Finite Groups In Which Every Two Elements Generate A Soluble Subgroup

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1 Introduction

We will prove the following result.

Theorem *Let G be a finite group in which every two elements generate a soluble subgroup. Then G is soluble.*

This result was first obtained by John Thompson as a by-product of his classification of N-groups [7]. The proof we present is short and direct. It does not use any classification theorem.

The possibility of obtaining results of this type by direct means was first realized by Martin Powell who proved that a finite group in which every three elements generate a soluble subgroup is soluble. An account of his work can be found in [2, pages 473-476]. Powell's argument uses the Hall-Higman Theorem B. We use a different strategy that we shall now describe.

Let G be a soluble group and p an odd prime divisor of $|G|$. If G contains an abelian p-subgroup that normalizes no nontrivial p'-subgroup of G then $O_{p'}(G) = 1$ so as G is soluble we must have $O_p(G) \neq 1$. If every abelian p-subgroup of G normalizes a nontrivial p'-subgroup then a result of Thompson [3, Theorem 1.12, page 19] implies that $O_{p'}(G) \neq 1$. The point of these observations is that they enable us to draw conclusions concerning the normal structure of G without explicitly mentioning normal subgroups in their hypotheses.

Now suppose that G is a minimal counterexample to our theorem. Then G is a simple group in which every proper subgroup is soluble. Let p be a prime divisor of $|G|$ with $p > 3$. If G contains an abelian p-subgroup that normalizes no nontrivial p'-subgroup then we try to force $O_p(G) \neq 1$. If every abelian p-subgroup of G normalizes a nontrivial p'-subgroup then we try to force $O_{p'}(G) \neq 1$.

The implementation of this strategy uses the theory of soluble groups that was developed in the 1960's together with some ideas developed by the author in [4].

2 Preliminaries

See [5] for any undefined notation. Throughout this paper, group means finite group. If A is a subgroup of the group G, define

$$\mathcal{N}_G(A) = \{Q \leq G \mid Q \text{ is normalized by } A \text{ and has order coprime to } |A|\}.$$

The set $\mathcal{N}_G(A)$ is partially ordered by inclusion and we let $\mathcal{N}^*_G(A)$ be the set of maximal members of $\mathcal{N}_G(A)$.

Glauberman's ZJ-Theorem *Let H be a soluble group, P a Sylow p-subgroup of H, $p > 3$, then $H = N_H(ZJ(P))O_{p'}(H)$.*
For the proof and the definition of $ZJ(P)$ see [5, Theorem 8.2.11, page 279] or [3, Cor. 3.9,page 26]. See also Stellmacher[6].

Theorem 2.1 (Thompson-Bender) *Let p be an odd prime, H a soluble group and A an abelian p-subgroup of H that contains every p-element of its centralizer. If Q is a p'-subgroup of H that is normalized by A then $Q \leq O_{p'}(H)$.*
[3, Theorem 1.12, page 19]

Lemma 2.2 *Let p be a prime and A a p-subgroup of a group G acting nontrivially on a p'-subgroup Q. Then $g \in \langle A, A^g \rangle$ for some $g \in Q^\sharp$.*

Proof. Let Q be a minimal counterexample. Then $[[g, A], A] = 1$ for all $g \in Q$ so $1 = [Q, A, A] = [Q, A]$, a contradiction.

Theorem 2.3 *Let H be a soluble group that admits a fixed point free automorphism of prime order. Then H is nilpotent.*
[5, Theorem 10.2.1, page 337]

Lemma 2.4 (Powell) *Let G be a group, p a prime, D a p'-subgroup of G and suppose that $D \leq O_{p'}(\langle D, g\rangle)$ for all $g \in G$. Then $D \leq O_{p'}(G)$.*

Proof. We show that if $d_1, \ldots, d_m \in D$ and $g_1, \ldots, g_m \in G$ then $d_1^{g_1} \ldots d_m^{g_m}$ is a p'-element. If $m = 1$ this is clear, so suppose that $m \geq 2$. Conjugating by g_1^{-1} we may suppose that $g_1 = 1$. By induction $h = d_2^{g_2} \ldots d_m^{g_m}$ is a p'-element and as $d_1^{g_1} \in O_{p'}(\langle D, h\rangle)$ we see that $d_1^{g_1} \ldots d_m^{g_m}$ is an element of the p'-group $O_{p'}(\langle D, h\rangle)\langle h\rangle$. It follows that the normal closure of D in G is a p'-group. Hence $D \leq O_{p'}(G)$.

3 Normal p-subgroups

We begin this section with a general result on minimal simple groups, which is based on ideas of Bender[1].

Lemma 3.1 *Assume G is a minimal simple group, $p > 3$ is a prime divisor of $|G|$, $P \in Syl_p(G)$, and $A \leq P$ is abelian with $\mathcal{N}_G(A) = \{1\}$. Then:*

(i) *$N_G(ZJ(P))$ is the unique maximal subgroup containing A, and*

(ii) *If $1 \neq B \leq A$ is contained in two maximal subgroups then B is cyclic and there is a nontrivial p'-subgroup that is normalized but not centralized by B.*

Proof. Let $M = N_G(ZJ(P))$. Let B be a nontrivial subgroup of A that is contained in a maximal subgroup distinct from M. Choose such a maximal subgroup H with $|H \cap M|_p$ as large as possible. Let Q be a Sylow p-subgroup of $H \cap M$ that contains B. Let $\tilde{Q}$ be a Sylow p-subgroup of M that contains Q. Since P and $\tilde{Q}$ are conjugate in M we see that $M = N_G(ZJ(\tilde{Q}))$.

Suppose $Q = \tilde{Q}$. Then as $\tilde{Q}$ is conjugate to P, it is a Sylow p-subgroup of G and hence of H. Now $\tilde{Q}$ contains a conjugate of A and since A normalizes no nontrivial p'-subgroup it follows that $O_{p'}(H) = 1$. The ZJ-Theorem implies that $ZJ(\tilde{Q}) \trianglelefteq H$. Then $H = M$, a contradiction. Thus $Q < \tilde{Q}$.

Now $Q < N_{\tilde{Q}}(Q)$ hence

$$|H \cap M|_p = |Q| < |N_{\tilde{Q}}(Q)| \leq |N_G(Q) \cap M|_p$$

so the maximal choice of $|H \cap M|_p$ implies that $N_G(Q) \leq M$. Similarly, as $N_{\tilde{Q}}(Q) \leq N_G(ZJ(Q))$ we also have $N_G(ZJ(Q)) \leq M$. Now $N_H(Q) \leq H \cap M$ so as Q is a Sylow p-subgroup of $H \cap M$, it follows that Q is a Sylow p-subgroup of H. The ZJ-Theorem implies that

$$H = N_H(ZJ(Q))O_{p'}(H),$$

so as $N_G(ZJ(Q)) \leq M$ we deduce that

$$1 \neq O_{p'}(H) \nleq M.$$

Since A normalizes no nontrivial p'-subgroups it follows that $B \neq A$. Thus M is the unique maximal subgroup that contains A. This proves (i). Now A is abelian so this implies $C_G(b) \leq M$ for all $b \in B^\sharp$. Hence B does not centralize $O_{p'}(H)$ and Theorem 6.2.4 of [5] implies B is cyclic.

Theorem 3.2 *Let G be a minimal simple group, $p > 3$ a prime divisor of $|G|$ and let A be an abelian p-subgroup of G. Suppose that A normalizes no nontrivial p'-subgroup of G. Then $G = \langle x, y \rangle$ for some $x, y \in G$.*

Proof. Assume false. Let P be a Sylow p-subgroup of G that contains A and let $M = N_G(ZJ(P))$. We may suppose that $Z(P) \leq A$. Let b be an element of order p in $Z(P)$. If A is cyclic then let a be a generator for A, otherwise let a be a member of A such that $\langle b, a \rangle$ is noncyclic. In either case, the previous lemma implies that M is the unique maximal subgroup that contains $\langle b, a \rangle$.

Choose $g \in G - M$. By assumption, $\langle b, g \rangle$ is a proper subgroup of G so as $g \notin M$ we see that $\langle b \rangle$ is contained in a maximal subgroup distinct from M. Let $B = \langle b \rangle$. The previous lemma implies that there exists a p'-subgroup R that is normalized but not centralized by B. Lemma 2.2 implies that there exists $h \in R^\sharp$ such that $h \in \langle B, B^h \rangle$. Using the fact that a centralizes B we have

$$h \in \langle B, B^h \rangle = \langle B, B^{ah} \rangle \leq \langle b, ah \rangle.$$

Then $a = (ah)h^{-1} \in \langle b, ah \rangle$ whence $\langle b, a \rangle \leq \langle b, ah \rangle$. By assumption, $\langle b, ah \rangle$ is a proper subgroup. Since M is the unique maximal subgroup that contains $\langle b, a \rangle$ we deduce that $\langle b, ah \rangle \leq M$. In particular, $h \in M$.

Since A normalizes no nontrivial p'-subgroup, it follows that $O_{p'}(M) = 1$. By Theorem 6.1.3 of [5] we have $Z(P) \leq O_p(M)$. In particular, $B \leq O_p(M)$. Now $h \in M$ so

$$h \in \langle B, B^h \rangle \leq O_p(M).$$

This is a contradiction since h is a nonidentity element of the p'-subgroup R. Thus there exist $x, y \in G$ such that $G = \langle x, y \rangle$.

4 Normal p'-subgroups

The aim of this section is to prove:

Theorem 4.1 *Let G be a minimal simple group and p an odd prime divisor of $|G|$. Suppose that every abelian p-subgroup of G normalizes a nontrivial p'-subgroup of G. Then $G = \langle x, y \rangle$ for some $x, y \in G$.*

Throughout the remainder of this section we assume the hypothesis of Theorem 4.1 but that

$$G \neq \langle x, y \rangle \text{ for all } x, y \in G.$$

We will analyze this situation in the following sequence of lemmas and eventually derive a contradiction.

Lemma 4.2 *Let A be an abelian p-subgroup of G that contains every p-element of its centralizer. If $R \in \mathcal{M}_G(A)$ and $a \in A^\sharp$ then $C_R(a) = 1$.*

Proof. By Theorem 2.1, $C_R(a) \leq O_{p'}(C_G(a))$. Let D be a cyclic subgroup of $C_R(a)$ and let $W = \langle D, a \rangle$. Since D and a have coprime orders, W is cyclic.

Let $g \in G$ and set $H = \langle W, g \rangle$. By assumption $H \neq G$ so H is soluble. Using Lemma X.1.6 of [3] we have

$$D \leq O_{p'}(C_G(a)) \cap H \leq O_{p'}(C_H(a)) \leq O_{p'}(H).$$

But $\langle D, g \rangle \leq H$ so $D \leq O_{p'}(\langle D, g \rangle)$. Lemma 2.4 forces $D \leq O_{p'}(G) = 1$. We deduce that $C_R(a) = 1$.

Throughout the remainder of this section we let P be a Sylow p-subgroup of G.

Lemma 4.3 *P is cyclic so $\mathcal{M}_G(P) \neq \{1\}$.*

Proof. Let A be a maximal abelian p-subgroup of G. Then A contains every p-element of its centralizer. By hypothesis there is $1 \neq R \in \mathcal{M}_G(A)$. Then Theorem 6.2.4 of [5] and Lemma 4.2 imply that A is cyclic. Then every abelian p-subgroup of G is cyclic and Theorem 5.4.10 of [5] implies that P is cyclic.

Lemma 4.4

(i) *If $Q, R \in \mathcal{M}^*_G(P)$ and if $Q \cap R \neq 1$ then $Q = R$.*

(ii) *If H is a maximal subgroup that contains P and if $O_{p'}(H) \neq 1$ then $O_{p'}(H) \in \mathcal{M}^*_G(P)$.*

Proof. Since P is a Sylow p-subgroup of G it contains every p-element of its centralizer. Now P is abelian so using Lemma 4.2 and Theorem 2.3 we see that every member of $\mathcal{M}_G(P)$ is nilpotent.

(i) Assume false and let Q, R be a counterexample in which $D = Q \cap R$ is maximal. Let $N = N_G(D)$. Note that $P \leq N$ so $O_{p'}(N) \in \mathcal{M}_G(P)$. Let T be a member of $\mathcal{M}^*_G(P)$ that contains $O_{p'}(N)$. Now $N_Q(D)$ is a subgroup of N that is normalized by P so Theorem 2.1 implies that $N_Q(D) \leq O_{p'}(N)$. Since Q, R is a counterexample it follows that $D < Q$, so as Q is nilpotent we have $D < N_Q(D)$. Then $D < Q \cap T$ so the maximal choice of $Q \cap R$ implies that $Q = T$. Similarly $R = T$ whence $Q = R$, a contradiction.

(ii) Let T be a member of $\text{И}^*_G(P)$ that contains $O_{p'}(H)$. Since H is a maximal subgroup of the simple group G and as $O_{p'}(H) \neq 1$ we have that $H = N_G(O_{p'}(H))$. In particular $N_T(O_{p'}(H))$ is a P-invariant subgroup of H so Theorem 2.1 implies that $N_T(O_{p'}(H)) \leq O_{p'}(H)$. But T is nilpotent so we see that $O_{p'}(H) = T$. Hence $O_{p'}(H) \in \text{И}^*_G(P)$.

The following lemma will enable us to determine the structure of the maximal subgroups that contain P.

Lemma 4.5 *$N_G(P)$ normalizes every member of $\text{И}^*_G(P)$.*

Proof. Let $R \in \text{И}^*_G(P)$ and let $n \in N_G(P)$. Then $R \neq 1$ and Lemma 4.2 implies that P does not centralize R. By Lemma 2.2 there exists $g \in R^\sharp$ such that $g \in \langle P, P^g \rangle$. We have

$$g \in \langle P, P^g \rangle = \langle P, P^{ng} \rangle \leq \langle P, ng \rangle.$$

Since P is cyclic, $\langle P, ng \rangle$ is a proper subgroup of G. Let H be a maximal subgroup that contains $\langle P, ng \rangle$. Now $R \cap H$ is a P-invariant p'-subgroup of H so Theorem 2.1 implies that $R \cap H \leq O_{p'}(H)$. But $g \in R \cap H$ so Lemma 4.4 implies that $R = O_{p'}(H)$. Now $g \in \langle P, ng \rangle \leq H$ so $n \in H$. Thus $n \in N_G(O_{p'}(H))$. We deduce that $N_G(P) \leq N_G(R)$.

Lemma 4.6 *Let M be a maximal subgroup of G that contains P. Then:*

(i) *$M = N_G(P)O_{p'}(M)$ and $O_{p'}(M) \in \text{И}^*_G(P)$.*

(ii) *If $K \leq N_G(P)$ and $K \trianglelefteq M$ then $K = 1$.*

(iii) *If H is a maximal subgroup of G that contains P and if $H \neq M$ then $H \cap M = N_G(P)$.*

Proof. (i) As P is abelian and M is soluble, $M = N_M(P)O_{p'}(M)$. As $\text{И}_G(P) \neq \{1\}$, Lemmas 4.2 and 4.5 imply $N_G(P)$ is not maximal in G, so $O_{p'}(M) \neq 1$. Then by Lemma 4.4, $O_{p'}(M) \in \text{И}^*_G(P)$ and $N_G(P) \leq M$ by Lemma 4.5.

(ii) Since $O_{p'}(M) \neq 1$ it follows from Lemma 4.2 that $O_p(M) = 1$. Then $O_p(K) = 1$ and K is a p'-group. Lemma 4.2 also implies $O_{p'}(N_G(P)) = 1$ so $K = 1$.

(iii) From (i) we have that $N_G(P) \leq H \cap M$ and since P is cyclic we have $H \cap M = N_G(P)O_{p'}(H \cap M)$. Theorem 2.1 implies that $O_{p'}(H \cap M) \leq O_{p'}(H) \cap O_{p'}(M)$. Using (i), Lemma 4.4(i) and the fact that $H \neq M$ we see that $O_{p'}(H) \cap O_{p'}(M) = 1$. Thus $H \cap M = N_G(P)$.

Next we study the embedding of $N_G(P)$ in G.

Lemma 4.7 *Let $1 \neq T \leq N_G(P)$. Then $N_G(T) \leq N_G(P)$.*

Proof. Let M be a maximal subgroup that contains P. First we show that $N_G(T) \leq M$. Assume false. Choose $n \in N_G(T) - M$. Let m be any member of M. Then $nm \notin M$. Since P is cyclic, $\langle P, nm \rangle$ is a proper subgroup of G. Let H be a maximal subgroup that contains $\langle P, nm \rangle$. Since $nm \notin M$ we have $H \neq M$, so Lemma 4.6(iii) implies that $H \cap M = N_G(P)$. Then

$$T^m = T^{nm} \leq M \cap H = N_G(P).$$

We deduce that

$$T \ \leq \ \bigcap \{N_G(P)^{m^{-1}} \mid m \in M\}.$$

Lemma 4.6(ii) now implies that $T = 1$, a contradiction. Thus $N_G(T) \leq M$.

Now choose $g \in G - M$. Since P is cyclic, $\langle P, g \rangle$ is a proper subgroup of G and hence is contained in a maximal subgroup L. Since $g \notin M$ we have $M \neq L$ so Lemma 4.6(iii) implies that $M \cap L = N_G(P)$. The previous paragraph with L in place of M implies that $N_G(T) \leq L$. Whence $N_G(T) \leq M \cap L = N_G(P)$ as claimed.

Proof of Theorem 4.1. Lemma 4.7 implies that $N_G(P)$ is a Frobenius complement in G. Frobenius' Theorem, or a transfer argument, implies that G is not simple, a contradiction.

5 Proof of Main Theorem

Theorem *Let G be a finite group in which every two elements generate a soluble subgroup. Then G is soluble.*

Proof. Assume false and let G be a minimal counterexample. Then G is a minimal simple group in which every two elements generate a proper subgroup. Burnside's $p^\alpha q^\beta$-Theorem implies that $|G|$ has a prime divisor $p > 3$. Theorem 3.1 implies that every abelian p-subgroup of G normalizes a nontrivial p'-subgroup. This contradicts Theorem 4.1.

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