AUTOMORPHISMS AND FUSION IN FINITE GROUPS

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Abstract

We study how the fixed point subgroup of an automorphism influences the structure of a group.

1. *Introduction*

We investigate how the fixed point subgroup of an automorphism influences the structure of a group. We shall prove:

Theorem A. *Let R be a group of prime order r that acts on the r'-group G. Let p be an odd prime and choose $S \in \mathrm{Syl}_p(G)$. Assume that $C_G(R)$ is a p'-group. Then $N_G(S)$ controls strong fusion in S with respect to G.*

Theorem A is a generalization of Thompson's Thesis, which asserts that if R is fixed point free then G is nilpotent. Indeed, Thompson's result follows from Theorem A and Frobenius' Normal p-Complement Theorem. Many authors have extended Thompson's work, notably Glauberman [**5**].

Collins [**2**, **3**] studied groups that admit an automorphism of prime order whose fixed point subgroup also has prime order. He realized that Glauberman's arguments could be modified to obtain the conclusion of Theorem A in his situation. We follow a similar path by proving:

Theorem B. *Let G be a group, p an odd prime, $S \in \mathrm{Syl}_p(G)$ and $T \leq Z(S)$. Suppose that $T \trianglelefteq N_G(J(S))$. Then at least one of the following holds:*

(a) *T is weakly closed in S with respect to G.*
(b) *There exists a cyclic p'-subgroup $X \leq N_G(T)$ such that X acts nontrivially on T and transitively on $[T, X]^{\#}$.*

Notice the similarity with the result of Collins [**3**, p.26], [**6**, Theorem 14.14, p.46].

If r is not a Fermat prime then Theorem A follows without much difficulty from Theorem B and a result of Shult on modules [**7**, Theorem 3.1,p.702], [**1**, (36.2), p.193]. However, if r is a Fermat prime there is an unavoidable and considerable obstacle. Perhaps this explains why Theorem A was not proved during the 1970s when there was much activity in this area. The author's recent result [**4**] is invoked to complete the proof of Theorem A.

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2. *Preliminaries*

Henceforth, group will mean finite group. Suppose that G, N and S are groups with $S \leq N \leq G$. We say N *controls strong fusion in* S *with respect to* G if for all $X \subseteq S$ and $g \in G$ satisfying $X^g \subseteq S$ we have $g = cn$ for some $c \in C_G(X)$ and $n \in N$. We say S *is weakly closed in* N *with respect to* G if for all $g \in G$, $S^g \leq N$ implies $S^g = S$.

Lemma 2.1. *Let G be a group, p a prime and $S \in \mathrm{Syl}_p(G)$. Suppose that T is a subgroup of $Z(S)$ that is weakly closed in S with respect to G. Then $N_G(T)$ controls strong fusion in S with respect to G.*

Proof. This is an elementary consequence of Sylow's Theorem. □

Lemma 2.2. *Let G be a group, p a prime, $S \in \mathrm{Syl}_p(G)$ and $N \leq G$ with $S \leq N$. Suppose that:*

$$\left.\begin{array}{l} \textit{whenever } W \leq S \textit{ and } g \in G \textit{ satisfy } W^g \leq S,\ O_p(G) \leq W \textit{ and} \\ N_S(W) \in \mathrm{Syl}_p(N_G(W)) \textit{ then } g = cn \textit{ for some } c \in C_G(W) \textit{ and } n \in N. \end{array}\right\} \quad (*)$$

Then N controls strong fusion in S with respect to G.

Proof. Suppose that $W \leq S$ and $g \in G$ satisfy $W^g \leq S$. We must show that $g = cn$ for some $c \in C_G(W)$ and $n \in N$. Without loss, $O_p(G) \leq W$. By Sylow's Theorem, there exists a conjugate W^h of W with $W^h \leq S$ and $N_S(W^h) \in \mathrm{Syl}_p(N_G(W^h))$. Note that $(W^h)^{h^{-1}} = W \leq S$ and $(W^h)^{h^{-1}g} = W^g \leq S$. Applying $(*)$ twice, with W^h in place of W, we have

$$h^{-1} = ax \quad \text{and} \quad h^{-1}g = by$$

for some $a, b \in C_G(W^h)$ and $x, y \in N$. Then $g = hby = x^{-1}a^{-1}by = (a^{-1}b)^x(x^{-1}y)$. Now $W^{hx} = W^{ha^{-1}h^{-1}} = W^{hh^{-1}} = W$ so as $a^{-1}b \in C_G(W^h)$ we have $(a^{-1}b)^x \in C_G(W)$. Put $c = (a^{-1}b)^x$ and $n = x^{-1}y$. □

Lemma 2.3. *Let G be a group, p a prime and $S \in \mathrm{Syl}_p(G)$. Then any of the following imply that $N_G(S)$ controls strong fusion in S with respect to G.*

(a) *$N_G(S) \leq M \leq G$, M controls strong fusion in S with respect to G and $N_G(S)$ controls strong fusion in S with respect to M.*
(b) *$S \leq K \trianglelefteq G$ and $N_K(S)$ controls strong fusion in S with respect to K.*
(c) *There is a subgroup $Z \leq Z(S) \cap Z(G)$ such that $N_{G/Z}(S/Z)$ controls strong fusion in S/Z with respect to G/Z.*

Proof. (a) is trivial and (b) follows from the Frattini Argument. To prove (c), set $\overline{G} = G/Z$. Assume the hypothesis of $(*)$ in Lemma 2.2. Then $Z \leq W$ so $N_{\overline{G}}(\overline{W}) = \overline{N_G(W)}$. Let C be the inverse image of $C_{\overline{G}}(\overline{W})$ in G. Then $C \trianglelefteq N_G(W)$ and $S \cap C \in \mathrm{Syl}_p(C)$. Now C acts trivially on each factor of the chain

$$1 \leq Z \leq W$$

so $C/C_G(W)$ is a p-group. It follows that $C = C_G(W)(S \cap C)$. Also $Z \leq S$ so $N_{\overline{G}}(\overline{S}) = \overline{N_G(S)}$. By hypothesis, there exist $c \in C$ and $n \in N_G(S)$ with $\overline{g} = \overline{cn}$.

Then

$$g = zcn$$

for some $z \in Z$. Now $C = C_G(W)(S \cap C)$ so $c = ds$ with $d \in C_G(W)$ and $s \in S \cap C$. Then $g = (zd)(sn)$, $zd \in C_G(W)$ and $sn \in N_G(S)$. This verifies $(*)$ and completes the proof. □

The following is well known, see for instance [**1**]

Theorem 2.4 (Coprime Action). *Let the r-group R act on the r'-group G.*
(a) *If p is a prime then G possesses an R-invariant Sylow p-subgroup.*
(b) *$G = C_G(R)[G, R]$, in particular, $[G, R] = [G, R, R]$.*
(c) *If G is abelian then $G = C_G(R) \times [G, R]$.*
(d) *If R is abelian and noncyclic then $G = \langle C_G(x) \mid x \in R^{\#} \rangle$.*

If S is a p-group then $d(S)$ is the largest of the orders of the abelian subgroups of S and $\mathcal{A}(S)$ is the set of abelian subgroups of S with order $d(S)$. The Thompson subgroup of S is defined by $J(S) = \langle \mathcal{A}(S) \rangle$. The following is a slight re-statement of a result of Glauberman [**5**, Theorem 5, p.10].

Theorem 2.5. *Suppose that G is a group and p is a prime. Assume:*
(i) *$M < G$.*
(ii) *$S \in \mathrm{Syl}_p(G)$.*
(iii) *$N_G(J(S)) \leq M$ and $C_G(O_p(G)) \leq M$.*
(iv) *Whenever H satisfies*

$$O_p(G) \leq H < G, \ S \cap H \in \mathrm{Syl}_p(H) \ \text{and} \ N_H(J(S \cap H)) \leq M \qquad (*)$$

then $H \leq M$.
Let $P = O_p(G)$, $C = C_G(Z(P))$ and $W = Z(P)/(Z(P) \cap Z(G))$. Then C/P is a p'-group, W is an elementary abelian group and G/C acts faithfully on W. Moreover:
(a) *There exists $A \in \mathcal{A}(S)$ such that $A \not\leq O_p(G)$.*
(b) *There exists a field K of endomorphisms of W such that W is a vector space of dimension 2 over K and the group of automorphisms of W induced by G is $SL(W, K)$.*
(c) *If p is odd or $|K| = 2$ then $Z(P) = (Z(P) \cap Z(G)) \times [Z(P), G]$.*
(d) *If A satisfies (a) and K satisfies (b) then $|K| = |AC/C|$ and $S = PA$.*

3. *Modules*

Throughout this section we assume:
– R is a group of prime order r that acts on the r'-group G.
– V is an RG-module over a field of characteristic p.

Theorem 3.1. *Suppose that RG is faithful and irreducible on V and that $C_V(R) = 0$. Then:*
(a) *Either $[G, R] = 1$ or r is a Fermat prime and $[G, R]$ is a nonabelian special 2-group.*
(b) *If $[G, R]$ is extraspecial then $G = C_G(R) * [G, R]$, where $*$ denotes a central product.*

Proof. (a) is [**4**, Theorem A] and (b) follows from Coprime Action and [**4**, Lemma 3.2(e)]. □

THEOREM 3.2. *Suppose that G contains a cyclic p'-subgroup X that acts nontrivially on V and transitively $[V,X]^{\#}$. Then $C_V(R) \neq 0$.*

Proof. Assume false and consider a counterexample with $|G| + \dim V$ minimal. Then RG is faithful on V and $C_V(R) = 0$. In particular, $r \neq p$. Set

$$T = [G,R].$$

CLAIM 1.
(a) *RG is irreducible on V.*
(b) *$[V,X]$ and X are not R-invariant.*
(c) *$G = XT$ and T is a nonabelian special 2-group.*
(d) *p and r are both odd.*

Proof. (a). Since X is a p'-group there is an RG-composition factor $\overline{W} = W/U$ on which X acts nontrivially. Then $[W,X] \neq 0$, so as X is transitive on $[V,X]^{\#}$ it follows that $[V,X] = [W,X]$, that $[W,X] \cap U = 0$ and then that X is transitive on $[\overline{W},X]^{\#}$. Since $r \neq p$ and $C_V(R) = 0$ we have $C_{\overline{W}}(R) = 0$. Apply the minimality of $\dim V$.

(b). Suppose that $[V,X]$ is R-invariant. Then R normalizes $N_G([V,X])$, which is an r'-group. Also $X \leq N_G([V,X])$ so $N_G([V,X])$ is transitive on $[V,X]^{\#}$. A Frattini Argument implies that $C_{[V,X]}(R) \neq 0$, a contradiction. Thus $[V,X]$ is not R-invariant. Then neither is X.

(c). The minimality of $|G|$ implies $G = \langle X^R \rangle$. Then $G = XT$. By (b), $T \neq 1$. Theorem 3.1 implies that T is a nonabelian special 2-group.

(d). By Theorem 3.1, r is a Fermat prime so r is odd. Since $1 \neq T \leq O_2(G)$ and RG is irreducible on V it follows that p is odd. □

CLAIM 2. *T is homogeneous on V.*

Proof. Assume false and let $V_1, \dots, V_m$ be the homogeneous components for T on V. Then

$$V = V_1 \oplus \cdots \oplus V_m$$

and $m \geq 2$. Note that R normalizes each V_i because $C_V(R) = 0$. As $G = XT$ we see that X permutes $\{V_1, \dots, V_m\}$ transitively.

Suppose that $V_1 x = V_1$ for some $x \in X^{\#}$. Since X is cyclic it follows that $V_i x = V_i$ for all i and then that x is nontrivial on V_1. Choose $v_1 \in V_1^{\#}$ with $v_1 x \neq v_1$ and choose $y \in X$ with $V_1 y = V_2$. But then $v_1 x - v_1$ and $v_1 y - v_1$ are not in the same X-orbit, a contradiction. We deduce that X is regular on $\{V_1, \dots, V_m\}$ and then that $|X| = m \geq 2$.

Suppose that $m \geq 3$. Choose $x, y \in X$ with $V_1 x = V_2$ and $V_1 y = V_3$. Choose $v_1 \in V_1^{\#}$. Then $v_1 x - v_1, v_1 y - v_1, v_1 x + v_1 y - 2v_1 \in [V,X]^{\#}$. By Claim 1(d), $p \neq 2$. Thus $v_1 x - v_1$ and $v_1 x + v_1 y - 2v_1$ are not in the same X-orbit. This contradiction forces $|X| = m = 2$, $V = V_1 \oplus V_2$, $|X| = |[V,X]^{\#}| = |[V_1,X]^{\#}|$ and $|V_1| = 3$. But R

normalizes V_1 and $C_V(R) = 0$ so $r = 2$. This contradicts Claim 1(d) and completes the proof of Claim 2. $\square$

By Claim 2, $Z(T)$ is cyclic so T is extraspecial. Theorem 3.1(b) implies that $G = C_G(R) * T$. Then $C_G(T) = C_G(R)$ and $G/C_G(R)$ is an elementary abelian 2-group.

Let x be a generator for X. Then $x^2 \in C_G(R) = C_G(T)$ so as $G = XT$ we obtain $x^2 \in Z(RG)$. Suppose that $x^2 \neq 1$. The irreducibility of RG forces $C_V(x^2) = 0$. Then $V = [V, x^2] = [V, X]$, contrary to Claim 1(b). We deduce that $x^2 = 1$ so $|X| = 2$ and $|[V, X]| = 3$.

Let $Z = Z(T)$. Now $[X, T] \leq [G, T] = Z$ so $XZ \trianglelefteq G$. If $X \leq Z(G)$ then $X \leq C_G(T) = C_G(R)$, contrary to Claim 1(b). Thus $X \not\leq Z(G)$. Let z be a generator for Z. Now $XZ \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ so x is conjugate to xz. By Coprime Action and the fact that $C_V(Z) = 0$ we have

$$V = C_V(x) \oplus C_V(xz).$$

Then $\dim C_V(x) = \frac{1}{2} \dim V$. Now $V/C_V(x) \cong [V, X]$ and $|[V, X]| = 3$ so $\dim V = 2, p = 3$ and $|V| = 9$. But $C_V(R) = 0$ so r divides $|V| - 1 = 8$. This contradicts Claim 1(d) and completes the proof. $\square$

4. *Proofs of Theorems*

Proof of Theorem B. Assume false and let G be a minimal counterexample. [**6**, Theorem 5.6, p.14], which is a consequence of the Alperin-Gorenstein Fusion Theorem, implies there exists $Q \leq S$ such that

- $T \leq Q$,
- $T \trianglelefteq N_G(J(N_S(Q)))$,
- $N_S(Q) \in \mathrm{Syl}_p(N_G(Q))$,
- $T \leq Z(N_S(Q))$ and
- T is not weakly closed in $N_S(Q)$ with respect to $N_G(Q)$.

The minimality of G yields $G = N_G(Q)$. Thus $T \leq O_p(G)$ and then $T \leq Z(O_p(G))$ because $T \leq Z(S)$. Alternatively, this conclusion can be reached using the reduction in [**5**].

Let $M = N_G(T)$. We claim the assumptions of Theorem 2.5 are satisfied. Now $M < G$ because T is not normal in G and $C_G(O_p(G)) \leq M$ because $T \leq O_p(G)$. Suppose that H satisfies $(*)$ in assumption (iv). Then $T \trianglelefteq N_H(J(S \cap H))$ so the minimality of G implies that T is weakly closed in $S \cap H$ with respect to H; since otherwise, $N_H(T)$ and hence $N_G(T)$ would possess a subgroup X satisfying (b). As $T \leq O_p(G) \leq S \cap H$ this implies $T \trianglelefteq H$ so $H \leq M$. Thus assumption (iv) holds. Adopt the notation defined in the conclusion of Theorem 2.5.

Let $\overline{G} = G/C$. By Theorem 2.5(b), $\overline{G}$ acts faithfully on W, $\dim_K W = 2$ and $\overline{G}$ induces $SL(W, K)$ on W. Note that $\overline{S} \in \mathrm{Syl}_p(\overline{G})$. Let $\overline{X}$ be a complement to $\overline{S}$ in $N_{\overline{G}}(\overline{S})$. Then

$$C_W(S) \cong K$$

and $\overline{X} \cong K^\times$ acts regularly in $C_W(S)^\#$.

Recall that $T \leq Z(P)$ and that $W = Z(P)/(Z(P) \cap Z(G))$. Let $\widetilde{T}$ be the image of T in W. Then $\widetilde{T} \neq 1$ because T is not normal in G. We have $\overline{X} \leq N_{\overline{G}}(\overline{S}) = \overline{N_G(S)} \leq \overline{N_G(J(S))} \leq \overline{M}$ so $\overline{X}$ normalizes $\widetilde{T}$. Now $T \leq Z(S)$ so $\widetilde{T} \leq C_W(\overline{S})$. Since

$\overline{X}$ acts regularly on $C_W(S)^\#$ it follows that

$$\widetilde{T} = C_W(\overline{S}) \cong K$$

and that $\overline{X}$ is regular on $\widetilde{T}^\#$. Since $p > 2$ we have $|\widetilde{T}| > 2$ so $\overline{X}$ acts nontrivially on $\widetilde{T}$.

Let X be a cyclic p'-subgroup of M that maps onto $\overline{X}$. Then X acts nontrivially on T and transitively on $\widetilde{T}^\#$. Now T is abelian, so by Coprime Action,

$$T = C_T(X) \times [T, X].$$

Let $T_0 = T \cap Z(G)$, so that $\widetilde{T} \cong T/T_0$. Clearly $T_0 \leq C_T(X)$. Moreover $C_T(X) \leq T_0$ because $C_{\widetilde{T}}(X) = 1$. Thus $C_T(X) = T_0$. We deduce that $\widetilde{T}$ is X-isomorphic to $[T, X]$. Consequently X acts transitively on $[T, X]^\#$. Then (b) is satisfied, contrary to the fact that G is a counterexample. The proof is complete. □

Proof of Theorem A. Assume false and let G be a minimal counterexample. By Coprime Action we may suppose that S is R-invariant.

Suppose that $O_p(G) \neq 1$. Let V be a minimal normal subgroup of RG contained in $O_p(G)$ and set $\overline{G} = G/C_G(V)$. Now $C_V(R) = 1$ so Theorem 3.1 implies that $[\overline{G}, R]$ is a 2-group. Since $C_S(R) = 1$ we have $S = [S, R]$. As p is odd, this forces $S \leq C_G(V)$. Lemma 2.3(b) and the minimality of G force $G = C_G(V)$. Using Lemma 2.3(c) we obtain a contradiction. Hence $O_p(G) = 1$.

Let $T = \Omega_1(Z(S))$. If T is weakly closed in S with respect to G then Lemma 2.1 implies that $N_G(T)$ controls strong fusion in S with respect to G. Now $N_G(S) \leq N_G(T) < G$ so Lemma 2.3(a) and the minimality of G supply a contradiction. We deduce that T is not weakly closed in S.

Now $N_G(S) \leq N_G(J(S)) < G$ so the minimality of G implies

$$N_G(J(S)) = C_G(J(S))N_G(S).$$

As $T = \Omega_1(Z(S)) \leq J(S)$ we have

$$T \trianglelefteq N_G(J(S)).$$

Theorem B implies there exists a cyclic p'-subgroup $X \leq N_G(T)$ such that X acts nontrivially on T and transitively on $[T, X]^\#$. Now T is an $RN_G(T)$-module over $GF(p)$ so Theorem 3.2 implies that $C_T(R) \neq 1$. This contradicts the fact that $C_G(R)$ is a p'-group and completes the proof of Theorem A. □

References

1. M. Aschbacher, *Finite Group Theory, first edition* (Cambridge University Press, 1986).
2. M.J. Collins, *Some Problems in the Theory of Finite Insoluble Groups*, D.Phil thesis, University of Oxford, 1969.
3. M.J. Collins, 'Finite groups admitting almost fixed point free automorphisms', *Proc. Symposia in Pure Math.* **21** 25-27 Amer. Math. Soc., Providence, R.I. 1971.
4. P. Flavell, 'A Hall-Higman-Shult type theorem for arbitrary finite groups', To appear, Invent. Math.
5. G. Glauberman, 'Weakly closed elements of Sylow subgroups', Math. Z. **107** (1968) 1-20.
6. G. Glauberman, *Global and Local Properties of Finite Groups* in *Finite Simple Groups*, eds. M.B. Powell and G. Higman, (Academic Press Inc., London, 1971)
7. E. Shult 'On groups admitting fixed point free operator groups', Illinois J. Math. **9** (1965) 701-720.

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