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A characterization of $\mathcal{N}$-constrained groups

By

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Abstract. We prove that a finite group G is $\mathcal{N}$-constrained if and only if it contains a nilpotent subgroup I satisfying $C_G(I \cap I^g) \leqq I \cap I^g$ for all $g \in G$.

A group G is said to be $\mathcal{N}$*-constrained* if $C_G(F(G)) \leqq F(G)$. It is well known that finite soluble groups are $\mathcal{N}$-constrained, see for example [1, Theorem 6.1.3, p. 218]. An $\mathcal{N}$-constrained group possesses a nilpotent subgroup I with the property

$$C_G(I \cap I^g) \leqq I \cap I^g \quad \text{for all} \quad g \in G.$$

For example, put $I = F(G)$.

We shall prove the converse, namely that if a finite group G possesses a nilpotent subgroup I with the above property then G must be $\mathcal{N}$-constrained. Throughout, all groups are finite.

Lemma 1. *Suppose that I is a maximal nilpotent subgroup of G satisfying $C_G(I \cap I^g) \leqq I \cap I^g$ for all $g \in G$. If K is a subgroup of G that is normalized by I and if $I \cap K \leqq Z(K)$ then $K \leqq I$.*

Proof. Assume false and minimize $|G| + |K|$. Then $G = IK$ and as $I \cap K \leqq Z(K)$ we have that $I \cap K \trianglelefteqq G$.

Let $\overline{G} = G/(I \cap K)$. The minimal choice of K implies that if K_0 is any I-invariant subgroup of K then either $K_0 \leqq I \cap K$ or $K_0 = K$. Then 1 and $\overline{K}$ are the only $\overline{I}$-invariant subgroups of $\overline{K}$. Let $\pi = \pi(\overline{K})$ and choose $p \in \pi$. Extending $O_p(\overline{I})$ to a Sylow p-subgroup of $\overline{G}$ we see that $O_p(\overline{I})$ normalizes a Sylow p-subgroup of $\overline{K}$ and then that

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$O_p(\overline{I})$ centralizes some nontrivial p-elements of $\overline{K}$. Thus $C_{\overline{K}}(O_p(\overline{I})) \neq 1$ so $C_{\overline{K}}(O_p(\overline{I})) = \overline{K}$. We deduce that $[O_\pi(\overline{I}), \overline{K}] = 1$.

Now $O_{\pi'}(\overline{I})$ is nilpotent and has order coprime to $\overline{K}$ so by Sylow's Theorem for groups with operators [1, Theorem 6.2.2, p. 224], there exists a Sylow p-subgroup $\overline{P}$ of $\overline{K}$ that is normalized by $O_{\pi'}(\overline{I})$. Since $[O_\pi(\overline{I}), \overline{K}] = 1$ we see that $\overline{I}$ normalizes $\overline{P}$. Consequently $\overline{K} = \overline{P}$. Since $I \cap K \leqq Z(K)$ we deduce that K is nilpotent. Then K is the direct product of its Sylow subgroups so the minimal choice of K implies that K is a p-group.

Next we argue that $O_{p'}(I)$ centralizes K. Suppose that $k \in C_K(O_p(I))$. Now

$$[O_{p'}(I) \cap O_{p'}(I)^k, k^{-1}] \leqq O_{p'}(I) \cap K = 1$$

and since I is nilpotent we have that $I \cap I^k = O_p(I)(O_{p'}(I) \cap O_{p'}(I)^k)$. Then $k^{-1} \in C_G(I \cap I^k)$ and we obtain that $k \in I$. We deduce that $C_K(O_p(I)) \leqq I \cap K \leqq O_p(I)$.

Now $I = O_p(I) \times O_{p'}(I)$ acts on K and $O_{p'}(I)$ acts trivially on $C_K(O_p(I))$. The $P \times Q$-Lemma [1, Theorem 5.3.4, p. 179] implies that $O_{p'}(I)$ also acts trivially on K. Then IK is nilpotent so as I is a maximal nilpotent subgroup of G, it follows that $K \leqq I$, a contradiction. □

Theorem 2. *Suppose that I is a maximal nilpotent subgroup of G satisfying $C_G(I \cap I^g) \leqq I \cap I^g$ for all $g \in G$. If K is a nilpotent subgroup of G that is normalized by I then $K \leqq I$.*

Proof. We may suppose that K is a p-group for some prime p. Since p-groups satisfy the normalizer condition, we may also suppose that $I \cap K \trianglelefteq K$.

Now $O_{p'}(I)$ acts trivially on $I \cap K$ whence $[K, O_{p'}(I)]$ also acts trivially on $I \cap K$. In particular, $[K, O_{p'}(I)] \cap I \leqq Z([K, O_{p'}(I)])$ so Lemma 1 yields $[K, O_{p'}(I)] \leqq I$. Since K is a p-group and I is nilpotent we have $[K, O_{p'}(I), O_{p'}(I)] = 1$. Then [1, Theorem 5.3.6, p. 181] implies $[K, O_{p'}(I)] = 1$ and hence KI is nilpotent. Since I is a maximal nilpotent subgroup of G we have that $K \leqq I$. □

Theorem 3. *Let G be a group. Then G is $\mathcal{N}$-constrained if and only if G possesses a nilpotent subgroup I satisfying $C_G(I \cap I^g) \leqq I \cap I^g$ for all $g \in G$.*

Proof. If G is $\mathcal{N}$-constrained then $C_G(F(G)) \leqq F(G)$ so $F(G)$ will do.

To prove the converse, suppose that G possesses a nilpotent subgroup I of G that satisfies $C_G(I \cap I^g) \leqq I \cap I^g$ for all $g \in G$. We may suppose that I is a maximal nilpotent subgroup of G.

Let $K = C_G(F(G))$ and $\overline{G} = G/Z(K)$. Then $F(\overline{K}) = 1$. Let $\overline{Z} = Z(\overline{I}) \cap \overline{K}$. Suppose that $k \in K$. Let $\overline{C} = C_{\overline{K}}(\overline{I} \cap \overline{I}^{\overline{k}})$ and let C be the inverse image of $\overline{C}$ in G. Then $[I \cap I^k, C] \leqq Z(K)$ so as $C \leqq K$ we obtain $[I \cap I^k, C, C] = 1$. Also $[C, I \cap I^k, C] = 1$ so the Three Subgroups Lemma implies that $[C, C, I \cap I^k] = 1$. We deduce that $C' \leqq C_G(I \cap I^k) \leqq I \cap I^k$ and then that $\overline{C}' \leqq Z(\overline{C})$. In particular, $\overline{C}$ is nilpotent. Now $\langle \overline{Z}, \overline{Z}^{\overline{k}} \rangle \leqq \overline{C}$ so since k was arbitrary, we may invoke the Baer-Suzuki Theorem [1, Theorem 3.8.2, p. 105] to conclude that $\overline{Z} \leqq F(\overline{K})$. But $F(\overline{K}) = 1$ whence $Z(\overline{I}) \cap \overline{K} = 1$.

Now $\overline{I} \cap \overline{K} \trianglelefteq \overline{I}$ and $\overline{I}$ is nilpotent so we have $\overline{I} \cap \overline{K} = 1$. Consequently $I \cap K \leqq Z(K)$ and then Lemma 1 implies that $K \leqq I$. Thus K is nilpotent and as $K \trianglelefteq G$ we have $C_G(F(G)) = K \leqq F(G)$. Hence G is $\mathcal{N}$-constrained. □

Corollary 4. *Let G be a group, p a prime and P a Sylow p-subgroup of G. Then $C_G(O_p(G)) \leqq O_p(G)$ if and only if $C_G(P \cap P^g) \leqq P \cap P^g$ for all $g \in G$.*

Proof. Suppose that $C_G(P \cap P^g) \leqq P \cap P^g$ for all $g \in G$. Putting $g = 1$ yields $C_G(P) \leqq P$. It follows that P is a maximal nilpotent subgroup of G. Theorem 2 implies that $F(G) \leqq P$, so that $F(G) = O_p(G)$. Theorem 3 implies that G is $\mathcal{N}$-constrained, whence $C_G(O_p(G)) \leqq O_p(G)$.

The reverse implication is trivial. □

References

[1] D. GORENSTEIN, Finite groups, 2nd edn. New York 1980.

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