitting subgroup of* H, is the largest normal nilpotent subgroup of H. If p is a prime then $\mathcal{O}_p(H)$ is the largest normal p-subgroup of H.

Apart from the following results, this paper is self contained.

THEOREM 2.1 (D. Bartels [1]). *Let P be a subgroup of the group G. Then*

$$\langle Q \mid P \text{ is conjugate to } Q \text{ in } \langle P, Q \rangle \rangle$$

is the smallest subnormal subgroup of G that contains P.

We shall need a detailed knowledge of Frobenius groups. Let A be a subgroup of a group G such that $1 < A < G$ and $A \cap A^g = 1$ for all $g \in G - A$. Then we say that G is a *Frobenius group* and that A is a *Frobenius complement* in G. The latter condition is easily seen to be equivalent to $N(Q) \leq A$ whenever $1 \neq Q \leq A$. By the theorems of Frobenius and Thompson, we have

$$G = AF(G) \quad \text{and} \quad A \cap F(G) = 1.$$

The subgroup $F(G)$ is called the *Frobenius kernel* of G. Its non-identity elements are characterized as being those elements of G that do not belong to any conjugate of A. It is a consequence of the Schur–Zassenhaus Theorem that G has exactly one conjugacy class of Frobenius complements. Moreover, we have $g \in \langle A, A^g \rangle$ for all $g \in G$.

LEMMA 2.2. *Let G be a Frobenius group with Frobenius complement A. Suppose $M \le G$, $M \cap A \ne 1$ and $M \not\le A$. Then M is a Frobenius group with Frobenius complement $M \cap A$. Moreover,*

$$F(M) = M \cap F(G).$$

Proof. Let $A_0 = M \cap A$, so that $1 < A_0 < M$. If $m \in M - A_0$ then $m \notin A$ so $A_0 \cap A_0^m \le A \cap A^m = 1$. Thus A_0 is a Frobenius complement in M.

Since $M \cap F(G)$ is a nilpotent normal subgroup of M we have $M \cap F(G) \le F(M)$. Let $f \in F(M)$ and suppose that $f \notin F(G)$. Then $f \in A^g$ for some $g \in G$. Thus $1 \ne A^g \cap M$. If $M \le A^g$ then $1 \ne A_0 \le A \cap A^g$ so $M \le A^g = A$, a contradiction. Thus $M \not\le A^g$ and then A^g is also a Frobenius complement in M. But then $f \in (A^g \cap M) \cap F(M) = 1$, a contradiction. We deduce that $f \in F(G)$. Thus $F(M) = M \cap F(G)$.

The reader who merely wishes to see that a group satisfying the hypotheses of Theorem B cannot be simple may omit the following two lemmas.

LEMMA 2.3. *Let V be a 2-dimensional vectorspace over a field F of characteristic $p \ne 0$. Suppose $H \le GL(V)$ is a Frobenius group and that $\mathcal{O}_p(H) = 1$. Then the Frobenius complement of H has order 2 and the kernel of H is cyclic of odd order. In particular, H is dihedral.*

Proof. Since H is a Frobenius group we have $Z(H) = 1$ and so $Z(GL(V)) \cap H = 1$. Also, $F(H) \le [H, H]$ and hence $F(H) \le SL(V)$. Now $F(H) \le C(Z(F(H)))$ so [3, (6.8)(i),(iii), p. 396] implies that $F(H)$ is cyclic. By diagonalizing a generator for $F(H)$ over an algebraic closure of F, we see that $|N_{GL(V)}(F(H))/C_{GL(V)}(F(H))| \le 2$. Let P be a Frobenius complement of H. Then $C_P(F(H)) = 1$ so then $|P| = 2$. Finally, $F(H)$ has order coprime to P and the non identity element of P must invert $F(H)$. Thus H is dihedral.

LEMMA 2.4. *Let V be a 2-dimensional vectorspace over a finite field F of order q. Assume the following*:

(i) $H <_1 K \le GL(V/F)$.

(ii) *H acts irreducibly on V considered as a vectorspace over the prime subfield of F.*

(iii) *H is a Frobenius group with Frobenius complement P.*

(iv) *If $1 \ne Q \trianglelefteq H$ then $N_K(Q) \le H$.*

Then $q = 2^n$ for some $n \ge 2$, $K = SL(V/F)$ and $H \cong D_{2(q+1)}$ or $D_{2(q-1)}$.

Proof. Let p be the characteristic of F. Now $C_V(\mathscr{O}_p(H)) \neq 1$ is H-invariant so (ii) forces $\mathscr{O}_p(H) = 1$. Similarly, $\mathscr{O}_p(K) = 1$. Then (iii) and (2.3) imply that $|P| = 2$ and that $F(H)$ is cyclic of odd order. Let $K_0 = K \cap SL(V/F) \trianglelefteq K$. Then $F(H) = [H, H] \leq K_0$. Also, (iv) and (i) imply $F(H) < K_0$.

Suppose that $F(H) = H \cap K_0$. Now $F(H)$ is cyclic so every subgroup of $F(H)$ is normal in H and then (iv) implies that $F(H)$ is a Frobenius complement in K_0. Now $\mathscr{O}_p(K_0) \leq \mathscr{O}_p(K) = 1$ so (2.3) implies $|F(H)| = 2$, a contradiction. We deduce that $F(H) < H \cap K_0$. Then as $|H : F(H)| = |P| = 2$ we see that $H \leq K_0$. Using (iv) and (i) it follows that $K_0 = K$, in particular, $K \leq SL(V/F)$.

If p is odd then the only subgroup of order two in $SL(V/F)$ is $Z(SL(V/F))$. But P does not centralize $F(H)$ so we deduce that $p = 2$. Thus $q = 2^n$ for some $n \geq 1$. The only non-abelian subgroup of $SL_2(2)$ is $SL_2(2)$ so as $H < K \leq SL(V/F)$ it follows that $n \geq 2$. We use [3, Theorem 6.17, p. 404] to identify K. Note that $p = 2$ divides $|K|$ and that (iv) implies that K is not dihedral. Also note that part (x) of [3, Theorem 6.17, p. 404] is not applicable since $p = 2$. It follows that $K \cong SL_2(F_1)$ for some subfield F_1 of F. By choosing a basis, we may identify $SL(V/F)$ with $SL_2(F)$. We then have $SL_2(F_1) \leq SL_2(F)$ and then [3, Theorem 6.20, p. 408] implies that K is conjugate to $SL_2(F_1)$ in $SL_2(V)$. But then (ii) forces $F_1 = F$ whence $K = SL(V/F)$. Using [3, (6.23), p. 410] and (iv) it follows that $H \cong D_{2(q+1)}$ or $D_{2(q-1)}$.

3. UNIONS OF SUBGROUPS

In this section, we study groups that can be written as a union of subgroups in a special way. These groups will occur as normalizers and as stablilzers of equivalence classes in groups that satisfy the hypotheses of Theorem B.

Hypothesis 3.1. $(X, \{X_i\}_{i=1}^{\alpha}, V)$ is a 3-tuple that satisfies:

(i) Each X_i and V are subgroups of the group X.

(ii) $X = X_1 \cup \cdots \cup X_\alpha$, $V < X_i < X$ for all i and $X_i \cap X_j = V$ for all $i \neq j$.

Hypothesis 3.2. $(X, \{X_i\}_{i=1}^{\alpha}, \{X_{ij}\}_{i=1}^{\alpha}{}_{j=1}^{\alpha_i}, V)$ is a 4-tuple that satisfies:

(i) $(X, \{X_i\}_{i=1}^{\alpha}, V)$ satisfies Hypothesis (3.1).

(ii) For each i, $(X, \{X_{ij}\}_{j=1}^{\alpha_i}, X_i)$ satisfies Hypothesis 3.1.

(iii) For all i, j there exists k such that $X_j \leq X_{ik}$.

LEMMA 3.3. *Assume Hypothesis* 3.1, *that* $V = 1$, *and that* $X_i \trianglelefteq X$ *for all* i. *Then* X *is an elementary abelian* p-*group*.

Proof. This is left as an exercise.

LEMMA 3.4. *Assume Hypothesis* 3.1. *Then*:

(i) $\alpha \geq 3$.

(ii) $V \cap V^x = V_X$ *for all* $x \in X - V$.

(iii) *Either* $V \trianglelefteq X$ *or* X/V_X *is a Frobenius group with cyclic complement* V/V_X *and whose kernel is a* p-*group*.

Proof. Hypothesis 3.1(ii) implies $\alpha \geq 2$. If $X = X_1 \cup X_2$ then $X = X_1$ or $X = X_2$, contrary to (3.1(ii)). Thus $\alpha \geq 3$. Next we claim:

$$\text{If } i \neq j,\ x \in X_i - V \quad \text{and} \quad y \in X_j - V \quad \text{then } V \cap V^x = V \cap V^y. \tag{1}$$

Choose k such that $xy^{-1} \in X_k$. If $k = j$ then $x = (xy^{-1})y \in X_i \cap X_j = V$, a contradiction. Thus $k \neq j$. We have

$$(V \cap V^x)^{y^{-1}} = V^{y^{-1}} \cap V^{xy^{-1}} \leq X_j \cap X_k = V$$

whence $V \cap V^x \leq V \cap V^y$. Then (1) follows by symmetry.

We prove (ii). Choose i such that $x \in X_i - V$. Let $y \in X - X_i$ and $z \in X_i - V$. Now $y \in X_j - V$ for some $j \neq i$ so applying (1) twice yields $V \cap V^x = V \cap V^y = V \cap V^z$. We deduce that $V \cap V^x \leq V^w$ for all $w \in X - V$. This proves (ii).

Next we prove (iii). Assume $V \ntrianglelefteq X$. We may factor out by V_X and assume that $V_X = 1$. Then $V \neq 1$ so (ii) implies that V is a Frobenius complement in X and hence in each X_i. Thus $X = VF(X)$. For each i, set $F_i = F(X) \cap X_i$. We see that $(F(X), \{F_i\}_{i=1}^{\alpha}, 1)$ satisfies Hypothesis 3.1.

Let $1 \leq i \leq \alpha$. Then $N(F_i) = N_{X_1}(F_i) \cup \cdots \cup N_{X_\alpha}(F_i)$ and as $N_{F(X)}(F_i) > F_i$ we have $N_{X_j}(F_i) > V$ for some $j \neq i$. Suppose $N(F_i) \neq X$. Then by induction, V is cyclic and $F(N(F_i))$ is a p-group. But $Z(F(X)) \leq F(N(F_i))$ whence $F(X)$ is a p-group also. Hence we may suppose $F_i \trianglelefteq X$ for all i.

Lemma 3.3 implies that $F(X)$ is an elementary abelian p-group. Let Z_1 and Z_2 be minimal V-invariant subgroups of F_1 and F_2 respectively. Set $G = VZ_1Z_2$. Now Z_1 and Z_2 are the V-composition factors of Z_1Z_2 so it follows that $V <_2 G$. Suppose $G = \langle V, V^z \rangle$ for some $z \in Z_1Z_2$. Choose k such that $z \in X_k$. Then $G \leq X_k$. But then $Z_1 \leq F(X) \cap X_1 \cap X_k$ and as $F(X) \cap V = 1$, this forces $k = 1$. Similarly $k = 2$, a contradiction. Thus G cannot be generated by two conjugates of V. Now V is a Frobenius complement in G so $G = \langle V^G \rangle$ and $V_G = 1$. Then Theorem A implies V is cyclic.

LEMMA 3.5. *Assume Hypothesis* 3.2 *and that* $V_X = 1$. *Then*:

(i) *If* $i \neq j$ *then* $X_{iX} \cap X_j = 1$.

(ii) *If* $i \neq j$, $g \in X$ *and* $X_i^g \cap X_j \neq 1$ *then* $X_i^g \cap X_j$ *is a Frobenius complement in* X.

(iii) *Either* X *is an elementary abelian* p-*group or there exist* $l, m \in \mathbf{N}$ *and* $g \in X$ *such that* $l \neq m$ *and* $X_l^g \cap X_m \neq 1$.

(iv) *For each* i, $F(X) \cap X_i = X_{iX}$.

(v) $F(X) = \bigcup\{X_{iX} \mid 1 \leq i \leq \alpha\}$.

Proof. (i) We have $X_{iX} \cap X_j \leq X_i \cap X_j = V$. Choose $g \in X_j - V$. Then $X_{iX} \cap X_j = (X_{iX})^g \cap X_j^g = (X_{iX} \cap X_j)^g \leq V^g$. Lemma 3.4 applied to $(X, \{X_i\}_{i=1}^{\alpha}, V)$ yields $V \cap V^g \leq V_X = 1$ and so $X_{iX} \cap X_j = 1$.

(ii) Let $D = X_i^g \cap X_j$. Let $x \in X - X_j$. Then (3.4) applied to $(X, \{X_{jk}\}_{k=1}^{\alpha_j}, X_j)$ implies $X_j \cap X_j^x = X_{jX}$. Thus

$$D \cap D^x \leq X_i^g \cap X_j \cap X_j^x = X_i^g \cap X_{jX} = (X_i \cap X_{jX})^g$$

and then (i) forces $D \cap D^x = 1$.

Now let $x \in X - X_i^g$. Then (3.4) applied to $(X, \{X_{ik}^g\}_{k=1}^{\alpha_i}, X_i^g)$ implies $X_i^g \cap X_i^{gx} = (X_i^g)_X = X_{iX}$. Thus

$$D \cap D^x \leq X_i^g \cap X_j \cap X_i^{gx} = X_{iX} \cap X_j$$

and then (i) forces $D \cap D^x = 1$. We deduce that $D \cap D^x = 1$ for all $x \in X - D$, so D is a Frobenius complement in X.

(iii) If $X_l \trianglelefteq X$ for all l then (i) implies $X_l \cap X_m = 1$ for all $l \neq m$ and then (3.3) implies that X is an elementary abelian p-group. Suppose $X_l \ntrianglelefteq X$ for some l. Choose $g \in X$ such that $X_l^g \neq X_l$. Now $X = X_1 \cup \cdots \cup X_\alpha$ so there exists $m \neq l$ such that $X_l^g \cap X_m \neq 1$.

(iv) This is visibly true in the case that X is elementary abelian, hence we may suppose that the second possibility of (iii) holds. Now $X_{iX} \cap (X_l^g \cap X_m) = (X_{iX} \cap X_l)^g \cap (X_{iX} \cap X_m)$. But $l \neq m$ so $i \neq l$ or $i \neq m$ and then (i) implies $X_{iX} \cap (X_l^g \cap X_m) = 1$. As $X_{iX} \trianglelefteq X$ we see that X_{iX} has trivial intersection with every conjugate of $X_l^g \cap X_m$. Now (ii) implies that $X_l^g \cap X_m$ is a Frobenius complement in X so we deduce that $X_{iX} \leq F(X)$. Conversely, let $f \in F(X)^\sharp \cap X_i$. Then f is not contained in any Frobenius complement of X so (ii) and the fact that $X = X_1 \cup \cdots \cup X_\alpha$ imply $f^h \in X_i$ for all $h \in X$. Thus $f \in X_{iX}$. We deduce that $F(X) \cap X_i = X_{iX}$.

(v) We have $F(X) = (F(X) \cap X_1) \cup \cdots \cup (F(X) \cap X_\alpha)$, so (v) follows from (iv).

THEOREM 3.6. *Assume Hypothesis* 3.2 *and that* $V \trianglelefteq X$. *Then*:

(i) *Each* X_{ij} *is a union of at least three of the* X_k.

(ii) *There exists* i *and* j *such that* $X_{ij} \trianglelefteq X$ *and such that each* X_k *contained in* X_{ij} *is normal in* X.

(iii) *Let* $\overline{X} = X/V$. *Then* $F(\overline{X}) = \bigcup\{\overline{X}_{i\overline{X}} \mid 1 \le i \le \alpha\}$ *and* $F(\overline{X})$ *is an elementary abelian* p-*group*. *Moreover, either* $\overline{X} = F(\overline{X})$ *or* $\overline{X}$ *is a Frobenius group*.

Proof. (i) We have $X_{ij} = (X_1 \cap X_{ij}) \cup \cdots \cup (X_\alpha \cap X_{ij})$. Suppose $V < X_k \cap X_{ij}$. If $k = i$ then $X_k \le X_{ij}$ by (3.2(ii)). Suppose $k \ne i$. Then by (3.2(iii)) there exists l such that $X_k \le X_{il}$. Now $X_k \cap X_i = V$ so $X_k \cap X_{ij} \not\le X_i$. However, $X_k \cap X_{ij} \le X_{il} \cap X_{ij}$ so (3.2(ii)) forces $l = j$, whence $X_k \le X_{ij}$. We deduce that X_{ij} is a union of some of the X_k.

Note that (3.2(ii)) implies $X_i < X_{ij}$. Suppose $X_{ij} = X_k$ for some k. Then $X_i \le X_k$ so (3.2(i)) forces $X_i = X_k = X_{ij}$, a contradiction. Suppose $X_{ij} = X_k \cup X_l$ for some k and l. Then $X_{ij} = X_k$ or X_l, which again leads to a contradiction. Thus X_{ij} is a union of at least three of the X_k.

Next we prove (ii) and (iii). By passing to the quotient X/V we may suppose that $V = 1$. In particular, $V_X = 1$. Lemma 3.5(iii) implies X is either an elementary abelian p-group or there exist $l, m \in \mathbf{N}$ and $g \in X$ such that $l \ne m$ and $X_l^g \cap X_m \ne 1$. If the first case holds then (ii) and (iii) are trivially true. Hence we will suppose the second case holds. Let $D = X_l^g \cap X_m$. Then (3.5(ii)) implies that D is a Frobenius complement in X.

We claim:

$$\text{For each } i,\ X_i \trianglelefteq X \text{ or } X_i \text{ contains a Frobenius complement of } X. \tag{2}$$

Indeed, suppose $X_i \ntrianglelefteq X$. Choose $h \in X$ such that $X_i \ne X_i^h$. As $X = X_1 \cup \cdots \cup X_\alpha$, there exists $j \ne i$ such that $X_i^h \cap X_j \ne 1$. Then (3.5(ii)) implies $X_i^h \cap X_j$ is a Frobenius complement. Conjugating by h^{-1} proves the claim.

Next we claim:

$$\text{There exists } i \text{ such that } X_i \text{ does not contain a Frobenius complement of } X. \tag{3}$$

Assume false. Using the fact that any two Frobenius complements are conjugate, it follows that each X_i contains $|X_i|/|D|$ Frobenius complements of X. Now $X_i \cap X_j = V = 1$ for all $i \ne j$ and X has precisely $|X|/|D|$ Frobenius complements. Thus $|X|/|D| \ge \sum_{i=1}^{\alpha} |X_i|/|D|$ and so

$$|X| \ \ge\ \sum_{i=1}^{\alpha} |X_i| \ \ge\ \sum_{i=1}^{\alpha} |X_i| - (\alpha - 1) \ = \left|\bigcup_{i=1}^{\alpha} X_i\right| \ = |X|.$$

This forces $\alpha = 1$, contradicting (3.4(i)) and proving (3). We now prove:

$$\text{There exist } i, j \text{ such that } X_{ij} \text{ does not contain a Frobenius complement of } X. \tag{4}$$

Assume false and choose i in accordance with (3). Now $X = X_{i1} \cup \cdots \cup X_{i\alpha_i}$ and $X_{ij} \cap X_{ik} = X_i$ for all $j \neq k$ so the same argument used in the proof of (3) yields

$$|X| \geq \sum_{j=1}^{\alpha_i} |X_{ij}| \geq \sum_{j=1}^{\alpha_i} |X_{ij}| - (\alpha_i - 1)|X_i| = \left| \bigcup_{j=1}^{\alpha_i} X_{ij} \right| = |X|.$$

This forces $\alpha_i = 1$ and so $X = X_{i1}$. But by hypothesis (3.2(ii)) we have $X_{i1} < X$, a contradiction. We have proved (4).

Choose i and j in accordance with (4). By (i), X_{ij} is a union of some of the X_k. Now (2) implies each of these X_k is normal in X and so $X_{ij} \trianglelefteq X$. We have proved (ii). Finally, we prove (iii). By (3.5(i)) and (3.5(v)) we have

$$F(X) = X_{1X} \cup \cdots \cup X_{\alpha X} \quad \text{and} \quad X_{sX} \cap X_{tX} = 1 \quad \text{for all } s \neq t.$$

Using (ii) and (i) we see that there are at least two values of k for which $X_k \trianglelefteq X$ and hence $1 \neq X_k = X_{kX}$. Then (3.3) implies that $F(X)$ is an elementary abelian p-group.

THEOREM 3.7. *Assume Hypothesis* 3.2 *and that* $V \ntrianglelefteq X$. *Set* $\overline{X} = X/V_X$. *Then*:

(i) $\overline{V}$ *is a cyclic Frobenius complement in* $\overline{X}$.

(ii) *The Frobenius kernel of* $\overline{X}$ *is an elementary abelian* p-*group*.

(iii) *For each* i, $\overline{X}_i = \overline{V}F(\overline{X}_i)$ *and* $1 \neq F(\overline{X}_i) = \overline{X}_i \cap F(\overline{X}) \trianglelefteq \overline{X}$.

Proof. Passing to X/V_X we may suppose that $V_X = 1$ and that $\overline{X} = X$. Now $V \neq 1$ so (i) follows from (3.4(iii)) applied to $(X, \{X_i\}_{i=1}^{\alpha}, V)$.

We prove (iii). Now $V < X_i$ so as V is a Frobenius complement, (2.2) implies $X_i = VF(X_i)$ and $1 \neq F(X_i) = X_i \cap F(X)$. By (3.5(iv)) we have $X_i \cap F(X) = X_{iX} \trianglelefteq X$. This proves (iii). To prove (ii), note that $F(X) = (F(X) \cap X_1) \cup \cdots \cup (F(X) \cap X_\alpha) = F(X_1) \cup \cdots \cup F(X_\alpha)$ and that $F(X_i) \cap F(X_j) \leq F(X) \cap V = 1$ for all $i \neq j$. We have just seen that $1 \neq F(X_i) \trianglelefteq X$ for all i so (3.3) implies that $F(X)$ is an elementary abelian p-group.

4. TRIVIAL INTERSECTION SUBGROUPS

A subgroup P of a group G is a *TI-subgroup* if for all $g \in G$, $P \cap P^g \neq 1$ implies $P = P^g$. The main aim of this section is to describe a method for constructing TI-subgroups.

LEMMA 4.1. *Let Σ be a collection of subgroups of the group G such that $|\Sigma| \geq 2$ and $A <_1 \langle A, B \rangle \neq \langle \Sigma \rangle$ whenever A and B are distinct members of Σ. Then*:

(i) *If A, B and C are distinct members of Σ such that $C \not\leq \langle A, B \rangle$ then $A \not\leq \langle B, C \rangle$ and $B \not\leq \langle C, A \rangle$.*

(ii) *If A and B are distinct members of Σ then $A \cap B = \bigcap \Sigma$.*

(iii) *If A, B and C are distinct members of Σ such that $C \not\leq \langle A, B \rangle$ then $C \cap \langle A, B \rangle = \bigcap \Sigma$.*

Proof. (i) If $A \leq \langle B, C \rangle$ then $\langle A, B \rangle = \langle B, C \rangle$ contrary to $C \not\leq \langle A, B \rangle$. Thus $A \not\leq \langle B, C \rangle$ and similarly $B \not\leq \langle C, A \rangle$.

(ii) Choose $C \in \Sigma$ with $C \not\leq \langle A, B \rangle$. By (i) we have $A \not\leq \langle C, B \rangle$ whence $C \leq \langle C, A \rangle \cap \langle C, B \rangle < \langle C, A \rangle$ and then $C = \langle C, A \rangle \cap \langle C, B \rangle$. Thus $A \cap B \leq C$. Now suppose $D \in \Sigma - \{A\}$ and $D \leq \langle A, B \rangle$. Then $C \not\leq \langle A, B \rangle = \langle A, D \rangle$ so $D \not\leq \langle A, C \rangle$ by (i). The preceding argument implies $A \cap C \leq D$ whence $A \cap B \leq A \cap C \leq D$. We deduce that $A \cap B \leq \bigcap \Sigma$, proving (ii).

(iii) If $C \cap \langle A, B \rangle \not\leq A$ then $A < \langle A, C \cap \langle A, B \rangle \rangle \leq \langle A, C \rangle$ whence $C \leq \langle A, B \rangle$ contrary to hypothesis. Thus $C \cap \langle A, B \rangle \leq A$. Similarly, $C \cap \langle A, B \rangle \leq B$. Now apply (ii).

THEOREM 4.2. *Let Ω be a collection of subgroups of the group G such that $A <_1 \langle A, B \rangle$ whenever A and B are distinct members of Ω with $A \cap B \neq 1$. Set*

$$\Delta = \{A \cap B \mid A, B \in \Omega,\ A \neq B,\ A \cap B \neq 1 \text{ and there exists } C \in \Omega \text{ such that } A \cap B \cap C \neq 1 \text{ and } C \not\leq \langle A, B \rangle\},$$

and for each $P \in \Delta$ set

$$\Delta(P) = \{A \in \Omega \mid P \leq A\}.$$

Then:

(i) *If $P, Q \in \Delta$ and $P \cap Q \neq 1$ then $P = Q$.*

(ii) *If $P \in \Delta$ and $D \in \Omega$ are such that $P \cap D \neq 1$ then $P \leq D$.*

(iii) *If $P \in \Delta$ and D, E are distinct members of $\Delta(P)$ then $P = D \cap E$.*
Assume further that $\Omega = H^G$ for some subgroup H of G with $N(H) = H$. Then:

(iv) *Each member of Δ is a TI-subgroup in G.*

(v) *If $P, Q \in \Delta$ are conjugate and if $\langle P, Q \rangle \leq A \in \Omega$ then P is conjugate to Q in A.*

(vi) *If $P \in \Delta$ then $N(P)$ acts transitively by conjugation on $\Delta(P)$.*

(vii) *Let $P \in \Delta$ and $A \in \Delta(P)$. Suppose that $N_A(P) > P$. Then*:

(a) $N_A(P)/P$ *is a cyclic Frobenius complement in* $N(P)/P$.

(b) *The Frobenius kernel of* $N(P)/P$ *is a* p-*group*.

(c) $n \in \langle A, A^n \rangle$ *for all* $n \in N(P)$.

(d) $N_B(P) > P$ *for all* $B \in \Delta(P)$.

Proof. Let $P \in \Delta$ and choose $A, B, C \in \Omega$ such that $P = A \cap B$, $A \neq B$, $A \cap B \cap C \neq 1$ and $C \not\leq \langle A, B \rangle$. Let Σ be a subset of Ω that contains A, B and C and such that any two members of Σ have non-trivial intersection. Suppose $\langle \Sigma \rangle = \langle X, Y \rangle$ for some $X, Y \in \Sigma$. Without loss of generality, $Y \neq A$ so then $\langle \Sigma \rangle = \langle A, Y \rangle$ and then $\langle \Sigma \rangle = \langle A, B \rangle$, contrary to $C \not\leq \langle A, B \rangle$. We deduce that $X <_1 \langle X, Y \rangle \neq \langle \Sigma \rangle$ for all $X, Y \in \Sigma$ with $X \neq Y$. Lemma 4.1 implies $P = A \cap B = \bigcap \Sigma = X \cap Y$ for all $X, Y \in \Sigma$ with $X \neq Y$. Putting $\Sigma = \{A, B, C\}$ implies $C \in \Delta(P)$ and then putting $\Sigma = \Delta(P)$ implies (iii). To prove (ii), let $\Sigma = \{A, B, C, D\}$. Observe that (ii) and (iii) imply (i).

(iv) This follows immediately from (i).

(v) Let $p \in \pi(P)$, $P_0 \in \mathrm{Syl}_p(P)$, $Q_0 \in \mathrm{Syl}_p(Q)$ and $S \in \mathrm{Syl}_p(N_A(P_0))$. Suppose there exists $B \in \Omega$ with $S \leq B \neq A$. Now $P_0 \leq S$ so (ii) and (iii) imply $A \cap B = P$ whence $S \leq P$. This implies $P_0 \in \mathrm{Syl}_p(A)$. Using Sylow's Theorem and (iv) we see that Q is conjugate to P in A. Hence we may suppose that A is the only member of Ω that contains S. Since $N(A) = A$ this implies $N(S) \leq A$ and it follows that $S \in \mathrm{Syl}_p(N(P_0))$. By (iv), $N(P_0) \leq N(P)$ so $S \in \mathrm{Syl}_p(N(P))$.

Let $T \in \mathrm{Syl}_p(N_A(Q_0))$. Then again we may suppose that $T \in \mathrm{Syl}_p(N(Q))$. Choose $g \in G$ such that $P^g = Q$. Then $S^g \in \mathrm{Syl}_p(N(Q))$ so there exists $m \in N(Q)$ such that $S^{gm} = T \leq A$. But now $S \leq A \cap A^{(gm)^{-1}}$ so $A = A^{(gm)^{-1}}$ and as $N(A) = A$ we have $gm \in A$. Then $P^{gm} = Q^m = Q$ which proves (v).

(vi) Let $A, B \in \Delta(P)$. Since $\Omega = H^G$, there exists $g \in G$ such that $A^g = B$. Then $P, P^{g^{-1}} \leq A$ so by (v), $P^{g^{-1}} = P^a$ for some $a \in A$. Then $ag \in N(P)$ and $B = A^{ag}$.

(vii) Using (vi), the definition of Δ and (ii) we may choose $B, C \in \Delta(P)$ such that $B \neq A$ and $C \not\leq \langle A, B \rangle$. Set $N = N(P)$ and $\overline{N} = N/P$. By (vi), $N \not\leq A$ whence $1 < \overline{N_A(P)} < \overline{N}$. Suppose $n \in N$ and $\overline{N_A(P)} \cap \overline{N_A(P)}^{\overline{n}} \neq 1$. Then $A \cap A^n > P$ so (iii) forces $A = A^n$. Since $N(A) = A$ we have $\overline{n} \in \overline{N_A(P)}$. We deduce that $\overline{N_A(P)}$ is a Frobenius complement in $\overline{N}$. This implies that $\overline{n} \in \langle \overline{N_A(P)}, \overline{N_A(P)}^{\overline{n}} \rangle$ for all $\overline{n} \in \overline{N}$. In particular, we have proved (c).

Let $K_1, \ldots, K_\alpha$ be the distinct subgroups of the form $\langle A, D\rangle$ as D ranges over $\Delta(P) - \{A\}$. Then $A <_1 K_i$ for all i and consequently $K_i \cap K_j = A$ for all $i \neq j$. Since $C \not\leq \langle A, B\rangle$ we have $\alpha \geq 2$. For each i set $N_i = N \cap K_i$. Then (vi) and (c) imply that $N = N_1 \cup \cdots \cup N_\alpha$ and that $N_A(P) < N_i$ for all i. If $i \neq j$ then $N_A(P) \leq N_i \cap N_j = N \cap K_i \cap K_j = N_A(P)$ so $N_i \cap N_j = N_A(P)$. This together with $\alpha \geq 2$ implies $N_A(P) < N_i < N$ for all i and we deduce that $(\overline{N}, \{\overline{N}_i\}_{i=1}^{\alpha}, \overline{N_A(P)})$ satisfies Hypothesis 3.1. Moreover, as $\overline{N_A(P)}$ is a Frobenius complement in $\overline{N}$ we have $\overline{N_A(P)}_{\overline{N}} = 1$. Lemma 3.4 implies (a) and (b). Finally, (d) follows from (vi).

5. IDENTIFICATION OF THE EXAMPLES

Throughout the remainder of this paper, we assume the hypothesis of Theorem B. In this section, we characterize the groups in the conclusion of Theorem B in terms of normal subgroups.

Of course, the difficult part of the proof of Theorem B is showing that G is not simple. The reader who wishes only to see this may omit all of this section.

LEMMA 5.1. *Suppose $X \trianglelefteq G, H \cap X \neq 1$ and $HX <_2 G$. Then G is of type* 1.

Proof. Assume false and set $\overline{G} = G/X$. Now $\overline{H} = \overline{HX} <_2 \overline{G}$ hence there exists $g \in G$ such that $\overline{G} = \langle \overline{H}, \overline{H}^{\overline{g}}\rangle$. Then $G = \langle H, H^g\rangle X$. Now $HX <_2 G$ and $H <_3 G$ so $H <_1 HX$. But $\langle H, H^g\rangle \neq G$ so $\langle H, H^g\rangle \cap HX = H$. Let $P = H \cap X \neq 1$. Then $P^g \leq \langle H, H^g\rangle \cap X \leq H \cap X = P$ so $g \in N(P)$. Let $h \in HX$. Then $\langle \overline{H}, \overline{H}^{\overline{hg}}\rangle = \langle \overline{H}, \overline{H}^{\overline{g}}\rangle = \overline{G}$, so again, $hg \in N(P)$. Then $G = \langle HX, g\rangle \leq N(P)$ and consequently $P \leq H_G = 1$, a contradiction.

LEMMA 5.2. *Suppose $G = HF(G)$ and $\langle H, g\rangle \neq G$ for all $g \in G$. Then G is of type* 1 *or* 2.

Proof. Assume that G is not of type 1. Let $U \neq 1$ be a minimal H-invariant normal subgroup of $F(G)$. Then $U \leq Z(F(G))$ so as $H_G = 1$ we have $H \cap U = 1$ and hence $H < HU$. Let $\overline{G} = G/U$. Since $H < HU$ and $\mathrm{cl}_G(H) = 3$ we have $\mathrm{cl}_{\overline{G}}(\overline{H}) \leq 2$. Then as G is not of type 1, $\overline{G}$ can be generated by two conjugates of $\overline{H}$. Consequently, there exists $g \in G$ such that $G = \langle H, H^g\rangle U$. Since $U \leq Z(F(G))$ we have $\langle H, H^g\rangle \cap U = 1$ and then $F(G) = (F(G) \cap \langle H, H^g\rangle) \times U$. Thus every minimal H-invariant subgroup of $F(G)$ has a H-invariant direct factor. It follows that $F(G) = V_1 \times V_2 \times \cdots \times V_n$ where each V_i is a minimal H-invariant normal subgroup of $F(G)$. Then $F(G)$ is abelian and is therefore a H-module. Now $H_G = 1$ forces $H \cap F(G) = 1$, so G is the semidirect product of H with $F(G)$. Also, $\mathrm{cl}_G(H) = 3$ implies $n = 3$.

Now $G = H(V_1 \times V_2 \times V_3)$ and $G/V_3 \cong H(V_1 \times V_2)$ so as G is not of type 1, there exists $x \in H(V_1 \times V_2)$ such that $H(V_1 \times V_2) = \langle H, H^x \rangle$. Let $x = hv_1v_2$ with $h \in H$, $v_1 \in V_1$ and $v_2 \in V_2$. Then $H(V_1 \times V_2) = \langle H, v_1v_2 \rangle$ and hence $V_1 \times V_2$ is generated as a H-module by v_1v_2.

Let $v_3 \in V_3$ and set $v = v_1v_2v_3$. Let $\alpha\colon \langle v^H \rangle \longrightarrow V_1 \times V_2$ and $\beta\colon \langle v^H \rangle \longrightarrow V_3$ be the projection maps. Note that α and β are H-homomorphisms; $v\alpha = v_1v_2$, a generator for $V_1 \times V_2$; and that by assumption $\langle H, v \rangle \neq G$ so $\langle v^H \rangle$ is a proper submodule of $V_1 \times V_2 \times V_3$. It follows that α is a H-isomorphism and then that $\alpha^{-1}\beta$ is a H-homomorphism $V_1 \times V_2 \longrightarrow V_3$ that maps v_1v_2 to v_3. Since v_3 was arbitrary and since V_3 is an irreducible H-module, we see that

$$\mathrm{Hom}_H(V_1 \times V_2, V_3) \cong V_3.$$

Also, if we choose $v_3 \neq 0$ and note that V_1 and V_2 are the H-composition factors of $V_1 \times V_2$, we see that V_3 is H-isomorphic to V_1 or V_2. Similarly $V_2 \cong_H V_1$ or V_3 and $V_1 \cong_H V_2$ or V_3. It follows that $V_1 \cong_H V_2 \cong_H V_3$.

Set $F = \mathrm{End}_H(V_3)$. Then F is a field since V_3 is an irreducible H-module. We have

$$V_3 \cong \mathrm{Hom}_H(V_1 \times V_2, V_3) \cong \mathrm{Hom}_H(V_3 \times V_3, V_3) \cong F \oplus F$$

whence $\dim_F(V_3) = 2$. Thus G is of type 2.

Hypothesis 5.3.

(i) $1 \neq P \leq H$.

(ii) $G = \langle H^n \mid n \in N(P) \rangle$.

(iii) $g \in \langle H, H^g \rangle$ for all $g \in G$.

(iv) If $n \in N(P)$ and $H \neq H^n$ then $\langle H, H^n \rangle <_2 G$.

(v) If $n, m \in N(P)$ then $\langle H, H^n, H^m \rangle \neq G$.

(vi) G is not of type 1.

LEMMA 5.4. *Assume Hypothesis* 5.3 *and that $U \trianglelefteq G$ is such that $HU <_2 G$. Then $|U| = |C_U(P)|^2$.*

Proof. Let $\overline{G} = G/U$. Since G is not of type 1, there exists $g \in G$ such that $\overline{G} = \langle \overline{H}, \overline{H}^{\overline{g}} \rangle$. Let $M = \langle H, H^g \rangle$ so $M \neq G = MU$. Now $HU <_2 G$ implies $H <_1 HU$ whence $M \cap U \leq H$ and then (5.1) yields $M \cap U = 1$. Thus M is a complement to U that contains H. We shall argue that the number of such complements is equal to both $|U|$ and to $|C_U(P)|^2$.

For each $u \in U$ set $M_u = \langle H, H^{gu^{-1}} \rangle$. Now $\overline{M}_u = \overline{M}$ so the above argument implies that M_u is a complement to U. Suppose L is a complement to U that contains H. Then $g = lu$ for some $l \in L$ and $u \in U$. Observe that $H^{gu^{-1}} = H^l \leq L$ whence $M_u \leq L$ and consequently $M_u = L$. Now let $u, u' \in U$ and suppose $M_u = M_{u'}$. Then (5.3(iii)) implies $gu^{-1}, gu'^{-1} \in M_u$ so $u'u^{-1} \in M_u \cap U = 1$ and then $u = u'$. We deduce that there are precisely $|U|$ complements to U that contain H.

Since M is a complement to $U \trianglelefteq G$ we have $N(P) = N_M(P)C_U(P)$, so by (5.3(ii)) there exists $c \in C_U(P)^\sharp$. Now $H <_1 HU$ so (5.3(iii)) implies $HU = \langle H, H^c \rangle$. By (5.3(ii)) there exists $n \in N(P)$ such that $H^n \not\leq HU$ and then (5.3(v)) implies $\langle HU, H^n \rangle \neq G$. Again by (5.3(ii)) there exists $m \in N(P)$ such that $H^m \not\leq \langle HU, H^n \rangle$. We have

$$H < HU < \langle HU, H^n \rangle < \langle HU, H^n, H^m \rangle = \langle H, H^n, H^m \rangle U.$$

Then $\mathrm{cl}_G(H) = 3$ implies $G = \langle H, H^n, H^m \rangle U$. Using (5.3(v)), we see that $\langle H, H^n, H^m \rangle$ is a complement to U in G.

For each $u, v \in C_U(P)$ set $K_{u,v} = \langle H, H^{nu^{-1}}, H^{mv^{-1}} \rangle$ and observe that $\overline{K_{u,v}} = \overline{\langle H, H^n, H^m \rangle} = \overline{G}$ and then that (5.3(v)) implies $K_{u,v}$ is a complement to U in G. Suppose L is a complement to U that contains H. Then there exist $u, v \in U$ and $a, b \in L$ such that $n = au$ and $m = bv$. Now $P = P^n = P^{au}$ so $P^{u^{-1}} = P^a \leq L$ whence $[P, u^{-1}] \leq \langle P, P^{u^{-1}} \rangle \cap U \leq L \cap U = 1$ and so $u \in C_U(P)$. Similarly $v \in C_U(P)$. Also,

$$K_{u,v} = \langle H, H^{nu^{-1}}, H^{mv^{-1}} \rangle = \langle H, H^a, H^b \rangle \leq L$$

whence $K_{u,v} = L$. As previously, if $u, u', v, v' \in C_U(P)$ are such that $K_{u,v} = K_{u',v'}$ then $(u, v) = (u', v')$. We deduce that there are precisely $|C_U(P)|^2$ complements to U that contain H, completing the proof.

LEMMA 5.5. *Assume the following*:

(i) *Hypothesis* 5.3.

(ii) *P is a Frobenius complement in H.*

(iii) *If $1 \neq Q \trianglelefteq H$ then $N(Q) = H$.*

(iv) *If $U \trianglelefteq G$ is such that $HU <_2 G$ then U is a p-group.*

(v) *There exist $V_1, V_2 \trianglelefteq G$ such that $HV_1 <_2 G, HV_2 <_2 G$ and $HV_1V_2 <_1 G$.*

Then G is of type 2 *or* 3.

Proof. Using (v), (5.3(vi)), and (5.1) we have $H \cap V_1 = H \cap V_2 = 1$. Also, by (v), H is maximal in HV_1 and in HV_2, so (iv) implies each V_i is an elementary abelian group on which H acts irreducibly. Again by (v), $V_1 \neq V_2$ whence $V_1 \cap V_2 = 1$ and $V_1V_2 = V_1 \times V_2$. Then (iii) and the fact that V_1V_2 is abelian imply $H \cap V_1V_2 = 1$. By (5.3(ii)) there exists $n \in N(P)$ such that $H^n \not\leq HV_1V_2$. Let $K = \langle H, H^n \rangle$, then (5.3(iv)) yields $H <_1 K <_2 G$.

Consequently, $K \cap HV_1V_2 = H$ and then $K \cap V_1V_2 \leq H \cap V_1V_2 = 1$. Now $K < KV_1 < KV_1V_2$ so as $K <_2 G$ we deduce that $KV_1V_2 = G$ and then that K is a complement to V_1V_2.

Lemma 5.4 implies $C_{V_1}(P), C_{V_2}(P) \neq 1$. Choose $v_1 \in C_{V_1}(P)^\sharp$, $v_2 \in C_{V_2}(P)^\sharp$ and let $M = \langle K, H^{v_1v_2}\rangle$. Recall that $K = \langle H, H^n\rangle$ so (5.3(v)) implies $M \neq G$. We have $M = KW$ where $W = M \cap (V_1 \times V_2) \neq V_1 \times V_2$. For $i = 1, 2$ let π_i: $W \longrightarrow V_i$ be the projection map. Note that each π_i is a K-homomorphism. By (5.3(iii)) we have $v_1v_2 \in W$ so $(v_1v_2)\pi_1 = v_1$ and $(v_1v_2)\pi_2 = v_2$. Using the fact that H is irreducible on each V_i and the fact that $W \neq V_1 \times V_2$, we see that each π_i is a K-isomorphism. Consequently, $\pi_1^{-1}\pi_2$ is a K-isomorphism $V_1 \longrightarrow V_2$ that maps v_1 to v_2. In particular, there is a prime p such that V_1 and V_2 are isomorphic irreducible $\mathrm{GF}(p)K$-modules.

Let $F = \mathrm{End}_K(V_1)$. Then F is a field and V_1 and V_2 are isomorphic irreducible FK-modules. Keeping v_2 fixed and letting v_1 range over $C_{V_1}(P)^\sharp$, we see that $F - \{0\}$ acts transitively on $C_{V_1}(P)^\sharp$. But F is semiregular on $V_1^\sharp$ whence $|F| = |C_{V_1}(P)|$. Lemma 5.4 implies that $\dim_F(V_1) = 2$.

Let $C = C_K(V_1 \times V_2)$. Then as $G = K(V_1 \times V_2)$ we have $C \trianglelefteq G$. Suppose $C \neq 1$. Then $C \not\leq H$ since $H_G = 1$ so as $H <_1 K$ we have $K = HC$ whence $G = HCV_1V_2$. Now (iv) implies that CV_1V_2 is nilpotent and so $G = HF(G)$. Using (5.3(iii)) and (5.2), we deduce that G is of type 2. Hence we may suppose that $C = 1$. Then as V_1 and V_2 are isomorphic FK-modules it follows that they are faithful FK-modules. Then (ii), (iii), and (2.4) imply that G is of type 3.

6. INTERSECTIONS

For the rest of this paper, we assume the hypotheses of Theorem B but that G is not of type 1 or 2. We fix the notation

$$\Delta = \{A \cap B \mid A, B \in H^G,\ A \neq B,\ \text{and } A \cap B \neq 1\}.$$

For $P \in \Delta$ and $M \leq G$ we set

$$\Delta(P) = \{A \in H^G \mid P \leq A\}$$

and

$$\Delta_M(P) = \{A \in \Delta(P) \mid A \leq M\}.$$

Whenever K and M are subgroups of G we define

$$K \leq * M \text{ and } K <_n *M$$

to mean K contains a conjugate of H and $K \leq M$ or $K <_n M$ respectively.

If $\Delta = \emptyset$ then as $N(H) = H$ it follows that H is a Frobenius complement in G. Then $G = HF(G)$ and $g \in \langle H, H^g \rangle$ for all $g \in G$. But then (5.2) implies G is of type 1 or 2, a contradiction. Thus

$$\Delta \neq \emptyset.$$

The first aim of this section is to show that the hypotheses of Theorem 4.2 are satisfied with $\Omega = H^G$ and that the set Δ just defined is identical to the set Δ defined in (4.2). Then we shall study the normalizers of elements of Δ.

The following result, which relies on Bartels's Theorem (2.1), is crucial. Notice that only the first paragraph of the proof is needed if we assume that G is simple.

LEMMA 6.1. *Let $A, B \in H^G$ with $A \neq B$ and $A \cap B \neq 1$. Then*

$$A <_1 \langle A, B \rangle <_2 G.$$

Proof. Set $M = \langle A, B \rangle \neq G$ and $P = A \cap B \neq 1$. Suppose there exists Q such that $Q \not\leq M$ and $Q = P^g$ for some $g \in \langle P, Q \rangle$. Then $g \in \langle P, Q \rangle \leq \langle A, B^g \rangle$ so $A < \langle A, B \rangle < \langle A, B^g \rangle < G$ and as $\mathrm{cl}_G(A) = 3$, the result follows. Hence we may assume there are no such Q. Then (2.1) implies that P is contained in a subnormal subgroup of G that is contained in M. Let X be the largest such subgroup. Then any subnormal subgroup of G that is contained in M must be contained in X. Consequently, $X = \langle X^M \rangle \trianglelefteq M$.

Next we argue that $X \ntrianglelefteq G$. Assume false, so $X \trianglelefteq G$. Now $1 \neq P \leq A \cap X$ and $AX \leq M < G$ so (5.1) implies $AX = M <_1 G$. Similarly, $BX = M$. Note that $P \ntrianglelefteq G$ since $A_G = 1$. Let $g \in G - M$. If $P = P^g$ then as $P \ntrianglelefteq G = \langle M, g \rangle$, there exists $m \in M$ such that $P^m \neq P$ and so $P \neq P^{gm}$. Thus we may select $g \in G - M$ such that $P^g \neq P$. Now $A < AX = M <_1 G = \langle A^G \rangle$ so $N(M) = M$. Then as $X \trianglelefteq G$ it follows that $A^g \not\leq M$. Similarly $B^g \not\leq M$. Also $P^g \neq P = A \cap B$ so without loss of generality, $P^g \not\leq A$. We have

$$A < \langle A, P^g \rangle \leq \langle A, A^g \rangle \cap M < \langle A, A^g \rangle < G$$

and

$$A < \langle A, P^g \rangle \leq \langle A, B^g \rangle \cap M < \langle A, B^g \rangle < G.$$

Since $\mathrm{cl}_G(A) = 3$, this implies

$$\langle A, P^g \rangle <_2 G \text{ and } \langle A, A^g \rangle \cap M = \langle A, P^g \rangle = \langle A, B^g \rangle \cap M.$$

Then $\langle A, A^g \rangle \cap X = \langle A, P^g \rangle \cap X = \langle A, B^g \rangle \cap X$ and so

$$\langle A, P^g \rangle \cap X \trianglelefteq \langle A, A^g, B^g \rangle = \langle A, M^g \rangle = \langle A, A^g, X \rangle = G.$$

But $1 \neq P \leq \langle A, P^g \rangle \cap X$ and $\langle A, \langle A, P^g \rangle \cap X \rangle = \langle A, P^g \rangle <_2 G$, contrary to (5.1). We deduce that $X \ntrianglelefteq G$.

Now $X \trianglelefteq \trianglelefteq G$ so there exists a conjugate Y of X such that $Y \neq X$ and $Y \leq N(X)$. Since X is the largest subnormal subgroup of G that contains P and is contained in M, it follows that $Y \nleq M$. As $A_G = 1$ we have $X \nleq A$ so $A < AX \leq M < N(X) < G$. Then $\mathrm{cl}_G(A) = 3$ implies $A <_1 M <_2 G$.

LEMMA 6.2. *Let $K <_{2*} G$. Then there exists $g \in G$ such that $G = \langle K, K^g \rangle$.*

Proof. Assume false. Replacing K by a suitable conjugate, we may suppose that $H < K$. Set $\overline{G} = G/K_G$. Then $\overline{G}$ cannot be generated by two conjugates of $\overline{K}$ and as $G = \langle H^G \rangle$ we have $\overline{G} = \langle \overline{K}^{\overline{G}} \rangle$. Theorem A implies that $\overline{K}$ is cyclic. But $N(H) = H < K$ so $K_G \neq 1$. Now $H_G = 1$ so $K_G \nleq H$ and then $H < HK_G \leq K <_2 G$. Then $\mathrm{cl}_G(H) = 3$ implies $K = HK_G$. Thus $\overline{H} = \overline{K} <_2 \overline{G}$ and hence G is of type 1, a contradiction.

LEMMA 6.3. *Let $A, B \in H^G$ with $A \neq B$ and $A \cap B \neq 1$. Then there exists $C \in H^G$ such that $A \cap B \cap C \neq 1$ and $C \nleq \langle A, B \rangle$.*

Proof. Assume false. Let $P = A \cap B$ and $K = \langle A, B \rangle$. Then $C \leq K$ whenever $C \in H^G$ and $C \cap P \neq 1$. Lemma 6.1 implies $A <_1 K <_2 G$ and then (6.2) implies there exists $g \in G$ such that $G = \langle K, K^g \rangle$. Let $M = \langle A, A^g \rangle \neq G$. Then $K \nleq M$ or $K^g \nleq M$. By replacing g with g^{-1} if necessary, we may suppose that $K \nleq M$. Then as $A <_1 K$ we have $K \cap M = A$.

Let $p \in \pi(P)$, $P_0 \in \mathrm{Syl}_p(P)$, and $P_0 \leq Q \in \mathrm{Syl}_p(A)$. If $n \in N_M(Q)$ then $P_0 \leq Q = Q^n \leq A^n$ whence $A^n \leq K \cap M = A = N(A)$. Thus $N_M(Q) \leq A$ and $Q \in \mathrm{Syl}_p(M)$. Choose $m \in M$ such that $Q^{gm} = Q$. Then $P_0 \leq A^{gm}$ whence $A^{gm} \leq M \cap K = A$, $gm \in A$ and then $g \in M$.

Let $L = \langle A, P^g \rangle \leq M$. Choose $l \in L$ such that $P_0^{gl} \leq Q$. Then $P_0 \leq A^{(gl)^{-1}} \leq M \cap K = A$ so $gl \in A$ and then $g \in L$. Now $P^g \leq B^g$ so $g \in L \leq \langle A, B^g \rangle$ whence $G = \langle K, K^g \rangle \leq \langle A, B^g \rangle \neq G$, a contradiction.

COROLLARY 6.4. *The hypotheses of Theorem* 4.2 *are satisfied with $\Omega = H^G$ and the set Δ defined at the beginning of this section is identical to the set Δ defined in* (4.2). *In particular, all the results in the conclusion of* (4.2) *apply.*

Proof. This follows from (6.1) and (6.3)

COROLLARY 6.5. *Let $P \in \Delta$ and $M \leq G$ with $P \leq M$ and $\Delta_M(P) \neq \varnothing$. If $A \in \Delta(P)$ then either $A \leq M$ or $A \cap M = P$.*

Proof. Let $B \in \Delta_M(P)$ and suppose that $A \nleq M$. Then $B \leq \langle B, A \cap M \rangle < \langle B, A \rangle$. But $B <_1 \langle B, A \rangle$ by (6.1) so $A \cap M \leq B$ and thus $A \cap M = A \cap B$. Now $A \cap B = P$ by (4.2(iii)), hence result.

COROLLARY 6.6. *Let $K <_2 * G$ and $A \in H^G$ be such that $A \cap K \neq 1$. Then $\langle A, K\rangle \neq G$.*

Proof. Assume false, so that $G = \langle A, K\rangle$. Let $P = A \cap K \neq 1$. Since $K <_2 * G$ there exists $B \in H^G$ such that $B <_1 K$. If $P \not\leq B$ then $K = \langle B, P\rangle$ so $\langle K, A\rangle \leq \langle B, A\rangle \neq G$. Thus $P \leq B$. Then similarly $P \leq B^k$ for all $k \in K$ whence $P \leq B_K \trianglelefteq K$. Let $l \in K - B$ so $P \leq B_K \leq B \cap B^l \in \Delta$. Then (4.2(ii)) implies that $B_K = B \cap B^l$ and that $B \cap B^l \leq A$. Then $P \leq B_K \leq A \cap K = P$ so $P = B_K$. Now $A, B \in \Delta(P)$ and $P \trianglelefteq B$ so (4.2(vi)) implies $P \trianglelefteq A$. Thus $P \trianglelefteq \langle A, K\rangle = G$, contrary to $A_G = 1$. Thus $\langle A, K\rangle \neq G$.

COROLLARY 6.7. *Let $P \in \Delta$ and $A, B, C \in \Delta(P)$. Then $\langle A, B, C\rangle \neq G$.*

Proof. This follows from (6.1) and (6.6).

COROLLARY 6.8. *Let $P \in \Delta$. Then either $\langle \Delta(P)\rangle <_1 G$ or $\langle \Delta(P)\rangle = G$. Also, $N(P) \leq \langle \Delta(P)\rangle$.*

Proof. Choose $A, B \in H^G$ such that $P = A \cap B$. By (6.3) there exists $C \in H^G$ such that $P \cap C \neq 1$ and $C \not\leq \langle A, B\rangle$. Theorem 4.2(ii) implies $C \in \Delta(P)$ so $A < \langle A, B\rangle < \langle A, B, C\rangle \leq \langle \Delta(P)\rangle$. As $\mathrm{cl}_G(A) = 3$, the first assertion follows. Note that N(P) normalizes $\langle \Delta(P)\rangle$ so the second assertion follows from the first and the fact that $G = \langle H^G\rangle$.

THEOREM 6.9. *Let $P \in \Delta$ be such that $P \leq H$ and $\langle \Delta(P)\rangle = G$. If $N_H(P) = P$ assume that $n \in \langle H, H^n\rangle$ for all $n \in N(P)$. Let $K_1, \ldots, K_\alpha$ be the distinct subgroups of the form $\langle H, A\rangle$ as A ranges over $\Delta(P) - \{H\}$. For each i let $M_{i1}, \ldots, M_{i\alpha_i}$ be the distinct subgroups of the form $\langle K_i, B\rangle$ as B ranges over the members of $\Delta(P)$ not contained in K_i. Set $N = N(P)$ and for each i, j set $N_i = N \cap K_i$ and $N_{ij} = N \cap M_{ij}$. Then $\alpha \geq 2$ and $\alpha_i \geq 2$ for all i and:*

(i) *$H <_1 K_i <_2 G$ and $K_i \cap K_j = H$ for all $i \neq j$.*

(ii) *$K_i <_1 M_{ij} <_1 G$ and $M_{ij} \cap M_{ik} = K_i$ for all $j \neq k$ and all i.*

(iii) *$(N, \{N_i\}_{i=1}^{\alpha}, \{N_{ij}\}_{i=1}^{\alpha}{}_{j=1}^{\alpha_i}, N_H(P))$ satisfies Hypothesis 3.2*

Proof. (i) This follows from (6.1). Note that $\alpha \geq 2$ since $\langle \Delta(P)\rangle = G$.

(ii) By (6.7) we have $M_{ij} \neq G$ for all j. Since $K_i <_2 G$, the result follows. Note that $\alpha_i \geq 2$ since $\langle \Delta(P)\rangle = G$.

(iii) Either by assumption or by (4.2(vii)) we have $N = N_1 \cup \cdots \cup N_\alpha$ and using (4.2(vi)) it follows that $N_i > N_H(P)$ for all i. Now (i) implies $N_i \cap N_j = N_H(P)$ for all $i \neq j$ so as $\alpha \geq 2$ we have $N > N_i$ for all i. Thus $(N, \{N_i\}_{i=1}^{\alpha}, N_H(P))$ satisfies Hypothesis 3.1.

Let $1 \leq i \leq \alpha$. Again, either by assumption or by (4.2(vii)) we have $N = N_{i1} \cup \cdots \cup N_{i\alpha_i}$ and using (4.2(vi)) it follows that $N_{ij} > N_i$ for all j.

Now (ii) implies $N_{ij} \cap N_{ik} = N_i$ for all $j \neq k$ so as $\alpha_i \geq 2$ we have $N > N_{ij}$ for all j. Thus $(N, \{N_{ij}\}_{j=1}^{\alpha_i}, N_i)$ satisfies Hypothesis 3.1.

By construction, for all i, j there exists k such that $K_j \leq M_{ik}$, so $N_j \leq N_{ik}$. Thus Hypothesis 3.2 is satisfied.

COROLLARY 6.10. *Let $P \in \Delta$ be such that $P \leq H$ and $N_H(P) > P$. If $\langle \Delta(P) \rangle \neq G$ assume that $P \trianglelefteq H$. Then $N(P)/P$ is a Frobenius group with cyclic complement $N_H(P)/P$ and elementary abelian kernel.*

Proof. Let $N = N(P)$. By (4.2(vii)), $N_H(P)/P$ is a cyclic Frobenius complement in N/P so in particular, $N_H(P) > P = N_H(P)_N$. If $\langle \Delta(P) \rangle = G$ then (6.9(iii)) and (3.7) imply that $F(N(P)/P)$ is elementary abelian. Suppose $\langle \Delta(P) \rangle \neq G$, so by assumption, $P \trianglelefteq H$. Now $N(P)$ is transitive on $\Delta(P)$ by (4.2(vi)) whence $P \trianglelefteq \langle \Delta(P) \rangle$ and then (6.8) implies $\langle \Delta(P) \rangle = N <_1 G$. Thus $H <_2 N$. If $n \in N$ then $P \leq H \cap H^n$ so (6.1) implies that N cannot be generated by two conjugates of H. Moreover, $N = \langle H^N \rangle$ by (4.2(vii)). Theorem A implies that $F(N/P)$ is elementary abelian.

COROLLARY 6.11. *Let $P \in \Delta$ be such that $\langle \Delta(P) \rangle = G$ and $P \leq H$. If $N_H(P) = P$ assume that $n \in \langle H, H^n \rangle$ for all $n \in N(P)$. Suppose that $U \trianglelefteq G$ is such that $HU <_2 G$. Then $C_U(P)$ is a p-group.*

Proof. Let $N = N(P)$ and $\overline{N} = N/P$. By (5.1) we have $H \cap U = 1$ so $C_U(P) \cong \overline{C_U(P)} \trianglelefteq \overline{N}$. Then also $\overline{N_H(P)}^{\overline{n}} \cap \overline{C_U(P)} = 1$ for all $\overline{n} \in \overline{N}$. If $N_H(P) > P$ then by (6.10), $\overline{N_H(P)}$ is a Frobenius complement in $\overline{N}$, so this implies that $\overline{C_U(P)} \leq F(\overline{N})$ and then (6.10) implies that $\overline{C_U(P)}$ is a p-group. Hence we may suppose that $N_H(P) = P$. If $C_U(P) = 1$ then there is nothing to prove, so suppose also that $C_U(P) \neq 1$.

Assume the notation of (6.9). Then $(N, \{N_i\}_{i=1}^{\alpha}, \{N_{ij}\}_{i=1}^{\alpha}{}_{j=1}^{\alpha_i}, P)$ satisfies Hypothesis 3.2. Since $C_U(P) \neq 1$ and $HU <_2 G$ we have $HU = K_i$ for some i. Now $C_U(P) \trianglelefteq N$ so $\overline{C_U(P)} \leq \overline{N_{i\overline{N}}}$ and thus (3.6(iii)) implies that $\overline{C_U(P)}$ is contained in the p-subgroup $F(\overline{N})$. Hence the result.

7. THE SUBGROUPS $\mathbf{I}(P, Q)$

In this section we begin the analysis of the intersections of second maximal subgroups. We are most interested in intersections that contain at least two members of Δ. Define a relation $\sim$ on Δ by

$$P \sim Q \text{ if and only if } \quad \text{(i)} \quad \langle P, Q \rangle \text{ is not contained in a conjugate of } H$$
$$\text{and} \quad \text{(ii)} \quad G = \langle \Delta(P) \cup \Delta(Q) \rangle.$$

Note that (i) forces $P \neq Q$, that $\sim$ is symmetric and that $N(P)$ permutes $\{Q \in \Delta | P \sim Q\}$. Whenever $P \sim Q$ define

$$\Sigma(P, Q) = \{K <_2 G \mid \langle P, Q \rangle \leq K \quad \text{and} \quad \Delta_K(P) \cup \Delta_K(Q) \neq \varnothing\}$$

and set

$$\mathrm{I}(P, Q) = \bigcap \Sigma(P, Q).$$

Note that $\Sigma(P, Q) = \Sigma(Q, P)$ and that $\mathrm{I}(P, Q) = \mathrm{I}(Q, P)$.

We briefly outline the objectives of this section, omitting some technical details. We will be interested in conditions that force $N(P)$ to normalize $\mathrm{I}(P, Q)$. As a first application we prove that if $1 \neq P \trianglelefteq H$ then $N(P) = H$. The conditions referred to are quite weak so we suspect that $N(P)$ normalizes $\mathrm{I}(P, Q)$ for many $Q \in \Delta$. However, we also establish the following:

If $P \sim Q$; $P \sim R$; $N(P)$ normalizes both $\mathrm{I}(P, Q)$ and $\mathrm{I}(P, R)$; and $\langle Q, R \rangle$ is contained in a conjugate of H then $Q = R$.

The tension thus created enables us to prove the following result.

Let $P, Q \in \Delta$ and suppose that $\langle P, Q \rangle \leq K \leq * \, G$. If P and Q are conjugate in G then thay are also conjugate in K.

We begin with some elementary properties of the subgroups $\mathrm{I}(P, Q)$.

LEMMA 7.1. *Let $P, Q \in \Delta$ with $P \sim Q$. Then*:

(i) *If $A \in \Delta(P)$ then $A <_1 \langle A, Q \rangle \in \Sigma(P, Q)$.*

(ii) $G = \langle \Sigma(P, Q) \rangle$.

(iii) *If K and L are distinct members of $\Sigma(P, Q)$ then $K <_1 \langle K, L \rangle <_1 G$ and $\mathrm{I}(P, Q) = K \cap L$.*

(iv) *For each $A \in \Delta(P)$ there exists $B \in \Delta(P)$ such that $B \not\leq \langle A, Q \rangle$, $A \not\leq \langle B, Q \rangle$ and $\mathrm{I}(P, Q) = \langle A, Q \rangle \cap \langle B, Q \rangle$.*

(v) *If $A \in \Delta(P)$ then $A \cap \mathrm{I}(P, Q) = P$.*

(vi) *If $M \leq G$, $P < \mathrm{I}(P, Q) \cap M$ and $\Delta_M(P) \neq \varnothing$ then $\mathrm{I}(P, Q) \leq M$.*

(vii) *If $R \in \Delta$, $P \sim R$ and $\mathrm{I}(P, Q) \cap \mathrm{I}(P, R) > P$ then $\mathrm{I}(P, Q) = \mathrm{I}(P, R)$.*

Proof. (i) By definition $Q \not\leq A$ so $A < \langle A, Q \rangle$ and either $\langle A, Q \rangle <_2 G$ or $\langle A, Q \rangle <_1 G$. Assume that $\langle A, Q \rangle <_1 G$. Then for all $B \in \Delta(Q)$ we have $\langle A, Q \rangle = \langle A, B \rangle$, so using (6.8) it follows that $\langle A, Q \rangle = \langle \Delta(Q) \rangle \neq G$. But $G = \langle \Delta(P) \cup \Delta(Q) \rangle$ so there exists $C \in \Delta(P)$ such that $C \not\leq \langle \Delta(Q) \rangle$. The preceding argument with C in place of A yields $\langle C, Q \rangle <_2 G$. Now $P \leq A \cap \langle C, Q \rangle$ so (6.6) implies $\langle A, C, Q \rangle \neq G$. Then as $\langle A, Q \rangle = \langle \Delta(Q) \rangle <_1 G$ we have $C \leq \langle \Delta(Q) \rangle$, a contradiction. Thus $\langle A, Q \rangle <_2 G$. Hence $\langle A, Q \rangle \in \Sigma(P, Q)$ and $A <_1 \langle A, Q \rangle$.

(ii) By (i) and the symmetry of $\sim$ we have $G = \langle \Delta(P) \cup \Delta(Q) \rangle \leq \langle \Sigma(P, Q) \rangle$.

(iii) Without loss of generality there exists $A \in \Delta_K(P)$. Now $P \leq A \cap L$ so (6.6) implies $\langle A, L\rangle \neq G$. Now $K <_2 G$ so $A <_1 K$. By definition, $\langle P, Q\rangle \not\leq A$ whence $K = \langle A, Q\rangle \leq \langle A, L\rangle$. Thus $K < \langle K, L\rangle < G$ and as $K <_2 G$ we have $K <_1 \langle K, L\rangle <_1 G$. What we have just done, together with (ii) implies that $\Sigma(P, Q)$ satisfies the hypotheses of (4.1). Then (4.1(ii)) yields $\mathrm{I}(P, Q) = K \cap L$.

(iv) By (i), $\langle A, Q\rangle <_2 G$ so by (6.8) there exists $B \in \Delta(P)$ such that $B \not\leq \langle A, Q\rangle$. Then also $\langle B, Q\rangle <_2 G$ and consequently $A \not\leq \langle B, Q\rangle$. The final assertion follows from (iii).

(v) Choose $B \in \Delta(P)$ in accordance with (iv). Then $A \cap \mathrm{I}(P, Q) = A \cap \langle B, Q\rangle$. But $A \not\leq \langle B, Q\rangle$ so (6.5) yields $A \cap \langle B, Q\rangle = P$.

(vi) Choose $A \in \Delta_M(P)$. Then (i) implies $\mathrm{I}(P, Q) \leq \langle A, Q\rangle$ and $A <_1 \langle A, Q\rangle$. By (v), $A \cap \mathrm{I}(P, Q) = P$ so $M \cap \mathrm{I}(P, Q) \not\leq A$. Thus $\langle A, Q\rangle = \langle A, M \cap \mathrm{I}(P, Q)\rangle \leq M$ and then $\mathrm{I}(P, Q) \leq M$.

(vii) Let $A \in \Delta(P)$ and choose $B \in \Delta(P)$ in accordance with (iv). Then $P < \mathrm{I}(P, R) \cap \mathrm{I}(P, Q) \leq \mathrm{I}(P, R) \cap \langle A, Q\rangle$ so applying (vi) with R in place of Q yields $\mathrm{I}(P, R) \leq \langle A, Q\rangle$. Similarly $\mathrm{I}(P, R) \leq \langle B, Q\rangle$ and then (iv) yields $\mathrm{I}(P, R) \leq \mathrm{I}(P, Q)$. By symmetry, $\mathrm{I}(P, Q) \leq \mathrm{I}(P, R)$, hence result.

LEMMA 7.2. *Let $P \in \Delta$, $A \in \Delta(P)$ and $A < K <_2 G$. Assume that $n \in \langle A, A^n\rangle$ for all $n \in N(P)$ and that $G = \langle K, \Delta(P)\rangle$. Then $K \neq K^n$, $\langle K, K^n\rangle <_1 G$ and $K \cap K^n = K_{N(P)}$ for all $n \in N(P) - K$.*

Proof. Let $\Sigma = K^{N(P)}$. Lemma 4.2(vi) implies $\langle \Delta(P)\rangle \leq \langle \Sigma\rangle$ so by assumption, $\langle \Sigma\rangle = G$. Let $n \in N(P) - K$. Then $n \in \langle A, A^n\rangle$, $K \neq K^n$ and $\langle K, A^n\rangle = \langle K, K^n\rangle$. Now $P \leq K \cap A^n$ so (6.6) implies $\langle K, A^n\rangle \neq G$ and hence $K <_1 \langle K, K^n\rangle <_1 G = \langle \Sigma\rangle$. The result now follows from (4.1(ii)).

LEMMA 7.3. *Suppose $P \sim Q$ and $A \in \Delta(P)$. Assume that $N_A(P) > P$ and that either $\langle \Delta(P)\rangle = G$ or $P \trianglelefteq A$. If $N(P) \cap N(\mathrm{I}(P, Q)) > P$ then $N(P)$ normalizes* $\mathrm{I}(P, Q)$.

Proof. Let $N = N(P)$, $\overline{N} = N/P$ and $\overline{F} = F(\overline{N})$. By (6.10), $\overline{N_A(P)}$ is a Frobenius complement in $\overline{N}$ and $\overline{F}$ is abelian. Let F be the inverse image of $\overline{F}$ in N. Then $F \trianglelefteq N$ and hence $N_A(P)$ normalizes $F \cap \langle A, Q\rangle$. Now F/P is abelian so F normalizes $F \cap \langle A, Q\rangle$ also. Since $\overline{N} = \overline{N_A(P)}\overline{F}$ we have $N = N_A(P)F$ and we deduce that $F \cap \langle A, Q\rangle \trianglelefteq N$.

Choose $B \in \Delta(P)$ in accordance with (7.1(iv)), so $\mathrm{I}(P, Q) = \langle A, Q\rangle \cap \langle B, Q\rangle$. Note that (4.2(vi)) implies that B satisfies the same assumptions as A, so in particular, $F \cap \langle B, Q\rangle \trianglelefteq N$. Let $U = N \cap \mathrm{I}(P, Q)$. If $n \in N$ then $A^n \in \Delta(P)$ so (7.1(v)) implies $A^n \cap \mathrm{I}(P, Q) = P$ and so $\overline{N_A(P)}^{\,n} \cap \overline{U} = 1$. Since $\overline{N_A(P)}$ is a Frobenius complement in $\overline{N}$ we deduce that $\overline{U} \leq \overline{F}$ and hence that $U \leq F$. Then $U = (F \cap \langle A, Q\rangle) \cap (F \cap \langle B, Q\rangle)$ and it follows that $U \trianglelefteq N$.

Suppose that $U > P$. Let $n \in N$. Note that $P \sim Q^n$ and that $\mathrm{I}(P, Q)^n = \mathrm{I}(P, Q^n)$. We have $P < U \leq \mathrm{I}(P, Q) \cap \mathrm{I}(P, Q^n)$ and then (7.1(vii)) forces $\mathrm{I}(P, Q) = \mathrm{I}(P, Q)^n$. Thus $N \leq N(\mathrm{I}(P, Q))$. Hence we may suppose that $U = P$.

We will use (7.2). By hypothesis, there exists $m \in N \cap N(\mathrm{I}(P, Q))$ with $m \notin P$. As $N \cap \mathrm{I}(P, Q) = U = P$ we have $m \notin \mathrm{I}(P, Q)$. But $\mathrm{I}(P, Q) = \langle A, Q\rangle \cap \langle B, Q\rangle$, so replacing A by B if necessary, we have $m \notin \langle A, Q\rangle$. Let $K = \langle A, Q\rangle$ so $K <_2 G$ by (7.1(i)). Note that $n \in \langle A, A^n\rangle$ for all $n \in N$ by (4.2(vii)). If $\langle\Delta(P)\rangle = G$ then certainly $\langle K, \Delta(P)\rangle = G$. If $\langle\Delta(P)\rangle \neq G$ then by assumption $P \trianglelefteq A$, so as previously, $\langle\Delta(P)\rangle = N <_1 G$. But $N \cap \mathrm{I}(P, Q) = P \neq Q$ so as $Q \leq K$ we have $\langle K, \Delta(P)\rangle = G$ in this case also.

Lemma 7.2 implies that $K \neq K^m$ and that $K \cap K^m = K_N$. Now $A \in \Delta_K(P)$ and $A^m \in \Delta_{K^m}(P)$ so $K, K^m \in \Sigma(P, Q)$. Then (7.1(iii)) implies $\mathrm{I}(P, Q) = K \cap K^m$. As $K \cap K^m = K_N$ we deduce that N normalizes $\mathrm{I}(P, Q)$.

THEOREM 7.4. *Let $1 \neq P \trianglelefteq A \in H^G$. Then $N(P) = A$. In particular, $P \notin \Delta$.*

Proof. Assume false. Choose $g \in N(P) - A$ and let $B = A^g$. Then $P \leq A \cap B \in \Delta$; (4.2(iv)) implies $A \cap B$ is a TI-subgroup so $g \in N(A \cap B)$; and as $P \trianglelefteq A$ we have $A \cap B \trianglelefteq A$. Hence we may suppose that $P = A \cap B \in \Delta$. Let $M = N(P)$. Using (4.2(vi)) and (6.8) we have $\langle\Delta(P)\rangle = M <_1 G$. Now $A <_2 M$, (6.1) implies that M cannot be generated by two conjugates of A, and (4.2(vii)) implies $M = \langle A^M\rangle$. If M is a Frobenius complement in G then $G = MF(G)$ and $G/F(G) \cong M$. This contradicts the assumption that G is not of type 1 so we deduce that M is not a Frobenius complement in G.

We can now choose $1 \neq T \leq M$ and $x \in N(T)$ such that $x \notin M$. Note that as $M <_1 G$ and $\langle A^G\rangle = G$ we have $N(M) = M$ and hence $G = \langle M, M^x\rangle$. Let $Q = P^x \neq P$. Note that $\langle\Delta(P) \cup \Delta(Q)\rangle = \langle M, M^x\rangle = G$. If $P \not\sim Q$ then $\langle P, Q\rangle \leq C$ for some $C \in H^G$ and then P is conjugate to Q in C by (4.2(v)). But $C \leq \langle\Delta(P)\rangle = M$ so $P \trianglelefteq C$, contrary to $P \neq Q$. We deduce that $P \sim Q$.

Note that $T \leq M = N(P)$ and $x \in N(T)$ so T normalizes both P and Q. Then T permutes the set $\Sigma(P, Q)$ whence $T \leq N(\mathrm{I}(P, Q))$. Recall that by (4.2(iv)), both P and Q are TI-subgroups. Now $x \notin M = N(P)$ so $T \not\leq P$ and $x \notin M^x = N(Q)$ so $T \not\leq Q$. Lemma 7.3 implies that both M and M^x normalize $\mathrm{I}(P, Q)$. Consequently, $\mathrm{I}(P, Q) \trianglelefteq G$. By (7.1(i)) we have $\mathrm{I}(P, Q) \leq \langle A, Q\rangle <_2 G$ whence $A\mathrm{I}(P, Q) <_2 G$. But as $P \leq A \cap \mathrm{I}(P, Q)$, (5.1) implies that G is of type 1, a contradiction. Thus $N(P) = A$.

If $P \in \Delta$ then by (4.2(vi)), $\Delta(P) = \{A\}$, contrary to the definition of Δ. Thus $P \notin \Delta$.

COROLLARY 7.5. *Let $P \in \Delta$ and $A \in \Delta(P)$ be such that $N_A(P) = P$. Then $g \in \langle A, A^g\rangle$ for all $g \in G$.*

Proof. Theorem 4.2(iv) implies P is a TI-subgroup in A, so as $N_A(P) = P$ it follows that P is a Frobenius complement in A. Choose $q \in \pi(F(A))$ and let $Q \in \mathrm{Syl}_q(A)$. Then $Q \trianglelefteq A$ so $N(Q) = A$ by (7.4) and consequently $Q \in \mathrm{Syl}_q(G)$. If $g \in G$ then by Sylow's Theorem there exists $x \in \langle A, A^g \rangle$ such that $Q^{gx} = Q$. Then $gx \in N(Q) = A$ so $g \in \langle A, A^g \rangle$.

LEMMA 7.6. *Suppose that* $P \sim Q$, $\langle \Delta(P) \rangle = G$ *and that* $N(\mathrm{I}(P, Q)) > \mathrm{I}(P, Q)$. *Then* $N(P)$ *normalizes* $\mathrm{I}(P, Q)$.

Proof. If $N_{\mathrm{I}(P,Q)}(P) > P$ then $N(P) \cap N(\mathrm{I}(P, Q)) > P$. If $N_{\mathrm{I}(P,Q)}(P) = P$ then as P is a TI-subgroup, it is a Frobenius complement in $\mathrm{I}(P, Q)$ so the Frattini Argument yields $N(\mathrm{I}(P, Q)) = (N(P) \cap N(\mathrm{I}(P, Q)))\mathrm{I}(P, Q)$. Thus in both cases we have $N(P) \cap N(\mathrm{I}(P, Q)) > P$. If $N_A(P) > P$ for some $A \in \Delta(P)$ then the result follows from (7.3). Hence we may suppose that $N_A(P) = P$ for all $A \in \Delta(P)$.

We will apply (4.1). Choose $g \in N(\mathrm{I}(P, Q)) - \mathrm{I}(P, Q)$. By (7.1(iv)) there exists $A \in \Delta(P)$ such that $g \notin \langle A, Q \rangle$. Set $K = \langle A, Q \rangle$, so by (7.1(i)) we have $\mathrm{I}(P, Q) \leq K <_2 G$. Now $P \leq \mathrm{I}(P, Q) \leq K^g$ so $P \leq A \cap K^g$ and then (6.6) implies $\langle A, K^g \rangle \neq G$. But $g \in \langle A, A^g \rangle$ by (7.5) hence $\langle A, K^g \rangle = \langle K, K^g \rangle <_1 G$. By (4.2(vi)), $N(P)$ is transitive on $\Delta(P)$ so as $\langle \Delta(P) \rangle = G$ there exists $n \in N(P)$ such that $A^n \not\leq \langle K, K^g \rangle$. Using (7.2) together with (7.5) we have $\langle K, K^n \rangle <_1 G$. We also have $P \leq A^n \cap K^g$ so again, (6.6) implies $\langle A^n, K^g \rangle <_1 G$. By (7.5) we have $g^{-1}n \in \langle A^{gg^{-1}n}, A^g \rangle = \langle A^n, A^g \rangle$ whence $\langle K^n, K^g \rangle = \langle A^n, K^g \rangle <_1 G$.

Let $\Sigma = \{K, K^n, K^g\}$. Note that as $A^n \not\leq \langle K, K^g \rangle$ we have $\langle \Sigma \rangle = G$. If D and E are distinct members of Σ, then the previous paragraph implies $\langle D, E \rangle <_1 G = \langle \Sigma \rangle$ and as $K <_2 G$ we also have $D <_1 \langle D, E \rangle$. Lemma 4.1(ii) implies $K \cap K^g = \bigcap \Sigma = K \cap K^n$. But $\langle P, Q \rangle \leq \mathrm{I}(P, Q) \leq K^g$ whence $\langle P, Q \rangle \leq K^n$. As $A^n \in \Delta_{K^n}(P)$ we have $K^n \in \Sigma(P, Q)$ so $\mathrm{I}(P, Q) = K \cap K^n$ by (7.1(iii)). Using (7.2) together with (7.5) we have $K \cap K^n = K_{N(P)}$. We deduce that $N(P)$ normalizes $\mathrm{I}(P, Q)$.

THEOREM 7.7. *Suppose that* $P \sim Q, P \sim R$, $\langle \Delta(P) \rangle = G$, $\langle Q, R \rangle$ *is contained in a conjugate of* H *and that* $N(P)$ *normalizes both* $\mathrm{I}(P, Q)$ *and* $\mathrm{I}(P, R)$. *Then* $Q = R$.

Proof. Assume false. Choose $A \in H^G$ such that $\langle Q, R \rangle \leq A$. Now $A \cap \mathrm{I}(P, Q) = Q$ by (7.1(v)) so as $R \neq Q$ we have $\mathrm{I}(P, R) \not\leq \mathrm{I}(P, Q)$. Note that if $B \in \Delta(P)$ then (7.1(i)) implies $\mathrm{I}(P, Q) \leq \langle B, Q \rangle$ so $\langle B, \mathrm{I}(P, Q) \rangle = \langle B, Q \rangle$. Similarly, $\langle B, \mathrm{I}(P, R) \rangle = \langle B, R \rangle$. By (7.1(iv)) there exists $B \in \Delta(P)$ such that $\mathrm{I}(P, R) \not\leq \langle B, \mathrm{I}(P, Q) \rangle$. Let $J = \langle \mathrm{I}(P, Q), \mathrm{I}(P, R) \rangle$. We have

$$B < \langle B, \mathrm{I}(P, Q) \rangle < \langle B, J \rangle = \langle B, Q, R \rangle \leq \langle B, A \rangle < G,$$

so as $\mathrm{cl}_G(B) = 3$ we deduce that $\langle B, J \rangle = \langle B, A \rangle <_1 G$.

Let $n \in N(P)$. Since n normalizes $\mathrm{I}(P,Q), \mathrm{I}(P,R)$ and J we have $\mathrm{I}(P,R) \not\leq \langle B^n, \mathrm{I}(P,Q)\rangle$ so again, $\langle B^n, J\rangle = \langle B^n, A\rangle$ and then $\langle B, J\rangle^n = \langle B^n, A\rangle$. Note that $P \leq B^n$ so if we set $K = \langle P, A\rangle$ we deduce that

$$A < K \leq \langle B, J\rangle_{N(P)} \leq \langle B, J\rangle <_1 G.$$

By (4.2(vi)), $N(P)$ is transitive on $\Delta(P)$ so as $\langle\Delta(P)\rangle = G$ and $B \in \Delta(P)$ we see that $N(P)$ does not normalize $\langle B, J\rangle$. Thus $\langle B, J\rangle_{N(P)} < \langle B, J\rangle$. Then as $\mathrm{cl}_G(A) = 3$ we have $K = \langle B, J\rangle_{N(P)}$ and we deduce that $N(P)$ normalizes K.

Since $\langle A^G\rangle = G$ we have $N(P) \leq N(K) \neq G$, so as $\langle\Delta(P)\rangle = G$ we may choose $C \in \Delta(P)$ such that $C \not\leq N(K)$. Now K and $\langle C, Q\rangle$ are distinct members of $\Sigma(P,Q)$ by (7.1(i)) so (7.1(iii)) implies

$$K <_1 \langle K, C\rangle <_1 G \text{ and } \mathrm{I}(P,Q) = K \cap \langle C, Q\rangle.$$

Then $N(K) \cap \langle K, C\rangle = K$ and $C \cap K = C \cap \mathrm{I}(P,Q)$. By (7.1(v)) we have $C \cap \mathrm{I}(P,Q) = P$ whence $N_C(P) \leq C \cap N(K) = C \cap K = P$ so $N_C(P) = P$. Then by (7.5), $g \in \langle C, C^g\rangle$ for all $g \in G$. Then as A is a conjugate of C we have $g \in \langle A, A^g\rangle$ for all $g \in G$. But $A \leq K$ whence $N(K) = K$ and then $N(P) \leq K < \langle K, C\rangle <_1 G$. Finally, as $N(P)$ is transitive on $\Delta(P)$ we then have $\langle\Delta(P)\rangle \leq \langle K, C\rangle \neq G$, a contradiction.

LEMMA 7.8. *Suppose that $P \sim Q$, P is conjugate to Q and that $N(\mathbf{I}(P,Q)) = \mathbf{I}(P,Q)$. Then P is conjugate to Q in* $\mathrm{I}(P,Q)$.

Proof. Let $t \in \pi(P)$, $T \in \mathrm{Syl}_t(P)$ and choose $\tilde{T} \in \mathrm{Syl}_t(N_{\mathrm{I}(P,Q)}(P))$ with $T \leq \tilde{T}$, so $T = \tilde{T} \cap P \trianglelefteq \tilde{T}$. Recall that P is a TI-subgroup by (4.2(iv)). In particular, $N_{\mathrm{I}(P,Q)}(T) \leq N_{\mathrm{I}(P,Q)}(P)$ whence $\tilde{T} \in \mathrm{Syl}_t(N_{\mathrm{I}(P,Q)}(T))$. If $T = \tilde{T}$ then $T \in \mathrm{Syl}_t(\mathrm{I}(P,Q))$ so there exists $x \in \mathrm{I}(P,Q)$ such that $T^x \leq Q$ whence $P^x = Q$. Hence we may suppose that $T < \tilde{T}$.

Let $n \in N(P) \cap N(\tilde{T})$. We have $P \sim Q^n$ and $\mathrm{I}(P, Q^n) = \mathrm{I}(P,Q)^n$. Thus

$$P < P\tilde{T} \leq \mathrm{I}(P,Q) \cap \mathrm{I}(P,Q^n)$$

and thus $\mathrm{I}(P,Q) = \mathrm{I}(P,Q)^n$ by (7.1(vii)). We deduce that $N(P) \cap N(\tilde{T}) \leq N(\mathrm{I}(P,Q)) = \mathrm{I}(P,Q)$ and then that $\tilde{T} \in \mathrm{Syl}_t(N(P))$.

Choose $S \in \mathrm{Syl}_t(Q)$ and $\tilde{S} \in \mathrm{Syl}_t(N_{\mathrm{I}(P,Q)}(Q))$ with $S \leq \tilde{S}$. Again, we may suppose that $S < \tilde{S}$ and then that $\tilde{S} \in \mathrm{Syl}_t(N(Q))$. Choose $g \in G$ such that $P = Q^g$. Then $\tilde{S}^g \in \mathrm{Syl}_t(N(P))$ so there exists $m \in N(P)$ such that $\tilde{S}^{gm} = \tilde{T}$. Note that $P = Q^{gm}$. We have

$$P < P\tilde{T} = (Q\tilde{S})^{gm} \leq \mathrm{I}(P,Q) \cap \mathrm{I}(P,Q)^{gm}$$

so as $\mathrm{I}(P,Q)^{gm} = \mathrm{I}(P^{gm}, P)$, it follows from (7.1(vii)) that $gm \in N(\mathrm{I}(P,Q)) = \mathrm{I}(P,Q)$. Hence the result.

THEOREM 7.9. *Let $P \in \Delta$ be such that $\langle \Delta(P) \rangle = G$ and let $A \in H^G$. Then there exists $g \in \langle P, A \rangle$ such that $P^g \le A$.*

Proof. Assume false and set $K = \langle P, A \rangle$. By the definition of Δ there exists a conjugate Q of P such that $Q \le A$. Note that P is not conjugate to Q in K. Suppose that $P \not\sim Q$. Then as $G = \langle \Delta(P) \rangle$, the definition of $\sim$ implies $\langle P, Q \rangle \le B$ for some $B \in H^G$. Now $A \in \Delta_K(Q)$ and $Q < \langle P, Q \rangle \le B \cap K$ so (6.5) implies $B \le K$. But P is conjugate to Q in B by (4.2(v)), a contradiction. We deduce that $P \sim Q$.

By (7.1(i)) we have $\mathrm{I}(P, Q) \le K$ so P is not conjugate to Q in $\mathrm{I}(P, Q)$ and then (7.8) implies $N(\mathrm{I}(P, Q)) > \mathrm{I}(P, Q)$. Hence $N(P)$ normalizes $\mathrm{I}(P, Q)$ by (7.6). Let $a \in A$. Then again, $P \sim Q^a$ and $N(P)$ normalizes $\mathrm{I}(P, Q^a)$. Now (7.7) implies $Q = Q^a$ and we deduce that $Q \trianglelefteq A$. This contradicts (7.4) and completes the proof.

COROLLARY 7.10. *Let $P \in \Delta$ and $K \le\!* \, G$ be such that $\langle \Delta(P) \rangle = G$ and $P \le K$. Then*:

(i) $\Delta_K(P) \neq \emptyset$.

(ii) *If Q is a conjugate of P that is contained in K then Q is conjugate to P in K.*

Proof. Since $K \le\!* \, G$ there exists $A \in H^G$ such that $A \le K$. By (7.9) there exists $g \in \langle P, A \rangle \le K$ such that $P^g \le A$. Then $A^{g^{-1}} \in \Delta_K(P)$. This proves (i). To prove (ii), note that there exists $h \in \langle Q, A \rangle \le K$ such that $Q^h \le A$ and then that (4.2(v)) implies that Q^h is conjugate to P^g in A.

8. NONDEGENERACY RESULTS

Many of the results in the previous section are applicable only to those $P \in \Delta$ that satisfy $\langle \Delta(P) \rangle = G$, for example (7.9). The main aim of this section is to prove the existance of such a P.

THEOREM 8.1. (i) *There exists $P \in \Delta$ such that $\langle \Delta(P) \rangle = G$.*

(ii) *Suppose that whenever $S \in \Delta$ satisfies $\langle \Delta(S) \rangle = G$ then $N_E(S) > S$ for all $E \in \Delta(S)$. Then $\langle \Delta(P) \rangle = G$ for all $P \in \Delta$.*

Proof. Let $P \in \Delta$ and set $M = \langle \Delta(P) \rangle$. We may assume that $M \neq G$ so by (6.8) we have $N(P) \le M <_1 G$. Note that $N(P^m) \le M = \langle \Delta(P^m) \rangle$ for all $m \in M$. Since $G = \langle H^G \rangle$ we may choose $A \in H^G$ such that $A \not\le M$. Let Q be a conjugate of P that is contained in A. Suppose that $P^m \not\sim Q$ for some $m \in M$. Now $G = \langle M, A \rangle \le \langle \Delta(P^m) \cup \Delta(Q) \rangle$ so by the definition of $\sim$, there exists $B \in H^G$ such that $\langle P^m, Q \rangle \le B$. Then by (4.2(v)) there exists $b \in B$ such that $Q^b = P^m$. Consequently $A^b \le \langle \Delta(P^m) \rangle = M$.

But $B \in \Delta(P^m)$, so then $A \le M$, a contradiction. We deduce that $P^m \sim Q$ for all $m \in M$.

Next we claim we can choose Q such that $Q \not\le M$. By (7.4), $Q \ntrianglelefteq A$, so there exists $a \in A$ such that $Q^a \ne Q$. Now $A \cap \mathrm{I}(P, Q) = Q$ by (7.1(v)), so using (7.1(iv)) we may choose $C \in \Delta(P)$ such that $Q^a \not\le \langle C, Q\rangle$. Thus $C < \langle C, Q\rangle < \langle C, Q, Q^a\rangle \le \langle C, A\rangle < G$ so as $\mathrm{cl}_G(C) = 3$ we deduce that $\langle C, Q, Q^a\rangle = \langle C, A\rangle$. But $C \le M$ and $A \not\le M$ hence $Q \not\le M$ or $Q^a \not\le M$. Replacing Q with Q^a if necessary, we have $Q \not\le M$.

Let $D \in \Delta(P)$ and set $K = \langle D, Q\rangle$. We will show that D is not a Frobenius complement in K. Now $D <_1 K$ by (7.1(i)) so $K \cap M = D$. By (7.1(i),(v)) we have $\mathrm{I}(P, Q) \le K$ and $D \cap \mathrm{I}(P, Q) = P$. Thus $N_{\mathrm{I}(P,Q)}(P) \le K \cap M = D$ and hence $N_{\mathrm{I}(P,Q)}(P) = P$. Now P is a TI-subgroup by (4.2(iv)) so it follows that P is a Frobenius complement in $\mathrm{I}(P, Q)$. Then Q is also a Frobenius complement in $\mathrm{I}(P, Q)$ since $|Q| = |P|$. Consequently, there exists $f \in F(\mathrm{I}(P, Q))$ such that $Q^f = P$. Now $P \ntrianglelefteq D$ by (7.4) so choose $d \in D$ such that $P^d \ne P$. Again, there exists $f' \in F(\mathrm{I}(P^d, Q))$ such that $Q^{f'} = P^d$. We have $P^{f^{-1}f'} = P^d$ so $f^{-1}f'd^{-1} \in N_K(P) \le K \cap M = D$ whence $f^{-1}f' \in D$. If D is a Frobenius complement in K then $F(\mathrm{I}(P, Q)), F(\mathrm{I}(P^d, Q)) \le F(K)$ by (2.2) whence $f^{-1}f' \in F(K) \cap D = 1$. This contradicts $P^d \ne P$ and we deduce that D is not a Frobenius complement in K.

Choose $E \in D^K$ such that $E \ne D$ and $E \cap D \ne 1$. Let $R = E \cap D \in \Delta$. Recall that $K \cap M = D$ so $E \cap M = E \cap D = R$. Also, $D <_1 K$ so $K = \langle D, E\rangle \le \langle \Delta(R)\rangle$ and as $K \cap M = D$, we see that P is not conjugate to R in D. Then (4.2(v)) implies that P is not conjugate to R in G. By (7.4) there exists $e \in E - N(R)$. Set $S = R^e$. We will show that $\langle \Delta(S)\rangle = G$. Now $S \cap R = 1$ by (4.2(iv)) so as $E \cap M = R$ we have $S \not\le M$. Thus $G = \langle S, M\rangle$ and it suffices to prove that $\langle \Delta(P)\rangle \le \langle \Delta(S)\rangle$.

Since $e \in K \le \langle \Delta(R)\rangle$ we have $K \le \langle \Delta(S)\rangle$. In particular, $P < D \le \langle \Delta(S)\rangle$. Let $F \in \Delta(P)$ and let $f \in F$. Now $M = \langle \Delta(P^f)\rangle$ so as $S \not\le M$ it follows that $\langle P^f, S\rangle$ is not contained in a conjugate of H. Moreover, as $G = \langle S, M\rangle$ we have $\langle \Delta(P^f) \cup \Delta(S)\rangle = G$ and hence $P^f \sim S$. Now $S \not\le M$ so $\mathrm{I}(P^f, S) \not\le M$ and then (7.1(vi)) implies $\mathrm{I}(P^f, S) \cap M = P^f$. As $N(P^f) \le M$ we deduce that $N_{\mathrm{I}(P^f,S)}(P^f) = P^f$ and then that P^f is a Frobenius complement in $\mathrm{I}(P^f, S)$. But P is not conjugate to $S = R^e$ so S cannot be a Frobenius complement in $\mathrm{I}(P^f, S)$. Thus $N_{\mathrm{I}(P^f,S)}(S) > S$. Now $N(S) \le \langle \Delta(S)\rangle$ so $S < \mathrm{I}(P^f, S) \cap \langle \Delta(S)\rangle$ and then (7.1(vi)) implies $\mathrm{I}(P^f, S) \le \langle \Delta(S)\rangle$.

If we choose $f \in F - N(P)$ then we have $P < \langle P, P^f\rangle \le F \cap \langle \Delta(S)\rangle$. Now $D \in \Delta_{\langle \Delta(S)\rangle}(P)$ so (6.5) forces $F \le \langle \Delta(S)\rangle$. We deduce that $\langle \Delta(P)\rangle \le \langle \Delta(S)\rangle$ and so $G = \langle M, S\rangle \le \langle \Delta(S)\rangle$. This proves (i).

Assume the hypothesis of (ii) and continue the notation of the last but one paragraph. Then $N_E(S) > S$. If $f \in F$ we have $P^f \sim S$ and

$N_{\mathrm{I}(P^f,S)}(S) > S$ so (7.3) implies that $N(S)$ normalizes $\mathrm{I}(P^f, S)$. But then (7.7) implies $P \trianglelefteq F$, which contradicts (7.4) and proves (ii).

COROLLARY 8.2. (i) *$g \in \langle H, H^g\rangle$ for all $g \in G$.*

(ii) *If $K \leq* G$ then $N(K) = K$.*

(iii) *If $P \in \Delta$ is such that $\langle \Delta(P)\rangle = G$ then Hypothesis 5.3 is satisfied.*

(vi) *If $U \trianglelefteq G$ is such that $HU <_2 G$ then U is a p-group.*

Proof. (i) Choose $P \in \Delta$ such that $\langle \Delta(P)\rangle = G$. Now H contains a conjugate of P by the definition of Δ so we may assume $P \leq H$. If $N_H(P) = P$ then the result follows from (7.5) so assume $N_H(P) > P$. Let $g \in G$. By (7.10(ii)) there exists $x \in \langle H, H^g\rangle$ such that $P^{gx} = P$. Then $gx \in N(P)$ so using (4.2(vii))(c) we have $gx \in \langle H, H^{gx}\rangle \leq \langle H, H^g\rangle$, whence $g \in \langle H, H^g\rangle$.

(ii) This follows from (i) and the fact that K contains a conjugate of H.

(iii) This follows from (4.2(vi)), (i), (6.1) and (6.7).

(vi) Choose P as in (8.1(i)). Then $C_U(P)$ is a p-group by (6.11) and then (5.4) implies that U is a p-group.

LEMMA 8.3. *Let $K <_2* G$. Then K is not a Frobenius group.*

Proof. Assume false and choose $A \in H^G$ with $A \leq K$ and note that $A <_1 K$. Since $F(K)$ is a nilpotent normal subgroup of K we have $N_K(A \cap F(K)) > A \cap F(K)$. However, $A \cap F(K) \trianglelefteq A$ so (7.4) implies $A \cap F(K) = 1$. Thus A is contained in a Frobenius complement of K. But $A <_1 K$, so A is a Frobenius complement in K.

By (8.1) there exists $P \in \Delta$ such that $P \leq A$ and $\langle \Delta(P)\rangle = G$. Choose $f \in K - A$ and let $Q = P^f$, so $Q \not\leq A$. If $P \not\sim Q$ then as $\langle \Delta(P)\rangle = G$, there exists $B \in H^G$ such that $\langle P, Q\rangle \leq B$. Then (6.5) forces $B \leq K$. As $|B| = |A|$ it follows that B is also a Frobenius complement in K and is therefore conjugate to A in K. But $P \leq A \cap B$ so $A = B$ and then $Q \leq A$, a contradiction. We deduce that $P \sim Q$.

By (7.1(v)) we have $A \cap \mathrm{I}(P, Q) = P$ and $A^f \cap \mathrm{I}(P, Q) = Q$. Also, $\mathrm{I}(P, Q) \leq K$ by (7.1(i)) so (2.2) implies that P and Q are Frobenius complements in $\mathrm{I}(P, Q)$ and that $F(\mathrm{I}(P, Q)) \leq F(K)$. Consequently, there exists $u \in F(K)$ such that $Q^u = P$. Then $P^{fu} = P$ and hence $fu \in A$.

Let $a \in A$. Then $af \in K - A$ so again there exists $v \in F(K)$ such that $P^{afv} = P$ and $afv \in A$. Then $fv \in A$ and so $u^{-1}v = (fu)^{-1}(fv) \in A \cap F(K) = 1$ so $u = v$. But $P^{fu} = P = P^{afv}$ whence $P = P^a$. We deduce that $P \trianglelefteq A$, contrary to (7.4). Thus K is not a Frobenius group.

COROLLARY 8.4. *Let $K <_2* G$ and $P \in \Delta$ be such that $\langle \Delta(P)\rangle = G$. Then $\langle K, P\rangle \neq G$.*

Proof. We may assume that $P \nleq K$. Choose $A \in H^G$ with $A \leq K$. If $\langle A, P\rangle <_1 G$ then $\langle A, P\rangle = \langle A, B\rangle$ for all $B \in \Delta(P)$ so $\langle \Delta(P)\rangle = \langle A, P\rangle \neq G$, a contradiction. Thus $\langle A, P\rangle <_2 G$. Since A is not a Frobenius complement in K, there exists $C \in A^K$ such that $C \neq A$ and $C \cap A \neq 1$. Now $K <_2 G$ so $A <_1 K$ and then $K = \langle C, A\rangle$. But $C \cap \langle A, P\rangle \neq 1$ so (6.6) implies $\langle C, A, P\rangle \neq G$. Thus $\langle K, P\rangle \neq G$.

9. MORE TI-SUBGROUPS

This section has two goals. First we consider $P \in \Delta$ that satisfy $\langle \Delta(P)\rangle = G$. We define and study an equivalence relation on P^G. This will be used later on to construct large subgroups of G. Second we consider subgroups K such that $K <_2 G$ and $K_G = 1$. We will show that if L is a conjugate of K that satisfies $K \cap L \neq 1$ then $\langle K, L\rangle \neq G$. The results of Section 4 are invoked again to construct more TI-subgroups.

LEMMA 9.1. *Let $P \in \Delta$ be such that $\langle \Delta(P)\rangle = G$ and suppose that $P \leq K <_{2*} G$. Then*:

(i) $A \cap K_{N(P)} = P$ *for all* $A \in \Delta(P)$.

(ii) *If $L \neq K$ is a conjugate of K that contains P then $L <_1 \langle K, L\rangle <_1 G$ and $K \cap L = K_{N(P)}$.*

(iii) *If $M \leq* G, P \leq K \cap M, K \nleq M$ and M contains a conjugate of K then $K \cap M = K_{N(P)}$.*

Proof. (i) By (4.2(vi)), $N(P)$ is transitive on $\Delta(P)$ so as $\langle \Delta(P)\rangle = G$, there exists $n \in N(P)$ such that $A \nleq K^n$. Now $P \leq A \cap K^n$ and $\Delta_{K^n}(P) \neq \varnothing$ by (7.10(i)). Then (6.5) implies $A \cap K^n = P$, so $P \leq A \cap K_{N(P)} \leq A \cap K^n = P$, hence the result.

(ii) Choose $g \in G$ such that $K^g = L$. Then $P^{g^{-1}} \leq K$ so by (7.10(ii)) there exists $k \in K$ such that $P^{g^{-1}} = P^k$. Let $n = kg \in N(P)$, so $K^n = L$. Now $n \notin K$ since $K \neq L$ and (7.10(i)) imply that there exists $A \in \Delta_K(P)$. Then (7.2) together with (8.2(i)) yield $\langle K, L\rangle <_1 G$ and $K \cap L = K_{N(P)}$. As $L <_2 G$ we have $L <_1 \langle K, L\rangle$.

(iii) Let $L \leq M$ be a conjugate of K. By (7.10(ii)) we may choose L such that $P \leq L$. Then by (ii), $L <_1 \langle K, L\rangle$ so as $K \nleq M$, it follows that $\langle K, L\rangle \cap M = L$. Then $K \cap M = K \cap L$, so by (ii) we have $K \cap M = K_{N(P)}$.

THEOREM 9.2. *Let $P \in \Delta$ be such that $\langle \Delta(P)\rangle = G$. Define a relation $\approx$ on P^G by*

$$Q \approx R \textit{ if and only if } Q = R \textit{ or } Q \sim R \textit{ and } N(\mathrm{I}(Q, R)) > \mathrm{I}(Q, R).$$

Then:

(i) $\approx$ *is a* G*-invariant equivalence relation.*

(ii) *Let* $P \leq K <_2* G$ *and* $Q \in P^G$ *with* $Q \leq K$. *Then* $P \approx Q$ *if and only if* $Q \leq K_{N(P)}$.

(iii) *Let* $x \in G$ *and suppose that* $P \approx P^x \neq P$. *Then* $x \in N(\mathrm{I}(P, P^x))$ *and* $\mathrm{I}(P, P^x) = K_{N(P)}$ *for all* $K \in \Sigma(P, P^x)$.

Let Γ *be the equivalence class containing* P. *For each* $M \leq* G$ *with* $P \leq M$ *set* $\Gamma_M = \{Q \in \Gamma \mid Q \leq M\}$.

(iv) *If* $P \leq M \leq* G$ *then* $N(\Gamma_M) \leq N(\Gamma)$ *and* $N(\Gamma_M) = N(P)N_M(\Gamma) = \{x \in N(\Gamma) \mid P^x \leq M\}$.

(v) *Suppose* $K \leq* G$ *and* $M \leq* G$ *are such that* $P \leq K \leq M$. *Then* $N(\Gamma_K) \leq N(\Gamma_M)$.

(vi) *Suppose* $M \leq* G$ *and* $L \leq* G$ *are such that* $P \leq M \cap L \leq* G$. *Then* $N(\Gamma_{M\cap L}) = N(\Gamma_M) \cap N(\Gamma_L)$.

(vii) *If* $P \leq K \leq* G$ *then* $N(\Gamma_K) = N(K_{N(P)}) = N(P)N_K(K_{N(P)})$.

Proof. (ii) We may assume that $Q \neq P$. By (7.10) there exists $A \in \Delta_K(P)$. Suppose that $P \approx Q$. Then $K \in \Sigma(P, Q)$ so by definition, $\mathrm{I}(P, Q) \leq K$. Now $N(\mathrm{I}(P, Q)) > \mathrm{I}(P, Q)$ so (7.6) implies that $N(P)$ normalizes $\mathrm{I}(P, Q)$. Thus $Q \leq \mathrm{I}(P, Q) \leq K_{N(P)}$.

Now suppose that $Q \leq K_{N(P)}$. Then (9.1(i)) implies that $\langle P, Q\rangle$ is not contained in a conjugate of H, so $P \sim Q$. As $G = \langle \Delta(P)\rangle$ we may choose $n \in N(P)$ such that $A^n \not\leq K$, so $K \neq K^n$. Now $\langle P, Q\rangle \leq K_{N(P)} \leq K \cap K^n$ so $K, K^n \in \Sigma(P, Q)$ and then $\mathrm{I}(P, Q) = K \cap K^n$ by (7.1(iii)). Applying (7.2) together with (8.2(i)) yields $K \cap K^n = K_{N(P)}$ whence $n \in N(\mathrm{I}(P, Q)) - \mathrm{I}(P, Q)$ and so $P \approx Q$.

(i) Clearly $\approx$ is G-invariant and symmetric. Let $Q, R \in P^G$ be such that P, Q, R are distinct and $P \approx R \approx Q$. We claim that $P \approx Q$. By (7.6), $N(R)$ normalizes both $\mathrm{I}(P, R)$ and $\mathrm{I}(R, Q)$, so (7.7) implies that $\langle P, Q\rangle$ is not contained in a conjugate of H. Thus $P \sim Q$. Choose $A \in \Delta(P)$, set $K = \langle A, Q\rangle$ and $M = \langle K, R\rangle$. Then $K <_2 G$ by (7.1(i)) and then $M \neq G$ by (8.4). Now $\Delta_M(R) \neq \varnothing$ by (7.10) so as $G = \langle \Delta(R)\rangle$, there exists $n \in N(R)$ such that $M \neq M^n$. Then $K \not\leq M^n$. Also, $\mathrm{I}(P, R), \mathrm{I}(R, Q) \leq M$ by (7.1(vi)), so as $N(R)$ normalizes $\mathrm{I}(P, R)$ and $\mathrm{I}(R, Q)$ we have $\langle P, Q\rangle \leq M^n$. But then $\langle P, Q\rangle \leq K \cap M^n = K_{N(P)}$ by (9.1(i)), so (ii) implies that $P \approx Q$. Since $\approx$ is G-invariant, this implies that $\approx$ is transitive. Hence $\approx$ is an equivalence relation.

(iii) Let $K \in \Sigma(P, P^x)$. By (7.10) there exists $k \in K$ such that $P^x = P^k$. Now $P \approx P^k$ so (7.6) implies $N(P) \leq N(\mathrm{I}(P, P^k))$. Also, $\mathrm{I}(P, P^k) \leq K$ by the definition of $\mathrm{I}(P, P^k)$ whence $P^k \leq \mathrm{I}(P, P^k) \leq K_{N(P)}$. Then (7.1(iii)) implies $\mathrm{I}(P, P^k) = K_{N(P)}$. Similarly, $\mathrm{I}(P, P^k) = K_{N(P^k)}$. But as $k \in K$ we have $K_{N(P^k)} = (K_{N(P)})^k$, whence $k \in N(\mathrm{I}(P, P^k))$. Since $P^x = P^k$ we have $xk^{-1} \in N(P) \leq N(\mathrm{I}(P, P^k))$, so $x \in N(\mathrm{I}(P, P^x))$ as claimed.

(iv) Since $\approx$ is an equivalence relation we have $N(P) \leq N(\Gamma)$ and $N(\Gamma_M) \leq N(\Gamma)$. Let $Q \in \Gamma_M - \{P\}$. Now $\Delta_M(P) \neq \emptyset$ by (7.10(i)) so (7.1(vi)) implies $\mathrm{I}(P, Q) \leq M$. By (7.6) we have $N(P) \leq N(\mathrm{I}(P, Q))$ so $Q^n \leq M$ for all $n \in N(P)$. Since $N(P) \leq N(\Gamma)$, we deduce that $N(P) \leq N(\Gamma_M)$. Corollary (7.10(ii)) implies that $N_M(\Gamma)$ is transitive on Γ_M so as $N_M(\Gamma) \leq N(\Gamma_M)$ we deduce that $N(\Gamma_M) = N(P)N_M(\Gamma)$. Suppose that $x \in N(\Gamma)$ is such that $P^x \leq M$. By (7.10(ii)) there exists $m \in M$ such that $P^x = P^m$. Then $P^m \approx P$ so $m \in N_M(\Gamma)$ and then $x \in N(P)N_M(\Gamma) = N(\Gamma_M)$. This proves (iv).

(v) This follows from (iv).

(vi) Using (iv), $N(\Gamma_{M \cap L}) = \{x \in N(\Gamma) \mid P^x \in M \cap L\} = N(\Gamma_M) \cap N(\Gamma_L)$.

(vii) Using (ii) and (iv) we have $N(K_{N(P)}) \leq N(\Gamma_K) = N(P)N_K(\Gamma_K)$. Let $k \in N_K(\Gamma_K) - N(P)$. Now $\Delta_K(P) \neq \emptyset$ by (7.10) so $K \in \Sigma(P, P^k)$. Then (iii) implies that $k \in N(K_{N(P)})$. This proves (vii).

The following result is crucial. It allows us to construct either normal subgroups or more TI-subgroups.

THEOREM 9.3. *Let $K <_2* G$ and suppose that L is a conjugate of K such that $G = \langle K, L\rangle$. Then $K \cap L \trianglelefteq G$.*

Proof. Choose $A \in H^G$ such that $A \leq K$. By (8.1) there exists $P \in \Delta$ such that $P \leq A$ and $\langle \Delta(P)\rangle = G$.

Since $\langle K, L\rangle = G$, we have $P \not\leq L$ by (9.1(ii)), so (8.4) implies $\langle L, P\rangle <_1 G$. Now $P \leq K \cap \langle L, P\rangle$; $K \not\leq \langle L, P\rangle$, since $\langle K, L\rangle = G$; and L is a conjugate of K. Then (9.1(iii)) implies $K \cap \langle L, P\rangle = K_{N(P)}$. Thus $K \cap L \leq K_{N(P)} \leq \langle L, P\rangle$. If $k \in K$ then $P^k \leq A^k \leq K$ so the same argument yields $K \cap L \leq K_{N(P^k)} \leq \langle L, P^k\rangle <_1 G$. We deduce that

$$K \cap L \leq \left(K_{N(P)}\right)_K \leq \bigcap \left\{\langle L, P^k\rangle \mid k \in K\right\}. \tag{5}$$

Now $A \cap \langle L, P\rangle \leq K \cap \langle L, P\rangle = K_{N(P)}$ so (9.1(i)) implies that $A \cap \langle L, P\rangle = P$. By (7.4) there exists $a \in A$ such that $P^a \neq P$ and thus $P^a \not\leq \langle L, P\rangle$. But $\langle L, P^a\rangle <_1 G$ whence $\langle L, P\rangle \cap \langle L, P^a\rangle = L$. Thus the right hand side of (5) is equal to L, so we deduce that $K \cap L \leq (K_{N(P)})_K \leq K \cap L$ and thus that $K \cap L = (K_{N(P)})_K \trianglelefteq K$. Similarly, $K \cap L \trianglelefteq L$, so $K \cap L \trianglelefteq \langle K, L\rangle = G$.

COROLLARY 9.4. *Let $K <_2* G$ be such that $K_G = 1$. Let $P, Q \in \Delta$ with $P, Q \leq K$ and suppose that $\langle \Delta(P)\rangle = \langle \Delta(Q)\rangle = G$. Then*

(i) *If* $g \in G$ *and* $K_{N(P)} \cap (K_{N(Q)})^g \neq 1$ *then* $K_{N(P)} = (K_{N(Q)})^g$.

(ii) $K_{N(P)}$ *is a TI-subgroup with normalizer* $N(P)N_K(K_{N(P)})$. *Moreover,* $N_K(K_{N(P)}) > K_{N(P)}$ *and* $N(K_{N(P)})/K_{N(P)}$ *is a Frobenius group with cyclic complement* $N_K(K_{N(P)})/K_{N(P)}$ *whose kernel is a* p*-group.*

Proof. We will use (4.2) with $\Omega = K^G$. Now $K_G = 1$ and $K <_2 * G$ so using (9.3) we have $A <_1 \langle A, B \rangle <_1 G$ whenever A and B are distinct members of Ω with $A \cap B \neq 1$. Thus the hypothesis of (4.2) is satisfied. Let $\widetilde{\Delta}$ be the set Δ defined in (4.2), so that

$$\widetilde{\Delta} = \{\ A \cap B \ \mid \ A, B \in K^G,\ A \neq B,\ A \cap B \neq 1 \text{ and there exists } C \in K^G \text{ such that } A \cap B \cap C \neq 1 \text{ and } C \not\leq \langle A, B \rangle \},$$

We claim that $K_{N(P)} \in \widetilde{\Delta}$. By (7.10(i)) we may choose $C \in \Delta_K(P)$. Now $\langle \Delta(P) \rangle = G$ so by (4.2(vi)) there exists $n \in N(P)$ such that $C^n \not\leq K$. Thus $K \neq K^n$. Now $P \leq K \cap K^n$ so $\langle K, K^n \rangle \neq G$ by (9.1(ii)). Again we may choose $m \in N(P)$ such that $C^m \not\leq \langle K, K^n \rangle$. Then $K^m \not\leq \langle K, K^n \rangle$. Since $P \leq K \cap K^n \cap K^m$, it follows from the definition of $\widetilde{\Delta}$ that $K \cap K^n \in \widetilde{\Delta}$. But $K \cap K^n = K_{N(P)}$ by (9.1(ii)) so $K_{N(P)} \in \widetilde{\Delta}$. Similarly $K_{N(Q)} \in \widetilde{\Delta}$.

Theorem 4.2(i) now implies (i), and in particular, $K_{N(P)}$ is a TI-subgroup. By (9.2(vii)) we have $N(K_{N(P)}) = N(P)N_K(K_{N(P)})$. Now $C \cap K_{N(P)} = P$ by (9.1(i)) so $P \leq K_{N(P)} < K$ and by (8.3), $K_{N(P)}$ is not a Frobenius complement in K. Thus $N_K(K_{N(P)}) > K_{N(P)}$. Note that $N(K) = K$ by (8.2(ii)) so the remainder of (ii) follows from (4.2(vii)).

10. THE CASE $N_H(P) = P$

We begin this section by outlining the remainder of the proof of Theorem B. Partition Δ as follows.

$$\begin{aligned}
\mathscr{A} &= \{P \in \Delta \ \mid \ \langle \Delta(P) \rangle = G \text{ and } N_A(P) > P\} \\
&\qquad \text{for all } A \in \Delta(P) \\
\mathscr{B} &= \{P \in \Delta \ \mid \ \langle \Delta(P) \rangle = G \text{ and } N_A(P) = P\} \\
&\qquad \text{for some } A \in \Delta(P)\} \\
\mathscr{C} &= \{P \in \Delta \ \mid \ \langle \Delta(P) \rangle \neq G\}.
\end{aligned}$$

The proof divides into two cases depending on whether $\mathscr{B}$ is empty or not.

Consider the case $\mathscr{B} = \varnothing$. Then by (8.1) we have $\Delta = \mathscr{A}$. Using the TI-subgroups constructed in the previous section and a counting argument, we will force G to be of type 2 or 3. Note that the members of $\mathscr{A}$ are particularly pleasant to deal with since the structure of their normalizers is given by (6.10). However, we do have to consider every conjugacy class of subgroups contained in Δ.

The more difficult case $\mathscr{B} \neq \varnothing$ is handled by considering the equivalence relation defined in the previous section. The results of Section 3 are used to analyze the stabilizer of an equivalence class. Surprisingly, we have only to consider one conjugacy class of subgroups contained in Δ. Note that groups of type 3 give examples with $\mathscr{B} \neq \varnothing$.

The remainder of this section considers the case $\mathscr{B} \neq \varnothing$.

THEOREM 10.1. *Let $P \in \Delta$ be such that $\langle \Delta(P) \rangle = G, P \leq H$ and $N_H(P) = P$. Then G is of type 2 or 3.*

Proof. Assume false. Note that P is a Frobenius complement in H since P is a TI-subgroup by (4.2(ii)) and since $N_H(P) = P$.

Let $K_1, \ldots, K_\alpha$ be the distinct subgroups of the form $\langle H, A \rangle$ as A ranges over $\Delta(P) - \{H\}$. For each i let $M_{i1}, \ldots, M_{i\alpha_i}$ be the distinct subgroups of the form $\langle K_i, B \rangle$ as B ranges over the members of $\Delta(P)$ that are not contained in K_i. Set $V = N(P)$ and for each i, j set $V_i = V \cap K_i$ and $V_{ij} = V \cap M_{ij}$. Then (6.9) together with (7.5) and the fact that $N_H(P) = P$ implies:

- $\alpha \geq 2$ and $\alpha_i \geq 2$ for all i.
- $H <_1 K_i <_2 G$ and $K_i \cap K_j = H$ for all $i \neq j$.
- $K_i <_1 M_{ij} <_1 G$ and $M_{ij} \cap M_{ik} = K_i$ for all i and all $j \neq k$.
- $(V, \{V_i\}_{i=1}^{\alpha}, \{V_{ij}\}_{i=1}^{\alpha}{}_{j=1}^{\alpha_i}, P)$ satisfies Hypothesis 3.2.

STEP 1. *There is at most one i such that $K_{iG} \neq 1$.*

Proof. Using (8.2) and (7.4) we see that hypotheses (i)…(iv) of (5.5) are satisfied. Suppose that $i \neq j$ and that $K_{iG} \neq 1 \neq K_{jG}$. Now $K_{iG} \not\leq H$ since $H_G = 1$ so as $H <_1 K_i$ we have $HK_{iG} = K_i <_2 G$. Similarly $HK_{jG} = K_j <_2 G$. By construction, there exists k such that $\langle K_i, K_j \rangle = M_{ik}$. Then $HK_{iG}K_{jG} = M_{ik} <_1 G$ so (v) of (5.5) is satisfied. Consequently G is of type 2 or 3, a contradiction. This proves Step 1.

Assume the notation of Theorem 9.2. Let $X = N(\Gamma)$ and for each i, j let $X_i = N(\Gamma_{K_i})$ and $X_{ij} = N(\Gamma_{M_{ij}})$. Note that $K_{iV} \trianglelefteq X_i$ by (9.2(vii)).

STEP 2. *Let $1 \leq i \leq \alpha$ be such that $K_{iG} = 1$ and $V_i \trianglelefteq V$. Set $X_i^* = X_i/K_{iV}$. Then*:

(i) $X_i = N(K_{iV}) = VN_{K_i}(K_{iV})$ *and* $V \cap N_{K_i}(K_{iV}) = V_i \leq K_{iV}$.

(ii) $N_{K_i}(K_{iV})^*$ *is a Frobenius complement in* X_i^* *and* $F(X_i^*) = V^*$.

(iii) V/V_i *is nilpotent*, $VK_{iV} \trianglelefteq X_i$, *and* $VK_{iV} < X_i$.

Proof. By (9.2(vii)) we have $X_i = N(K_{iV}) = VN_{K_i}(K_{iV})$. Now $V \cap N_{K_i}(K_{iV}) = V_i$ and $V_i \leq K_{iV}$ since $V_i \trianglelefteq V$. Then $X_i^* = V^* N_{K_i}(K_{iV})^*$ and $V^* \cap N_{K_i}(K_{iV})^* = 1$. By (9.4(ii)), $N_{K_i}(K_{iV})^*$ is a Frobenius complement in X_i^* so $X_i^* = F(X_i^*)N_{K_i}(K_{iV})^*$ and $F(X_i^*) \cap N_{K_i}(K_{iV})^* = 1$. It follows that $V^* = F(X_i^*)$. Now $V^* \cong V/V \cap K_{iV} = V/V_i$ so V/V_i is nilpotent. Also, $V^* \trianglelefteq X_i^*$ and $V^* \neq X_i^*$, which proves (iii).

STEP 3. *There is a prime p such that V/P is an elementary abelian p-group.*

Proof. We have already noted that $(V, \{V_i\}_{i=1}^{\alpha}, \{V_{ij}\}_{i=1}^{\alpha}{}_{j=1}^{\alpha_i}, P)$ satisfies Hypothesis 3.2, so using ((3.6)(i),(ii)) and Step 1 we can choose $i \neq j$ such that $V_i \trianglelefteq V$, $V_j \trianglelefteq V$, and $K_{iG} = K_{jG} = 1$. Step 2 implies that V/V_i and V/V_j are nilpotent. Now $V_i \cap V_j = P$ so V/P is nilpotent. In particular, V/P is not a Frobenius group so (3.6(iii)) implies that V/P is elementary abelian.

STEP 4. $(X, \{X_i\}_{i=1}^{\alpha}, V)$ *satisfies Hypothesis* 3.1.

Proof. Note that $V \leq X_i \leq X$ for all i by (9.2(iv)). Let $x \in X - V$, so $P \approx P^x \neq P$. Let $K = \langle H, P^x \rangle$. Then $H <_1 K <_2 G$ by (7.1(i)). Now P is a TI-subgroup in K but by (8.3), it is not a Frobenius complement in K. Thus there exists $n \in N_K(P) - P$. Since $N_H(P) = P$ we have $n \notin H$, so as $N(H) = H <_1 K$ we have $K = \langle H, H^n \rangle$ and thus $K = K_i$ for some i. Then (9.2(iv)) implies $x \in X_i$. We deduce that $X = X_1 \cup \ldots \cup X_{\alpha}$. If $i \neq j$ then $K_i \cap K_j = H$, so by (9.2(vi)) we have $N(\Gamma_H) = X_i \cap X_j$. Now $\Gamma_H = \{P\}$ by the definition of $\approx$ so $V = X_i \cap X_j$.

Let $1 \leq i \leq \alpha$. We claim that $V < X_i$. Note that $V_i \trianglelefteq V$ by Step 3, so if $K_{iG} = 1$ then $V < X_i$ by Step 2. Hence we may suppose that $K_{iG} \neq 1$. By (9.2(vii)) we have $X_i = N(K_{iV})$ so $1 \neq K_{iG} \leq K_{iV} \leq X_i$. Recall that P is a Frobenius complement in H so $H = \langle P^H \rangle$, so as $N(H) = H$ and $H_G = 1$ we deduce that $V_G = 1$. In particular, $K_{iG} \not\leq V$ and then $V < X_i$ in this case also.

Now $\alpha \geq 2$ so we may choose $j \neq i$. Then $V < X_j \leq X$ and as $X_i \cap X_j = V$ it follows that $V < X_i < X$. Thus Hypothesis (3.1) is satisfied.

STEP 5. *$V \trianglelefteq X$ and V is an elementary abelian p-group.*

Proof. Set $\overline{X} = X/V_X$. By Step 1 we may choose i such that $K_{iG} = 1$. Now Step 3 implies $V_i \trianglelefteq V$ so by Step 2, VK_{iV} is a proper normal subgroup of X_i. Then as $V_X \leq V$ we see that $\overline{VK_{iV}}$ is a proper normal subgroup of $\overline{X_i}$, whence $\langle \overline{V} \rangle^{\overline{X_i}} \neq \overline{X_i}$. It follows that $\overline{V}$ is not a Frobenius complement in $\overline{X}$. Then Step 4 and (3.4) imply that $V \trianglelefteq X$.

Let $P_0 = [V, V]\langle v^p \mid v \in V \rangle$. Then $P_0 \leq P$ by Step 3. Now P_0 char $V \trianglelefteq X$ so $P_0 \trianglelefteq X$. If $P_0 \neq 1$ then as P is a TI-subgroup we have $X_i \leq N(P) = V$, contrary to $V < X_i$. Thus $P_0 = 1$ and it follows that V is an elementary abelian p-group.

STEP 6. $(X, \{X_i\}_{i=1}^{\alpha}, \{X_{ij}\}_{i=1}^{\alpha}{}_{j=1}^{\alpha_i}, V)$ *satisfies Hypothesis* 3.2.

Proof. Let $1 \leq i \leq \alpha$. Note that $X_i \leq X_{ij} \leq X$ for all j by (9.2(v)). If $1 \leq j \leq \alpha$ then by construction there exists k such that $\langle K_i, K_j \rangle \leq M_{ik}$. Then (9.2(v)) implies $X_j \leq X_{ik}$. This verifies (iii) of Hypothesis 3.2. By Step 4, $X = X_1 \cup \ldots \cup X_\alpha$ so it follows also that $X = X_{i1} \cup \ldots \cup X_{i\alpha_i}$. By (9.2(vi)) we have $X_{ij} \cap X_{ik} = X_i$ for all $j \neq k$.

Now let $1 \leq j \leq \alpha_i$. By construction, there exists $k \neq i$ such that $M_{ij} = \langle K_i, K_k \rangle$ so by (9.2(v)) we have $X_k \leq X_{ij}$. By Step 4, $X_i \cap X_k = V < X_k$ so $X_i < X_{ij}$. Now $\alpha_i \geq 2$, so if we choose $l \neq j$ we also have $X_i < X_{il}$ and $X_{ij} \cap X_{il} = X_i$ so $X_{ij} < X$. We deduce that $(X, \{X_{ij}\}_{j=1}^{\alpha_i}, X_i)$ satisfies Hypothesis 3.1, which completes the proof of this step.

STEP 7. *If* $X_i \leq X_{rs}$ *then* $V_i \leq V_{rs}$.

Proof. Since $V_i = V \cap K_i$ and $V_{rs} = V \cap M_{rs}$, it suffices to prove that $K_i \leq M_{rs}$. If $i = r$ then this follows from the definition of M_{rs}. Hence we may suppose that $i \neq r$. By construction, there exists t such that $\langle K_r, K_i \rangle = M_{rt}$. By (9.2(v)) we have $\langle X_r, X_i \rangle \leq X_{rt}$ and also $\langle X_r, X_i \rangle \leq X_{rs}$. But $r \neq i$ so Step 4 implies $X_r < \langle X_r, X_i \rangle$ and then Step 6 forces $t = s$. Then $K_i \leq M_{rs}$ as desired.

Now we finish the proof of Theorem 10.1. Applying Step 5, (3.6) and renumbering, we have

$$X_{11} \trianglelefteq X, X_{11} = X_1 \cup \ldots \cup X_\beta, \beta \geq 3, \quad \text{and} \quad X_i \trianglelefteq X \quad \text{for all } i \leq \beta.$$

Set $\overline{X} = X/V$. By (3.6(iii)), there is a prime q such that $\overline{X_{11}}$ is an elementary abelian q-group. Recall that p is the prime such that V is an elementary abelian p-group.

We claim $q \neq p$. By Step 1 there exists $i \leq \beta$ such that $K_{iG} = 1$. Now $V_i \trianglelefteq V$ so Step 2 implies $\overline{X_i} \cong N_{K_i}(K_{iV})/V_i$. Set $X_i^* = X_i/K_{iV}$. Then $N_{K_i}(K_{iV})^*$ is a q-group since $V_i \leq K_{iV}$. Then as V^* is a p-group, it follows from Step 2(ii) that $q \neq p$.

Since V is an abelian normal p-subgroup of X_{11} it follows that $\overline{X_{11}}$ acts as a group of automorphisms on V. Now $\overline{X_i} \cap \overline{X_j} = \overline{V} = 1 < \overline{X_i}$ for all $i \neq j$ so as $\beta \geq 2$ we see that $\overline{X_{11}}$ is noncyclic. Then as $q \neq p$ we have

$$V = \langle C_V(x) \mid x \in X_{11} - V \rangle.$$

If $K_{iG} = 1$ for all $i \leq \alpha$ then set $l = 1$. Otherwise, by Step 1 there is a unique l such that $K_{lG} \neq 1$.

Let $x \in X_{11} - V$ and let $v \in C_V(x)^\sharp$. Recall that $V = V_1 \cup \ldots \cup V_\alpha$ so $v \in V_i$ for some i. If $i = l$ then $v \in V_l$. Suppose that $i \neq l$. Then K_{iV} is a TI-subgroup by (9.4) and $v \in V_i \leq K_{iV}$ since V is abelian. Thus $x \in N(K_{iV})$ and then Step 2(i) implies $x \in X_i$. Now $x \in X_{11} = X_1 \cup \ldots \cup X_\beta$ so there exists $j \le \beta$ such that $x \in X_j$. Then $x \in X_i \cap X_j - V$ so Step 4 implies $i = j$. Thus $X_i \le X_{11}$ and then Step 7 implies $v \in V_{11}$. We deduce that $C_V(x) \le V_{11} \cup V_l$ and then that $V = V_{11} \cup V_l$. Thus $V = V_{11}$ or $V = V_l$. Both of these possibilities contradict the fact that $(V, \{V_i\}_{i=1}^{\alpha}, \{V_{ij}\}_{i=1}^{\alpha}{}_{j=1}^{\alpha_i}, P)$ satisfies Hypothesis 3.2, completing the proof of this theorem.

11. THE PROOF OF THEOREM B

Throughout this section we assume that G is a counterexample to Theorem B.

LEMMA 11.1. (i) *There exists K such that $H <_1 K <_2 G$ and $K_G = 1$.*
(ii) *If $P \in \Delta$ then $\langle \Delta(P) \rangle = G$ and $N_A(P) > P$ for all $A \in \Delta(P)$.*

Proof. (ii) follows from (10.1) and (8.1(ii)). Choose $P \in \Delta$ such that $P \le H$. Let $K_1, \dots, K_\alpha$ be the distinct subgroups of the form $\langle H, A \rangle$ as A ranges over $\Delta(P) - \{H\}$. By (6.9) we have $H <_1 K_i <_2 G$ and $K_i \cap K_j = H$ for all $i \ne j$. Suppose that $K_{iG} \ne 1$ for all i. Then as $H_G = 1$ and $H <_1 K_i$ we have $K_i = HK_{iG}$ and then (8.2(iv)) implies that $K_{iG} \le F(G)$. Now $G = \langle \Delta(P) \rangle$ so $G = \langle K_1, \dots, K_\alpha \rangle$ and consequently $G = HF(G)$. By (8.2(i)) we have $\langle H, g \rangle \ne G$ for all $g \in G$ and then (5.2) implies that G is of type 1 or 2, a contradiction. We deduce that there exists i such that $K_{iG} = 1$.

In what follows, we let K be the subgroup whose existence was proved in (11.1(i)).

LEMMA 11.2. *Let $P \in \Delta$ with $P \le H$, let $t \in \pi(N_H(P)/P)$ and $T \in \mathrm{Syl}_t(N_H(P))$. Then $N_K(T) \le H$.*

Proof. Assume false. Note that $T \not\le P$ and that $N_H(P)/P$ is a Frobenius complement in $N(P)/P$ by (4.2(vii)). Thus $N_{N(P)}(T) \le H$.

Let $x \in N_K(T) - H$. Then $P \ne P^x$. We claim that $P \approx P^x$ where $\approx$ is the relation defined in (9.2). Suppose that $\langle P, P^x \rangle \le A$ for some $A \in H^G$. Now T normalizes $\langle P, P^x \rangle$ and by (4.2(i)), $\langle P, P^x \rangle$ is not contained in a member of Δ. As $N(A) = A$, this implies that $T \le A$. Then $P < PT \le A \cap H$ and $P^x < P^xT \le A \cap H^x$. But by (4.2(i)) neither PT nor P^xT is contained in a member of Δ so $H = A = H^x$ and then $x \in H$, contrary to $x \notin H$. Thus $\langle P, P^x \rangle$ is not contained in a conjugate of H, so $P \sim P^x$. Since T normalizes both P and P^x it permutes the set $\Sigma(P, P^x)$ so $T \le N(\mathrm{I}(P, P^x))$. Now $H \cap \mathrm{I}(P, P^x) = P$ by (7.1(v)) so $N(\mathrm{I}(P, P^x)) > \mathrm{I}(P, P^x)$ whence $P \approx P^x$. Lemma 9.2(iii) implies $x \in N(K_{N(P)})$. We deduce that $N_K(T) = (N_K(T) \cap N(K_{N(P)})) \cup N_H(T)$, so as we are assuming $N_K(T) \not\le H$, it follows that

$$N_K(T) \le N_K(K_{N(P)}). \tag{6}$$

Choose $n \in N_K(T) - H$ and set $\widetilde{T} = H \cap H^n$. Then $T \leq \widetilde{T} \in \Delta$, so (4.2(iv)) implies that $\widetilde{T}$ is a TI-subgroup and hence $N_K(T) \leq N_K(\widetilde{T}) \leq N_K(K_{N(\widetilde{T})})$. Note that $\langle \Delta(P) \rangle = \langle \Delta(\widetilde{T}) \rangle = G$ by (11.1(ii)). Corollary 9.4(ii) implies that both $N_K(K_{N(P)})/K_{N(P)}$ and $N_K(K_{N(\widetilde{T})})/K_{N(\widetilde{T})}$ are cyclic. Using (6) we see that $[N_K(T), N_K(T)] \leq K_{N(P)} \cap K_{N(\widetilde{T})}$. Now $H \cap K_{N(P)} = P$ by (9.1(i)) so as $T \not\leq P$ we have $T \not\leq K_{N(P)}$. Then (9.4(i)) implies that $K_{N(P)} \cap K_{N(\widetilde{T})} = 1$. We deduce that $N_K(T)$ is abelian.

Now we argue that $N_K(T)$ is nonabelian. Set $N = N_K(\widetilde{T})$ and $\overline{N} = N/\widetilde{T}$. Now $N_K(T) \leq N$ and $N_K(T) \not\leq H$ so $\overline{N_H(\widetilde{T})} < \overline{N}$. Theorems 10.1 and (4.2(vii)) imply that $N_H(\widetilde{T})/\widetilde{T}$ is a Frobenius complement in $N(\widetilde{T})/\widetilde{T}$ so $\overline{N}$ is nonabelian. Since $H \cap K_{N(P)} = P$ we have $N_H(K_{N(P)}) \leq N_H(P)$ so (6) implies $N_H(T) \leq N_H(P)$. As $T \in \mathrm{Syl}_t(N_H(P))$ we deduce that $T \in \mathrm{Syl}_t(H)$ and then that $T \in \mathrm{Syl}_t(\widetilde{T})$. The Frattini Argument implies $N = \widetilde{T} N_K(T)$ so $N_K(T)$ is nonabelian also. This contradiction completes the proof.

COROLLARY 11.3. *Let $P \in \Delta$ with $P \leq H$. Then*

$$N_K(K_{N(P)}) = N_H(P)K_{N(P)} \quad \text{and} \quad N_H(P) \cap K_{N(P)} = P.$$

Proof. Let $N = N_K(K_{N(P)})$. Now $N_H(P) > P$ by (11.1(ii)) so choose $t \in \pi(N_H(P)/P)$ and $T \in \mathrm{Syl}_t(N_H(P))$. By (9.1(i)), $H \cap K_{N(P)} = P$ so $N_H(P) \cap K_{N(P)} = P$ and $H \cap N = N_H(P)$. Then (11.2) implies $N_N(T) \leq N_H(P)$ whence $T \in \mathrm{Syl}_t(N)$. By (9.4(ii)), $N/K_{N(P)}$ is cyclic so $TK_{N(P)} \trianglelefteq N$. The Frattini Argument yields $N = N_N(T)K_{N(P)} = N_H(P)K_{N(P)}$.

LEMMA 11.4. *Let $P, Q \in \Delta$ with $P, Q \leq K$. Let $k \in K$ and suppose that $K_{N(P)} \cap (K_{N(Q)})^k \neq 1$. Then P is conjugate to Q.*

Proof. Assume false. By (7.9) we may suppose that $P \leq H$. Now $K_{N(P)} = (K_{N(Q)})^k$ by (9.4(i)) so replacing Q by Q^k we may suppose $K_{N(P)} = K_{N(Q)}$. Let $I = K_{N(P)}$ and $N = N_K(I)$. By (7.10(i)) we may choose $A \in \Delta_K(Q)$.

Corollary 11.3 implies $N = N_H(P)I = N_A(Q)I, N_H(P) \cap I = P$ and $N_A(Q) \cap I = Q$. Then $N/I \cong N_H(P)/P \cong N_A(Q)/Q$. By (11.1), we may choose $t \in \pi(N/I)$. Let $T \in \mathrm{Syl}_t(N_H(P))$, so $T \not\leq P$. By (9.1(i)), $H \cap K_{N(P)} = P$ so $H \cap N = N_H(P)$. Then (11.2) implies $N_N(T) \leq N_H(P)$ so $T \in \mathrm{Syl}_t(N)$. Similarly, $N_A(Q)$ contains a Sylow t-subgroup of N. Conjugating by a suitable element of N, we may suppose that $T \leq A$ also. Now $H \cap K_{N(P)} = P \neq Q = A \cap K_{N(Q)}$, so $H \neq A$.

We claim that I is nilpotent. Let $\widetilde{T} = H \cap A$, so $T \leq \widetilde{T} \in \Delta$ and $\widetilde{T}$ is a TI-subgroup. Also, $H \cap I = P$ so as $T \not\leq P$, (4.2(i)) implies $\widetilde{T} \cap I = 1$. Let $g \in T$ have order t. Then $[C_I(g), T] \leq \widetilde{T} \cap I = 1$ so $C_I(g) \leq N(T)$. But $N(T) \leq H \cap A$ by (11.2), so $C_I(g) \leq \widetilde{T} \cap I = 1$ and then Thompson's Theorem implies that I is nilpotent.

Let $X = N_G(I)$ and $\overline{X} = X/I$. By (9.4(ii)), $\overline{N_K(I)}$ is a Frobenius complement in $\overline{X}$, the kernel of $\overline{X}$ is a p-group for some prime p, and $X = N(P)N_K(I)$. We will show that I is a p-group. Let F be the inverse image of $F(N(P)/P)$ in $N(P)$. Then (4.2(vii)) implies $N(P) = FN_H(P)$, so as $N_H(P) \le N_K(I)$ we deduce that $\overline{X} = \overline{FN_K(I)}$. Consequently, $F(\overline{X}) \le \overline{F}$. But F/P has prime power order by (4.2(vii)), so it follows that F/P is a p-group. Now $N_H(P) \cap I = P$ by (9.1(i)) and $N_I(P) \trianglelefteq N(P)$. Since $N_H(P)/P$ is a Frobenius complement in $N(P)/P$, we see that $N_I(P) \le F$. Then $\mathcal{O}_{p'}(Z(I)) \le P$. Similarly $\mathcal{O}_{p'}(Z(I)) \le Q$. But $P \cap Q = 1$ by (4.2(i)), so as I is nilpotent, we deduce that I is a p-group.

Now $N_H(P)/P$ is a Frobenius complement in $N(P)/P$ and F/P is a p-group, so $N_H(P)/P$ is a p'-group. But $P \le I$ so P is a p-group. It follows that $P \in \mathrm{Syl}_p(H)$. Similarly $Q \in \mathrm{Syl}_p(A)$. But H is conjugate to A, so P is conjugate to Q, a contradiction.

LEMMA 11.5. *Let $P \in \Delta$ with $P \le H$ and let $k \in K$ be such that $H^k \cap K_{N(P)} \neq 1$. Then $H^k \cap K_{N(P)}$ is conjugate to P.*

Proof. Let $A = H^k$ and $R = A \cap K_{N(P)}$. We claim there exists $Q \in \Delta$ such that $Q \cap R \neq 1$ and $Q \le A$. If $A \le K_{N(P)}$ then let Q be a conjugate of P contained in A. Thus we may suppose that $A \not\le K^n$ for some $n \in N(P)$. Let $L = K^n$ and $B = H^n$, so $R \le A \cap L$ and $B \le L$. Choose $s \in \pi(R)$, $S \in \mathrm{Syl}_s(R)$ and $\widetilde{S} \in \mathrm{Syl}_s(A \cap L)$ with $S \le \widetilde{S}$. If there exists $m \in N_L(\widetilde{S}) - A$ then $A \neq A^m$ so setting $Q = A \cap A^m$ we have $Q \in \Delta$ and $S \le Q \cap R$. If $N_L(\widetilde{S}) \le A$ then $\widetilde{S} \in \mathrm{Syl}_s(L)$, so as $|B| = |A|$, there exists $l \in L$ such that $\widetilde{S} \le B^l$. Set $Q = A \cap B^l$. Now $A \not\le L$ so $B^l \neq A$ and we have $S \le Q \in \Delta$.

Now $1 \neq R \cap Q \le K_{N(P)} \cap K_{N(Q)}$ so (9.4(i)) implies $K_{N(P)} = K_{N(Q)}$ and thus (11.4) implies that P is conjugate to Q. Also, $R = A \cap K_{N(P)} = A \cap K_{N(Q)}$ and as (9.1(i)) implies $A \cap K_{N(Q)} = Q$, we deduce that R is conjugate to P.

We are now in a position to derive a contradiction. By its definition, Δ is a union of conjugacy classes of subgroups and each member of Δ is conjugate to a subgroup of H. Let $\{P_1, \ldots, P_r\}$ be a set of representatives for the conjugacy classes of Δ with $P_i \le H$ for all i. Let

$$H_0 = H - \bigcup \left\{ P_i^h \mid 1 \le i \le r, h \in H \right\} \quad \text{and} \quad \widetilde{H} = \bigcup H_0^K.$$

If $k \in K - H$ and $H \cap H^k \neq 1$ then $H \cap H^k \in \Delta$ so by (4.2(v)), $H \cap H^k$ is conjugate in H to some P_i. We deduce that $H_0 \cap H_0^k = \varnothing$ for all $k \in K - H$ and then that $|\widetilde{H}| = |K : H||H_0|$. Now $|H_0| = |H| - 1 - \sum_{i=1}^r |H : N_H(P_i)|(|P_i| - 1)$ by (4.2(i)), whence

$$|\widetilde{H}| = |K| - |K : H| - \sum_{i=1}^r \frac{|K|}{|N_H(P_i) : P_i|} + \sum_{i=1}^r \frac{|K|}{|N_H(P_i)|}. \tag{7}$$

For each i, let

$$I_i = K_{N(P_i)} \text{ and } \widetilde{I}_i = \bigcup I_i^K - \{1\}.$$

By (9.4(ii)), I_i is a TI-subgroup of K so $|\widetilde{I}_i| = |K : N_K(I_i)|(|I_i| - 1)$. By (11.3), $N_K(I_i) = N_H(P_i)I_i$ and $N_H(P_i) \cap I_i = P_i$. Then $|N_K(I_i)| = |N_H(P_i) : P_i||I_i|$ and we deduce that

$$|\widetilde{I}_i| = \frac{|K|}{|N_H(P_i) : P_i|} - \frac{|K|}{|N_H(P_i)||I_i : P_i|}. \tag{8}$$

Let $1 \le i \le r, k \in K$ and suppose that $H^k \cap I_i \ne 1$. Then (11.5) implies $H^k \cap I_i$ is conjugate to P_i. By (7.10) this conjugation takes place in K, so $H_0^k \cap I_i = \emptyset$. We deduce that $\widetilde{H} \cap \widetilde{I}_i = \emptyset$ for all i. Also, by (9.4(i)) and (11.4) we have $\widetilde{I}_i \cap \widetilde{I}_j = \emptyset$ for all $i \ne j$. Noting that $1 \notin \widetilde{H}$ and $1 \notin \widetilde{I}_i$ we have

$$|K| > |\widetilde{H}| + \sum_{i=1}^{r} |\widetilde{I}_i|.$$

Using (7) and (8) we obtain

$$|K| > |K| - |K : H| + \sum_{i=1}^{r} \frac{|K|}{|N_H(P_i)|} - \sum_{i=1}^{r} \frac{|K|}{|N_H(P_i)||I_i : P_i|}.$$

Rearranging and dividing through by $|K : H|$, we have

$$1 > \sum_{i=1}^{r} |H : N_H(P_i)| \left(1 - \frac{1}{|I_i : P_i|}\right).$$

Let $1 \le i \le r$. Now $P_i \ntrianglelefteq H$ by (7.4) so $|H : N_H(P_i)| \ge 2$. Then $1 > 2(1 - 1/|I_i : P_i|)$ whence $2 > |I_i : P_i|$ and then $I_i = P_i$. Recall that $N_K(I_i) = N_H(P_i)I_i$ by (11.3) so $N_K(I_i) = N_H(P_i)$. Now $N_K(P_i) \le N_K(I_i)$ so we have $N_K(P_i) \le H$. We deduce that $N_K(T) \le H$ for all $1 \ne T \le H$ and then that H is a Frobenius complement in K. This contradicts (8.3) and completes the proof of Theorem B.

REFERENCES

1. D. Bartels, *Subnormality and invariant relations on conjugacy classes in finite groups, Mat. Z.* 157 (1977), 13–17.
2. P. J. Flavell, *Generating finite groups with conjugates of a subgroup, Comm. Algebra* **23** No. 11 (1995), 3947–3959.
3. M. Suzuki, "Group Theory I," Grundlehren der Mathematischen Wissenschaften, Vol. 247, Springer-Verlag, New York/Berlin, 1982.