Overgroups of second maximal subgroups

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1 Introduction

Let G be a finite group and let

$$\Lambda = \{H \mid H \neq G \text{ and } H \text{ is not a maximal subgroup of } G\}.$$

The *second maximal subgroups* of G are the maximal members of Λ. We see that if H is a second maximal subgroup of G and if $H < M < G$ (proper inclusions) then H is maximal in M and M is maximal in G. The obvious question to ask is:

> *Given a second maximal subgroup H, how many maximal subgroups is H contained in?*

This question has been studied by Feit [2], Köhler [3], Lucchini[4, 5, 6],Pálfy [7], and Pálfy and Pudlák [8] in the context of determining which lattices can arise as interval lattices of subgroups in finite groups. In particular Pálfy and Pudlák [8] have shown that if G is soluble then H is contained in $1+q$ maximal subgroups for some prime power q. Feit [2] and Lucchini [6] have constructed examples which show this result to be false without the solubility assumption on G. In this paper we will obtain an upper bound for the number of maximal subgroups containing H and classify the extremal examples. We shall prove.

Theorem A *Let H be a second maximal subgroup of a finite group G. Then*

$$\textit{the number of maximal subgroups containing } H$$
$$\leq$$
$$1 + \max\{|G : M| \mid M \textit{ is a maximal subgroup containing } H\}.$$

Moreover, equality holds if and only if G has one of the following structures, modulo H_G.

1. *There exist primes p and q such that $p|q-1$, G is the nonabelian group of order pq and $H = 1$;*

2. *There exists a prime p such that $G \cong Z_p \times Z_p$ and $H = 1$;*

3. *G is a Frobenius group with cyclic Frobenius complement H. The Frobenius kernel of G is the direct product of two H-invariant elementary abelian p-groups which are irreducible and isomorphic when considered as H-modules.*

Theorem A raises the following questions.

1. The extremal examples are soluble modulo H_G. This leaves the problem of finding a better inequality in the case that G/H_G is insoluble.

2. Define a *weak second maximal subgroup* to be a maximal subgroup of a maximal subgroup. It is certainly true that a second maximal subgroup is a weak second maximal subgroup. However, the converse is false. For instance $\langle(1\,2)\rangle$ is maximal in S_3 which is maximal in S_4 so $\langle(1\,2)\rangle$ is a weak second maximal subgroup of S_4. However, $\langle(1\,2)\rangle$ is not a second maximal subgroup of S_4 since it is not maximal in a Sylow 2-subgroup of S_4. Does the inequality proved in Theorem A still hold if second maximal is replaced by weak second maximal?

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2 Preliminaries

Throughout this paper, the word group means finite group, $A \leq B$ means A is a subgroup of B and $A < B$ means A is a proper subgroup of B. If $H \leq M$ we let

$$H_M = \bigcap \{H^m \mid m \in M\} = \text{ the core of } H \text{ in } M$$

and note that H_M is the largest subgroup of H that is normal in M. The following lemma is elementary and will be quoted without reference.

Lemma 2.1 *Suppose that $H \leq G$, that P is a Sylow p-subgroup of H and that $N_G(P) \leq H$. Then P is a Sylow p-subgroup of G.*

Frobenius' Theorem *Suppose that $1 < H < G$ and that $H \cap H^g = 1$ for all $g \in G - H$. Then G contains a normal subgroup K such that $G = HK$ and $H \cap K = 1$.*

[1, (35.24) page 191]

Groups satisfying the Hypotheses of Frobenius' theorem are called Frobenius groups, H being the Frobenius complement and K the Frobenius kernel. The following well known lemma provides a useful characterization of Frobenius groups.

Lemma 2.2 *Suppose that $1 < H < G$ and that $N_G(P) \leq H$ whenever P is a nontrivial subgroup of H with prime power order. Then G is a Frobenius group with complement H.*

Proof Let $g \in G$ and suppose $H \cap H^g \neq 1$. Let $D = H \cap H^g$, let p be a prime divisor of $|D|$ and let P be a Sylow p-subgroup of D. Then by hypothesis $N_{H^g}(P) \leq H \cap H^g = D$ so P is a Sylow p-subgroup of H^g and hence $P^{g^{-1}}$ is a Sylow p-subgroup of H. Now P is also a Sylow p-subgroup of H since it has the same order as $P^{g^{-1}}$ so by Sylow's Theorem, there exists $h \in H$ such that $P^h = P^{g^{-1}}$. Then $hg \in N_G(P) \leq H$ whence $g \in H$. We deduce that $H \cap H^g = 1$ for all $g \in G - H$.

Thompson's Theorem *Frobenius kernels are nilpotent.*

[1, (40.8) page 207]

Burnside's Normal p-Complement Theorem *If P is a Sylow p-subgroup of a group G and if $P \leq Z(N_G(P))$, then G has a normal p-complement.*

[1, (39.1) page 202]

3 Proof of Theorem A

Throughout this section we assume the following hypothesis.

1. H is a second maximal subgroup of a finite group G;
2. $M_1, M_2, \ldots, M_r$ is a complete list of the maximal subgroups of G that contain H ordered so that $|M_1| \leq |M_2| \leq \ldots \leq |M_r|$;
3. $\Omega = \{1, 2, \ldots, r\}$;
4. $\Gamma = \bigcup\{M_i \mid i \in \Omega\}$;
5. $H_G = 1$;
6. $r \geq 1 + |G : M_1|$.

We note that in proving Theorem A that there is no loss in assuming $H_G = 1$. In the following sequence of lemmas we will prove that $r = 1 + |G : M_1|$ and that G has one of the structures listed in the conclusion of Theorem A. Observe that $r \geq 3$ and that $M_i \cap M_j = H$ whenever $i \neq j$.

Lemma 3.1 $|\Gamma| \geq |G| + |M_1| - (r-1)|H|$.

Proof We have

$$\begin{aligned}
|\Gamma| &= |H| + \textstyle\sum_{i=1}^{r}(|M_i| - |H|) \\
&= \textstyle\sum_{i=1}^{r} |M_i| - (r-1)|H| \\
&\geq (1 + |G : M_1|)|M_1| - (r-1)|H| \\
&= |G| + |M_1| - (r-1)|H|.
\end{aligned}$$

Lemma 3.2 *If P is a nontrivial subgroup of H then*

$$N_G(P) \leq H.$$

Proof Assume false and let P be a maximal counterexample. Choose $n \in N_G(P) - H$. Let

$$\Sigma = \{i \in \Omega \mid P \not\leq H_{M_i} \text{ and } n \notin M_i\} \quad \text{and} \quad \Delta = \{i \in \Omega \mid P \leq H_{M_i}\}.$$

For each $i \in \Sigma$ choose $m_i \in M_i$ such that $P^{m_i} \not\leq H$. We will show that the sets nm_iH, $i \in \Sigma$ are pairwise disjoint and disjoint from Γ. Let $i, j \in \Sigma$. Suppose that $nm_iH \cap nm_jH \neq \emptyset$, then $nm_iH = nm_jH$ so $m_i \in m_jH \subseteq M_j$ whence $P^{m_j} \leq M_i \cap M_j$. Since $P^{m_j} \not\leq H$ we must have $i = j$. Now suppose that $nm_iH \cap \Gamma \neq \emptyset$, so there exists $k \in \Omega$ such that $nm_i \in M_k$. Then $P^{m_i} = P^{nm_i} \leq M_i \cap M_k$ which again forces $i = k$. Since $m_i \in M_i$ it follows that $n \in M_i$, contradicting $i \in \Sigma$. Thus $nm_i \cap \Gamma = \emptyset$.

What we have just done implies that

$$|G| \geq |\Gamma| + \sum_{i \in \Sigma} |nm_iH| = |\Gamma| + |\Sigma||H|.$$

Using Lemma 3.1 we obtain

$$|G| \geq |G| + |M_1| - (r-1)|H| + |\Sigma||H|$$

and then

$$r - 1 - |\Sigma| \geq |M_1 : H| \geq 2.$$

Since $n \notin H$ it follows that n is contained in at most one maximal subgroup containing H. Thus

$$|\Sigma| + |\Delta| + 1 \geq r.$$

The previous two inequalities imply that

$$|\Delta| \geq 2.$$

Let i and j be distinct members of Δ. Then $P \leq H_{M_i}$ so $H_{M_i} \neq 1$. Since $H_{M_i} \trianglelefteq M_i$ we see that $N_G(H_{M_i}) \not\leq H$ so the maximality of P implies that $P = H_{M_i}$. In particular $P \trianglelefteq M_i$. Similarly $P \trianglelefteq M_j$. Now M_i and M_j are distinct maximal subgroups of G so $P \trianglelefteq G$ and it follows that $P \leq H_G$. This is a contradiction since by hypothesis, $H_G = 1$.

Lemma 3.3 *Assume that $H = 1$. Then*

$$r = 1 + |G : M_1|$$

and one of the following holds.

1. *There exist primes p and q such that $p|q-1$ and G is the nonabelian group of order pq;*
2. *There exists a prime p such that $G \cong Z_p \times Z_p$.*

Proof Since $H = 1$ it follows that 1 is a maximal subgroup of every nontrivial proper subgroup of G. Thus every such subgroup is cyclic of prime order.

Case 1 Assume that there exists $i \in \Omega$ such that $N_G(M_i) = M_i$. Let p be the prime such that $M_i \cong Z_p$. Then M_i is a Sylow p-subgroup of G and Burnside's Normal p-Complement Theorem implies that G has a normal p-complement K. Then $K \cong Z_q$ for some prime $q \neq p$ and $G = M_i K$. Thus $|G| = pq$ and as $N_G(M_i) = M_i$ we see that G is nonabelian. It follows that $p|q-1$ and that $r = 1 + |G : M_1|$.

Case 2 Assume that $M_i \trianglelefteq G$ for all $i \in \Omega$. Now $M_1 \cap M_2 = 1$ so we have $G \cong M_1 \times M_2$. Let $q \geq p$ be primes such that $M_1 \cong Z_p$ and $M_2 \cong Z_q$. If $p \neq q$ then the maximal subgroups of G are M_1 and M_2 contrary to the fact that $r \geq 3$. Thus $p = q$ and $G \cong Z_p \times Z_p$. Then G has $p+1$ maximal subgroups and as $|G : M_1| = p$ we are done.

Lemma 3.4 *Assume that $H \neq 1$. Then G is a Frobenius group with complement H. Let K be the kernel of G. Then K is an elementary abelian p-group which when considered as an H-module is the direct product of two irreducible isomorphic submodules. Moreover*

$$r = 1 + |G : M_1|$$

and H is cyclic.

Proof Lemmas 3.2 and 2.2 imply that G is a Frobenius group with complement H. By Frobenius' Theorem, G contains a normal subgroup K such that

$G = HK$ and $H \cap K = 1$. Let i be any member of Ω. Define $K_i = K \cap M_i$ so that $M_i = HK_i$ and $K \neq K_i \neq 1$. Thompson's Theorem implies that K is nilpotent whence $N_K(K_i) > K_i$. However, M_i is a maximal subgroup of G so K_i is a maximal H-invariant subgroup of K. Thus $K_i \trianglelefteq K$. Moreover, if $i \neq j$ then $K_i \cap K_j = 1$ since $M_i \cap M_j = H$.

Let i, j, k be distinct members of Ω. The previous paragraph implies that $K = K_i \times K_j = K_i \times K_k = K_j \times K_k$. Thus K_j is H-isomorphic to K/K_i which is H-isomorphic to K_k. Hence K is the direct product of two H-invariant H-isomorphic subgroups. Now $K_j \trianglelefteq M_j = HK_j, H \cap K_j = 1$ and H is maximal in M_j. Thus K_j is characteristically simple. Since K_j is nilpotent, this implies that K_j is an elementary abelian p-group for some prime p, on which H acts irreducibly. As $K_j \cong K_k$ it follows that K is an elementary abelian p-group. In particular, the action of H by conjugation on K makes K into an H-module over $\mathrm{GF}(p)$.

Now
$$r \geq 1 + |G : M_1| = 1 + |K : K_1|$$
and as $K = K_1 \times K_2$ and $K_1 \cong K_2$ we have
$$r \geq 1 + |K_1|.$$

It follows that

$$\begin{aligned} |K| &\geq |\Gamma \cap K| \\ &= |\bigcup\{K_i \mid i \in \Omega\}| \\ &\geq r(|K_1| - 1) + 1 \\ &\geq (1 + |K_1|)(|K_1| - 1) + 1 \\ &= |K_1|^2 = |K|. \end{aligned}$$

We deduce that
$$r = |K_1| + 1 = 1 + |G : M_1| \text{ and that } K \subseteq \Gamma.$$

It remains to prove that H is cyclic. Let $E = \mathrm{End}_H(K_1)$ and suppose that E acts transitively on $K_1^\sharp$. Then E is irreducible on K_1 so $\mathrm{End}_E(K_1)$ is a field. Now H is a subgroup of the multiplicative group of $\mathrm{End}_E(K_1)$ so H is cyclic. Thus it is sufficient to prove that E is transitive on $K_1^\sharp$.

Consider the decomposition $K = K_1 \times K_2$. Let $a \in K_1^\sharp, b \in K_2^\sharp$ and let $c = ab$. Now $c \in K \subseteq \Gamma$ so there exists $l \in \Omega$ such that $c \in K_l$. the projection maps

$$\pi_1 : K_l \longrightarrow K_1 \text{ and } \pi_2 : K_l \longrightarrow K_2$$

are both H-homomorphisms. Since $a \neq 1$ and $b \neq 1$, they are both nontrivial and as K_1, K_2 and K_l are irreducible, they are in fact H-isomorphisms. Thus $\pi_1^{-1}\pi_2$ is an H-isomorphism $K_1 \to K_2$ that maps a to b. By keeping b fixed and letting a range over the members of $K_1^\sharp$ we see that E is transitive on $K_1^\sharp$.

The preceding lemmas complete the proof of Theorem A except for showing that a group with the structure described in the last possibility of Theorem A must necessarily satisfy $r = 1 + |G : M_1|$. This is proved by reversing the proof of the previous lemma.

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