RESILIENT DEGREE SEQUENCES WITH RESPECT TO HAMILTON CYCLES AND MATCHINGS IN RANDOM GRAPHS

PADRAIG CONDON, ALBERTO ESPUNY DÍAZ, JAEHOON KIM, DANIELA KÜHN,
AND DERYK OSTHUS

Abstract. Pósa's theorem states that any graph G whose degree sequence $d_1 \leq \ldots \leq d_n$ satisfies $d_i \geq i+1$ for all $i < n/2$ has a Hamilton cycle. This degree condition is best possible. We show that a similar result holds for suitable subgraphs G of random graphs, i.e. we prove a 'resilient' version of Pósa's theorem: if $pn \geq C \log n$ and the i-th vertex degree (ordered increasingly) of $G \subseteq G_{n,p}$ is at least $(i+o(n))p$ for all $i < n/2$, then G has a Hamilton cycle. This is essentially best possible and strengthens a resilient version of Dirac's theorem obtained by Lee and Sudakov.

Chvátal's theorem generalises Pósa's theorem and characterises all degree sequences which ensure the existence of a Hamilton cycle. We show that a natural guess for a resilient version of Chvátal's theorem fails to be true. We formulate a conjecture which would repair this guess, and show that the corresponding degree conditions ensure the existence of a perfect matching in any subgraph of $G_{n,p}$ which satisfies these conditions. This provides an asymptotic characterisation of all degree sequences which resiliently guarantee the existence of a perfect matching.

1. Introduction

One of the most well-known and well-studied properties in graph theory is *Hamiltonicity*. We say that a graph G is *Hamiltonian* whenever it contains a cycle which covers all of the vertices of G. We refer to such a cycle as a *Hamilton cycle*. The problem of determining whether or not a graph is Hamiltonian is NP-complete [17]. Thus, the study of Hamiltonicity focuses on finding sufficient conditions, particularly in the form of degree conditions.

In 1952, Dirac [9] proved that every n-vertex graph G with minimum degree at least $n/2$ is Hamiltonian. Pósa [26] strengthened this result. More specifically, a graph G with degree sequence $d_1 \leq \ldots \leq d_n$ such that $d_i \geq i+1$ for all $i < n/2$ is Hamiltonian. This is best possible in the sense that the condition $d_i \geq i+1$ cannot be reduced for any i. Chvátal [8] generalised this further by essentially characterising all degree sequences which guarantee Hamiltonicity: a graph with degree sequence $d_1 \leq \ldots \leq d_n$ is Hamiltonian if for all $i < n/2$ we have $d_i \geq i+1$ or $d_{n-i} \geq n-i$.

The search for Hamilton cycles in random graphs has also been at the core of the subject (as well as the closely related problem of finding perfect matchings). Erdős and Rényi [10, 11] showed that the random graph $G_{n,p}$ with $p \geq C \log n/n$ a.a.s. contains a perfect matching (if n is even and C is large enough). Pósa [27] and Koršunov [20] independently showed that for the same threshold $G_{n,p}$ is a.a.s. Hamiltonian, and Komlós and Szemerédi [19] determined the exact threshold for p. Remarkably, one can strengthen these results to obtain the following hitting time results. Consider the following random graph process: given a vertex set of size n, add each of the $\binom{n}{2}$ possible edges, one by one, chosen uniformly at random among all edges that have not been added yet. Then, Bollobás and Thomason [7] showed that a.a.s. a perfect matching appears as soon as every vertex has degree at least 1, and Ajtai, Komlós and Szemerédi [1] and

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Bollobás [6] independently proved that a.a.s. a Hamilton cycle appears as soon as this graph has minimum degree 2.

One more recent approach to extend the classical extremal results to random graphs is based on the following concept of *resilience*. The *local resilience* of a graph G with respect to some property $\mathcal{P}$ is the maximum number r such that for any subgraph $H \subseteq G$ with $\Delta(H) < r$, the graph $G \setminus H$ satisfies $\mathcal{P}$. One may view this concept as a measure of the damage an adversary can commit at each vertex of G, without destroying the property $\mathcal{P}$. The systematic study of local resilience was initiated by Sudakov and Vu [29]. Restated in this terminology, Dirac's theorem says that the complete graph K_n is $\lfloor n/2 \rfloor$-resilient with respect to Hamiltonicity.

This concept of resilience naturally suggests a generalisation of Dirac's theorem in the setting of random graphs. Lee and Sudakov [23] proved that, when $p = C \log n/n$ and C is sufficiently large, the random graph $G_{n,p}$ is a.a.s. $(1/2 - \varepsilon)np$-resilient with respect to Hamiltonicity, extending Dirac's theorem to random graphs. This improved on earlier bounds [5, 13, 29]. Very recently, Montgomery [24] as well as Nenadov, Steger and Trujić [25] independently obtained a hitting time version of this result (Nenadov, Steger and Trujić also obtained such a hitting time version for perfect matchings [25]).

Resilience of random graphs with respect to other properties has also been extensively studied. In particular, the containment of directed Hamilton cycles [12, 14], cycles of all possible lengths [21], k-th powers of cycles of all possible lengths [31], bounded degree trees [3], triangle factors [4], and bounded degree graphs [2, 15] have been considered.

Lee and Sudakov [23] asked for a characterisation of the degree sequences for which the random graph $G_{n,p}$ is resilient with respect to Hamiltonicity, for p close to $\log n/n$. In this paper, we partially answer this question by extending Pósa's theorem to the setting of random graphs. We also prove that the obvious extension to a Chvátal-type degree condition is false, while some modifications to those conditions suffice to force at least the containment of a perfect matching. We conjecture that such a modification is also sufficient for Hamiltonicity.

To state our results precisely, we start with the following definition, which generalises the class of graphs whose degree sequences satisfy Pósa's condition to the setting of random graphs.

Definition 1.1 (Pósa-resilience). *Let $G = G_{n,p}$ and $\varepsilon > 0$. Let $\mathcal{H}^{\varepsilon}_{n,p}$ be the collection of all n-vertex graphs H which satisfy the following property: there is an ordering $v_1, \ldots, v_n$ of the vertices with $d_H(v_1) \geq \ldots \geq d_H(v_n)$ such that, for all $i < n/2$,*

$$d_H(v_i) \leq (n-i)p - \varepsilon np. \tag{1.1}$$

*We denote $\mathcal{H}^{\varepsilon}_{n,p}(G) := \{H \in \mathcal{H}^{\varepsilon}_{n,p} : H \subseteq G\}$. We say that G is ε-*Pósa-resilient *with respect to a property $\mathcal{P}$ if $G \setminus H \in \mathcal{P}$ for all $H \in \mathcal{H}^{\varepsilon}_{n,p}(G)$.*

We can now state our first main result.

Theorem 1.2. *For every $\varepsilon > 0$, there exists $C > 0$ such that, for $p \geq C \log n/n$, a.a.s. the random graph $G_{n,p}$ is ε-Pósa-resilient with respect to Hamiltonicity.*

Next, we consider the following definition, which generalises the class of graphs whose degree sequences satisfy Chvátal's condition to the setting of random graphs.

Definition 1.3 (Chvátal-resilience). *Let $G = G_{n,p}$ and $\varepsilon > 0$. Let $\mathcal{H}^{\varepsilon,0}_{n,p}$ be the collection of all n-vertex graphs H which satisfy the following property: there is an ordering $v_1, \ldots, v_n$ of the vertices with $d_H(v_1) \geq \ldots \geq d_H(v_n)$ such that, for all $i < n/2$, either*

$$d_H(v_i) \leq (n-i)p - \varepsilon np \qquad \text{or} \qquad d_H(v_{n-i}) \leq ip - \varepsilon np.$$

*We denote $\mathcal{H}^{\varepsilon,0}_{n,p}(G) := \{H \in \mathcal{H}^{\varepsilon,0}_{n,p} : H \subseteq G\}$. We say that G is ε-*Chvátal-resilient *with respect to a property $\mathcal{P}$ if $G \setminus H \in \mathcal{P}$ for all $H \in \mathcal{H}^{\varepsilon,0}_{n,p}(G)$.*

Surprisingly, unlike the case of Pósa-resilience, random graphs are not Chvátal-resilient with respect to even the containment of perfect matchings. (We actually prove a stronger result, see Theorem 3.1.)

Theorem 1.4. *For every $0 < \varepsilon < 10^{-6}$ there exists $C > 0$ such that, for $C \log n/n \leq p \leq 1/20$, a.a.s. the random graph $G_{n,p}$ is not ε-Chvátal-resilient with respect to containing a perfect matching.*

This leads to the following modified version of Definition 1.3. A related concept (i.e. a shift in the Chvátal condition) was considered by Kühn, Osthus and Treglown [22] in the setting of directed Hamilton cycles.

Definition 1.5 (Shifted Chvátal-resilience)**.** *Let $G = G_{n,p}$ and let $\varepsilon, \delta > 0$. Let $\mathcal{H}^{\varepsilon,\delta}_{n,p}$ be the collection of all n-vertex graphs H which satisfy the following property: there is an ordering $v_1, \dots, v_n$ of the vertices with $d_H(v_1) \geq \ldots \geq d_H(v_n)$ such that, for all $i < n/2$, either*

$$d_H(v_i) \leq (n - i)p - \varepsilon np \tag{1.2}$$

or

$$d_H(v_{n-i-\delta n}) \leq ip - \varepsilon np. \tag{1.3}$$

We denote $\mathcal{H}^{\varepsilon,\delta}_{n,p}(G) := \{H \in \mathcal{H}^{\varepsilon,\delta}_{n,p} : H \subseteq G\}$. We say that G is (ε, δ)-Chvátal-resilient *with respect to a property $\mathcal{P}$ if $G \setminus H \in \mathcal{P}$ for all $H \in \mathcal{H}^{\varepsilon,\delta}_{n,p}(G)$.*

Note that (1.3) is never satisfied for $i < \varepsilon n$. The conditions (1.2) and (1.3) together imply that

$$d_H(v) \leq (1 - \varepsilon)np \tag{1.4}$$

for all $H \in \mathcal{H}^{\varepsilon,\delta}_{n,p}$ and all vertices v of H. As $\mathcal{H}^{\varepsilon}_{n,p} \subseteq \mathcal{H}^{\varepsilon,\delta}_{n,p}$, the same bound holds when considering ε-Pósa-resilience.

With this new definition of shifted Chvátal-resilience we can obtain the following version of Chvátal's theorem for random graphs with respect to the containment of perfect matchings.

Theorem 1.6. *For every $\varepsilon > 0$, there exists $C > 0$ such that, for $p \geq C \log n/n$, a.a.s. the random graph $G_{n,p}$ is $(\varepsilon, \varepsilon)$-Chvátal-resilient with respect to containing a perfect matching if n is even.*

We conjecture that Theorem 1.6 also holds if perfect matchings are replaced by Hamilton cycles. The following simple construction shows that this statement, if true, is essentially best possible. Let $G = G_{n,p}$ with $p \geq C \log n/n$ for some sufficiently large C. Given any $\varepsilon n \leq i < n/2$, fix disjoint sets $X, Y \subseteq V$ of sizes i and $n - i$, respectively, and let H be the induced bipartite subgraph between X and Y. One can then prove that a.a.s.

$$d_H(x) \leq (n - i)p + \varepsilon np \qquad \text{and} \qquad d_H(y) \leq ip + \varepsilon np$$

for all $x \in X$ and $y \in Y$. Thus, H is 'close' to satisfying the conditions of Definition 1.5, and it is clear that $G \setminus H$ is not Hamiltonian since it is disconnected. The same construction shows that Theorem 1.2 is essentially best possible (in the sense that we cannot significantly relax the degree condition) and that Theorem 1.6 is essentially best possible when considering odd i.

Investigating resilience with respect to degree sequences is natural not only for perfect matchings and Hamilton cycles, but also for other properties. Several results on degree sequences forcing given substructures have been obtained in the classical setting (see e.g. [28, 30] for such results involving Pósa-type degree sequences and [18] for Chvátal-type degree sequences). It would be interesting to see if one can obtain 'resilient' versions (for random graphs) of some of these results.

2. PRELIMINARIES

2.1. **Notation.** For $n \in \mathbb{N}$, we denote $[n] := \{1, \dots, n\}$. The constants which appear in hierarchies are chosen from right to left. That is, whenever we use the hierarchy $0 < 1/n \ll a \ll b \leq 1$, we mean that there exist non-decreasing functions $f\colon [0, 1) \to [0, 1)$ and $g\colon [0, 1) \to [0, 1)$ such that the result holds for all $0 \leq a, b \leq 1$ and all $n \in \mathbb{N}$ with $a \leq f(b)$ and $1/n \leq g(a)$. We will not calculate these functions explicitly.

We use *a.a.s.* as an abbreviation for *asymptotically almost surely*. Whenever we claim that a result holds a.a.s. for $G_{n,p}$, we mean that the probability that our result holds tends to one as n tends to infinity. For the purpose of clarity, we will ignore rounding issues when dealing with asymptotic statements, whenever the values we consider tend to infinity with n.

Given an n-vertex graph G we define $e(G) \coloneqq |E(G)|$. Given a set $A \subseteq V(G)$ we denote by $e_G(A)$ the number of edges in G whose endpoints are both in A. Given another set $B \subseteq V(G)$ we denote by $E_G(A,B)$ the set of edges of G with one endpoint in A and the other in B (note that A and B are allowed to have nonempty intersection), and $e_G(A,B) \coloneqq |E_G(A,B)|$. Sometimes it will be useful to consider $e'_G(A,B) \coloneqq e_G(A,B) + e_G(A \cap B)$. We will often refer to the graph $G[V(G) \setminus A]$, which we denote as $G - A$. If A and B are disjoint, the notation $G[A,B]$ will refer to the induced bipartite subgraph with vertex classes A and B. We denote the *neighbourhood* of A as $N_G(A) \coloneqq \{v \in V(G) : e_G(\{v\},A) > 0\}$. Given a vertex $v \in V(G)$ we define its *degree* as $d_G(v) \coloneqq |N_G(\{v\})|$. We denote the minimum degree in a set of vertices as $\delta_G(A) \coloneqq \min\{d_G(v) : v \in A\}$, and the maximum degree as $\Delta(G) \coloneqq \max\{d_G(v) : v \in V(G)\}$. We often consider the sequence of degrees of the vertices of G ordered increasingly, and refer to it as the *degree sequence* of G.

The binomial random graph $G_{n,p}$ is obtained by adding each of the edges of a complete graph on n vertices with probability p, independently of the other edges. We will always denote the vertex set of $G_{n,p}$ by V. We use $G_{n,m,p}$ for a random bipartite graph with vertex classes of size n and m, respectively; each edge between the classes is added with probability p independently of every other edge, as above. Whenever we consider a random bipartite graph between vertex sets A and B, we also refer to this model as $G_{A,B,p}$.

2.2. **Tools for random graphs.**

We will need the following Chernoff bound (see e.g. [16, Corollary 2.3]).

Lemma 2.1. *Let X be the sum of n independent Bernoulli random variables and let $\mu \coloneqq \mathbb{E}[X]$. Then, for all $0 \leq \delta \leq 1$ we have that $\mathbb{P}[X \neq (1 \pm \delta)\mu] \leq 2e^{-\delta^2\mu/3}$.*

The following lemmas are standard results for random graphs. They can be proved using Chernoff bounds and the fact that the considered random variables follow binomial distributions.

Lemma 2.2. *There exist constants $C, c > 0$ such that for any $p \geq C \log n/n$ the random graph $G = G_{n,p}$ a.a.s. satisfies that for all $X, Y \subseteq V$ we have*

$$|e_G(X,Y) - |X||Y|p + |X \cap Y|^2 p/2| \leq c\sqrt{|X||Y|np}$$

and

$$|e'_G(X,Y) - |X||Y|p| \leq c\sqrt{|X||Y|np}.$$

Lemma 2.3. *For every $\eta > 0$, there exists a constant C such that for $p \geq C \log n/n$ the random graph $G = G_{n,p}$ a.a.s. satisfies that $d_G(v) = (1 \pm \eta)np$ for all $v \in V$.*

Lemma 2.4. *Let A and B be two disjoint sets of vertices with $|A| = n$, $|B| = m$ and $m = \Theta(n)$. For every $\eta > 0$, there exists a constant C such that, for $p \geq C \log n/n$, the random graph $G = G_{A,B,p}$ a.a.s. satisfies that for each $v \in A$ we have $d_G(v) = (1 \pm \eta)mp$.*

We now prove some properties of the subgraphs of the random graphs which satisfy the conditions of Definition 1.5.

Proposition 2.5. *For every $0 < \varepsilon < 1$, there exists $C > 0$ such that for $p \geq C \log n/n$ the random graph $G = G_{n,p}$ a.a.s. satisfies that, for all $H \in \mathcal{H}^{\varepsilon,\varepsilon}_{n,p}(G)$ and $G' \coloneqq G \setminus H$, the following hold:*

- *(i) For each $X \subseteq V$, we have $|N_{G'}(X)| \geq \min\{\varepsilon|X|np/2,\ \varepsilon n(\log n)^{-1/4}/2\}$.*
- *(ii) For each $X \subseteq V$ with $|X| \geq n(\log n)^{-1/2}$, we have that $|N_{G'}(X)| > (1-\varepsilon^2/10)p^{-1}\delta_{G'}(X)$. In particular, $|N_{G'}(X)| \geq \varepsilon n/2$.*

(iii) G' is connected.

Proof. Choose a number $0 < \eta \ll \varepsilon$. Consider the event that for all $v \in V$ we have

$$d_G(v) = (1 \pm \eta)np \tag{2.1}$$

and for all $X, Y \subseteq V$ with $|X| \geq n(\log n)^{-1/2}$ and $|Y| \geq \eta n$ we have

$$e'_G(X, Y) = (1 \pm \eta)|X||Y|p. \tag{2.2}$$

Throughout the proof, we condition on the event that (2.1) and (2.2) hold. Note that Lemmas 2.2 and 2.3 imply that such an event a.a.s. occurs.

(i). To prove (i), a simple calculation (see e.g. [23, Proposition 2.5(i)]) shows that a.a.s. for all $X \subseteq V$ of size at most $\lceil (\log n)^{-1/4} p^{-1} \rceil$,

$$|N_G(X)| \geq (1 - \varepsilon/2)|X|np. \tag{2.3}$$

As $H \in \mathcal{H}^{\varepsilon,\varepsilon}_{n,p}$, (1.4) together with (2.3) implies

$$|N_{G'}(X)| \geq |N_G(X)| - (1 - \varepsilon)np|X| \geq \varepsilon np|X|/2.$$

Given a set $X \subseteq V$ of size at least $(\log n)^{-1/4} p^{-1}$, we can choose a subset $X' \subseteq X$ of size $\lceil (\log n)^{-1/4} p^{-1} \rceil$, and apply the bound above to obtain $|N_{G'}(X)| \geq |N_{G'}(X')| \geq \varepsilon n(\log n)^{-1/4}/2$. This proves (i).

(ii). As $H \in \mathcal{H}^{\varepsilon,\varepsilon}_{n,p}$, (2.1) together with (1.4) implies that $\delta_{G'}(X) \geq (\varepsilon - \eta)np$. For each $X \subseteq V$, we have

$$e'_{G'}(X, V) \geq |X|\delta_{G'}(X). \tag{2.4}$$

Suppose that there is a set $X \subseteq V$ with $|X| \geq n(\log n)^{-1/2}$ and $|N_{G'}(X)| \leq (1 - \varepsilon^2/10)p^{-1}\delta_{G'}(X)$. Let $Y \subseteq V$ be a set containing $N_{G'}(X)$ with $|Y| = (1 - \varepsilon^2/10)p^{-1}\delta_{G'}(X) \geq \eta n$. Hence, (2.2) implies that

$$e'_G(X, Y) \leq (1 + \eta)p|X|(1 - \varepsilon^2/10)p^{-1}\delta_{G'}(X) \leq (1 - \varepsilon^2/20)|X|\delta_{G'}(X) \overset{(2.4)}{<} e'_{G'}(X, V),$$

a contradiction to the fact that $N_{G'}(X) \subseteq Y$. In particular, as $\delta_{G'}(X) \geq (\varepsilon - \eta)np$, we have $|N_{G'}(X)| \geq (1 - \varepsilon^2/10)(\varepsilon - \eta)n \geq \varepsilon n/2$. This proves (ii).

(iii). Condition on the event that statements (i) and (ii) hold, in addition to (2.1) and (2.2). Assume that G' is not connected, and let $X \subsetneq V$ be a (connected) component of G' such that $|X| \leq n/2$. Note that $|N_G(X)| = |X|$. As (i) and (ii) both hold, it is easy to see that $|X| \geq \varepsilon n/2$. Let $m := |X| - \varepsilon n/4 \geq \varepsilon n/4$.

As $H \in \mathcal{H}^{\varepsilon,\varepsilon}_{n,p}$, by Definition 1.5 there exists a labelling $v_1, \ldots, v_n$ of V with $d_H(v_1) \geq \ldots \geq d_H(v_n)$ such that we have either

$$d_H(v_m) \leq (n - m)p - \varepsilon np \qquad \text{or} \qquad d_H(v_{n-m-\varepsilon n}) \leq mp - \varepsilon np. \tag{2.5}$$

If the former is true, then there exists a set $X' \subseteq X \cap \{v_m, \ldots, v_n\}$ with $|X'| = \varepsilon n/4$ and

$$\delta_{G'}(X') \overset{(2.1)}{\geq} (1 - \eta)np - (n - m)p + \varepsilon np \geq mp + \varepsilon np/2.$$

Then, (ii) ensures that $|N_{G'}(X')| \geq (1 - \varepsilon^2/10)(m + \varepsilon n/2) \geq m + \varepsilon n/3 > |X|$, a contradiction to the fact that X is a component of G'.

Hence, we may assume that the latter of (2.5) holds. In this case, there are at least $m + \varepsilon n \geq |X| + \varepsilon n/2$ vertices v with $d_H(v) \leq mp - \varepsilon np$, hence there exists a set $Y \subseteq \{v_{n-m-\varepsilon n}, \ldots, v_n\} \setminus X$ with $|Y| \geq \varepsilon n/2$ and

$$\delta_{G'}(Y) \overset{(2.1)}{\geq} (1 - \eta)np - mp + \varepsilon np \geq (n - m)p + \varepsilon np/2.$$

Then, (ii) ensures that $|N_{G'}(V \setminus X)| \geq |N_{G'}(Y)| \geq (1 - \varepsilon^2/10)(n - m + \varepsilon n/2) \geq n - m + \varepsilon n/3 > |V \setminus X|$, a contradiction to the fact that X is a component of G'. □

3. CHVÁTAL-TYPE RESILIENCE FOR MATCHINGS IN RANDOM GRAPHS

Proof of Theorem 1.6. Let $0 < 1/n \ll 1/C \ll \eta \ll \varepsilon \ll 1$ and $1/c < 1$, where n is even and c is the constant given by Lemma 2.2. We condition on the event that $G = G_{n,p}$ satisfies the assertions of Lemma 2.2, Lemma 2.3 and Proposition 2.5 with the chosen constants ε, η, C and c, which happens a.a.s. We will show that all such G are $(\varepsilon, \varepsilon)$-Chvátal-resilient with respect to containing a perfect matching. Let $H \in \mathcal{H}^{\varepsilon,\varepsilon}_{n,p}(G)$ and let $G' \coloneqq G \setminus H$. Let $v_1, \dots, v_n$ be an ordering of the vertices as in Definition 1.5. Let $D(H) \coloneqq \{v_{\lceil n/2 \rceil}, \dots, v_n\}$. In particular, by Lemma 2.3 we have that

$$\delta_{G'}(D(H)) \geq (1+\varepsilon)np/2. \tag{3.1}$$

By Tutte's Theorem, it suffices to show that, for any vertex set $U \subseteq V$, the number of odd components of $G' - U$ is at most $|U|$ (here a component is odd if it contains an odd number of vertices). As we conditioned on the assertion of Proposition 2.5(iii) and since n is even, this holds if U is the empty set.

Hence, we will prove that, for any non-empty $U \subseteq V$, the number of (not necessarily odd) components of $G' - U$ is at most $|U|$. As each component of $G' - U$ has at least one vertex, we may further assume that $|U| < n/2$.

Let $U \subseteq V$ with $|U| < n/2$ and let k be the total number of components of $G' - U$. To derive a contradiction, assume that $k > |U|$; in particular, $k \geq 2$. Enumerate the components in $G' - U$ as $C_1, \dots, C_k$ with $|C_1| \leq |C_2| \leq \dots \leq |C_k|$. For each $S \subseteq [k]$, let $C_S \coloneqq \bigcup_{i \in S} C_i$. We consider the cases where $|U|$ is small and large separately.

Case 1: $|U| \leq \varepsilon n/10$.

First, we prove that C_k is large in this case.

Claim 1. *We have* $|C_k| > n/2$.

Proof. Suppose otherwise that $|C_k| \leq n/2$. Let

$$\mathcal{S} \coloneqq \{S \subseteq [k] : |C_S \cap D(H)| \geq \varepsilon n\}.$$

Let $S^* \in \mathcal{S}$ be a set in $\mathcal{S}$ with the minimum $|C_{S^*}|$. We claim that $|C_{S^*}| \leq n/2$. Indeed, suppose this is not the case. Then, we have $|S^*| \geq 2$. As a partition of S^* into two non-empty sets yields two disjoint sets not in $\mathcal{S}$, we have $|C_{S^*} \cap D(H)| < 2\varepsilon n$. Thus $C_{[k]\setminus S^*}$ satisfies that $|C_{[k]\setminus S^*}| \leq n/2$ and $|C_{[k]\setminus S^*} \cap D(H)| \geq n/2 - 3\varepsilon n$ so we have $[k] \setminus S^* \in \mathcal{S}$, which contradicts the minimality of C_{S^*}. Hence we have $|C_{S^*}| \leq n/2$.

Let $D \coloneqq C_{S^*} \cap D(H)$. As we have $|D| \geq \varepsilon n$, by (3.1) and Proposition 2.5(ii) we have

$$|N_{G'}(D)| > (1 - \varepsilon^2/10)(1+\varepsilon)n/2 > n/2 + |U|.$$

It follows that at least one vertex $v \in D \subseteq C_{S^*}$ is adjacent to a vertex $u \in C_{[k]\setminus S^*}$, a contradiction. This proves the claim. □

Let $\ell \coloneqq |C_{[k-1]}|$. Note that $\ell < n/2$.

Claim 2. *We have* $\ell < \varepsilon n/6$.

Proof. Assume otherwise that $\ell \geq \varepsilon n/6$.

First, assume that H satisfies (1.2) for all $i \in [\ell] \setminus [\ell - \varepsilon n/8]$. Note that the set $C' \coloneqq C_{[k-1]} \setminus \{v_1, \dots, v_{\ell-\varepsilon n/8}\}$ satisfies $|C'| \geq \varepsilon n/8$. Because G satisfies the assertion of Lemma 2.3 and $v_{\ell-\varepsilon n/8+1}$ satisfies (1.2) for H, we have

$$\delta_{G'}(C') \geq \delta_G(V) - d_H(v_{\ell-\varepsilon n/8+1}) \geq (1-\eta)np - ((n - \ell + \varepsilon n/8 - 1)p - \varepsilon np) \geq \ell p + 3\varepsilon np/4.$$

As G' satisfies the assertion of Proposition 2.5(ii), we have

$$|N_{G'}(C_{[k-1]})| \geq (1 - \varepsilon^2/10)(\ell + 3\varepsilon n/4) \geq \ell + \varepsilon n/2 > |C_{[k-1]}| + |U|,$$

a contradiction as C_k and $C_{[k-1]}$ are disconnected in $G' - U$.

So suppose that there is an index $j \in [\ell] \setminus [\ell - \varepsilon n/8]$ such that H does not satisfy (1.2) for j. We have that the set $C'' := C_k \setminus \{v_1, \dots, v_{n-j-\varepsilon n}\}$ satisfies

$$|C''| \geq |C_k| - (n - j - \varepsilon n) = n - \ell - |U| - (n - j - \varepsilon n) \geq \varepsilon n/4.$$

Here, we obtain the final inequality as $|U| \leq \varepsilon n/10$ and $j \geq \ell - \varepsilon n/8$. Moreover, because G satisfies the assertion of Lemma 2.3, the fact that (1.3) holds for j implies that

$$\delta_{G'}(C'') \geq \delta_G(V) - d_H(v_{n-j-\varepsilon n+1}) \geq (1-\eta)np - (j - \varepsilon n)p \geq (n - j + \varepsilon n/2)p.$$

As G' satisfies the assertion of Proposition 2.5(ii), this shows that

$$|N_{G'}(C'')| > (1 - \varepsilon^2/10)(n - j + \varepsilon n/2) \geq n - \ell + \varepsilon n/6 > |C_k| + |U|,$$

a contradiction to the fact that C_k is a component of $G' - U$. This proves the claim. □

It follows from the previous two claims that $G' - U$ has one 'giant' component C_k, containing more than $(1 - \varepsilon/3)n$ vertices. The following claim will give us the desired contradiction.

Claim 3. *For any set $W \subseteq V$ with $|W| < \varepsilon n/6$, we have that $|N_{G'}(W)| > 2|W|$.*

Proof. If $|W| \leq n(\log n)^{-1/2}$, then, as G' satisfies the assertion of Proposition 2.5(i), we have

$$|N_{G'}(W)| \geq \min\left\{\frac{1}{2}\varepsilon|W|np,\ \frac{1}{2}\varepsilon n(\log n)^{-1/4}\right\} > 2|W|.$$

If we have $n(\log n)^{-1/2} \leq |W| < \varepsilon n/6$, then, because G' satisfies the assertion of Proposition 2.5(ii), we have

$$|N_{G'}(W)| \geq \varepsilon n/2 > 2|W|.$$ □

Recall that $|U| \leq k - 1 \leq \ell$. As C_k and $C_{[k-1]}$ are disconnected in $G' - U$, we have $|N_{G'}(C_{[k-1]})| \leq |C_{[k-1]}| + |U| \leq 2\ell$. However, by Claim 2 and Claim 3, we have $|N_{G'}(C_{[k-1]})| > 2\ell$, a contradiction. This concludes Case 1.

Case 2: $|U| > \varepsilon n/10$.

Let $S := \{i \in [k] : |C_i| < 2\sqrt{n}\}$ and $t := |U|$. We first claim that

$$t - \sqrt{n} \leq |C_S|. \tag{3.2}$$

Indeed, suppose otherwise. As $k > t$ and each component of $G' - U$ contains at least one vertex, we have

$$|[k] \setminus S| > t - |S| \geq t - |C_S| > \sqrt{n}.$$

Hence, $|C_{[k]\setminus S}| \geq 2\sqrt{n} \cdot |[k] \setminus S| > 2n$, a contradiction. Thus $t - \sqrt{n} \leq |C_S|$.

As G satisfies the assertion of Lemma 2.2, by the definition of S we have

$$\begin{aligned} e_{G'}(C_S) &\leq \sum_{i \in S} e_G(C_i) \leq \sum_{i \in S} \left(|C_i|^2 p + c|C_i|\sqrt{np}\right) \leq \left(\sum_{i \in S} |C_i|\right)(2\sqrt{n}p + c\sqrt{np}) \\ &\leq |C_S| \cdot 4c\sqrt{np} \leq 4cn^{3/2}p^{1/2} \leq \eta n^2 p. \end{aligned} \tag{3.3}$$

We also claim that

$$C_S \text{ does not contain any set } C' \text{ with } |C'| \geq \varepsilon n/20 \text{ and } \delta_{G'}(C') \geq tp + \varepsilon np/2. \tag{3.4}$$

Indeed, suppose C_S contains such a set C'. By (3.3) we have that

$$e_{G'}(C', U) \geq |C'|\delta_{G'}(C') - 2e_{G'}(C_S) > |C'|tp + \varepsilon^2 n^2 p/50.$$

On the other hand, as G satisfies the assertion of Lemma 2.2, we have

$$e_{G'}(C', U) \leq e_G(C', U) \leq |C'|tp + c\sqrt{|C'|tnp} \leq |C'|tp + \varepsilon^2 n^2 p/100,$$

a contradiction. Hence, such a set C' does not exist.

Suppose that H satisfies (1.2) for all $i \in [t] \setminus [t - \varepsilon n/10]$. As G satisfies the assertion of Lemma 2.3 and by (3.2), the set $C'' := C_S \setminus \{v_1, \dots, v_{t-\varepsilon n/10}\}$ satisfies $|C''| \geq |C_S| - t + \varepsilon n/10 \geq \varepsilon n/20$ and

$$\delta_{G'}(C'') \geq (1-\eta)np - (n - t - 9\varepsilon n/10)p \geq tp + \varepsilon np/2,$$

a contradiction to (3.4).

Hence, there exists $j \in [t] \setminus [t - \varepsilon n/10]$ such that H does not satisfy (1.2) for j. By (1.3) and Lemma 2.3, this means that

$$\delta_G(V) - d_H(v_{n-t-9\varepsilon n/10-1}) \geq (1-\eta)np - (j - \varepsilon n)p \geq (n - t + 4\varepsilon/5)p.$$

Therefore, the set $R := (V \setminus U) \setminus \{v_1, \dots, v_{n-t-9\varepsilon n/10}\}$ satisfies $|R| \geq 9\varepsilon n/10$ and $\delta_{G'}(R) > (n-t+4\varepsilon n/5)p$. As $t \leq n/2$, we have $\delta_{G'}(R) \geq tp + \varepsilon np/2$. Hence, we conclude $|R \cap C_S| < \varepsilon n/20$, otherwise we have a contradiction to (3.4).

Hence, $R' := R \cap C_{[k]\setminus S}$ satisfies $|R'| \geq 4\varepsilon n/5$. As G' satisfies the assertion of Proposition 2.5(ii), we conclude that

$$|N_{G'}(R')| \geq (1 - \varepsilon^2/10)(n - t + 4\varepsilon n/5) \geq n - t + \varepsilon n/2 \overset{(3.2)}{>} |V \setminus C_S|.$$

This is a contradiction as R' lies inside $C_{[k]\setminus S}$, which is disconnected from C_S in $G' - U$. □

We now show that Theorem 1.6 is best possible in the sense that $(\varepsilon, \varepsilon)$-Chvátal-resilience cannot be improved to allow for $(\varepsilon, (3np)^{-1})$-Chvátal-resilience. That is, unlike the classical theorem of Chvátal, the random graphs analogue requires an extra shift in the indices whenever we veer from a Pósa degree sequence.

Given an n-vertex graph G, we say that G contains an *optimal matching* if it has a matching of size $\lfloor n/2 \rfloor$. In particular, if G does not contain an optimal matching, then G cannot be Hamiltonian. Note that Theorem 3.1 implies Theorem 1.4.

Theorem 3.1. *For every $0 < \varepsilon < 10^{-6}$ there exists $C > 0$ such that, for any $C \log n/n \leq p \leq 1/20$, the random graph $G = G_{n,p}$ is a.a.s. not $(\varepsilon, \lceil (3p)^{-1} \rceil/n)$-Chvátal-resilient with respect to containing an optimal matching.*

The proof strategy is as follows. We consider $G_{n,p}$ and remove appropriate edges to create a graph G' having an independent set X with $|N_{G'}(X)| < |X| - 1$. This ensures that G' does not contain an optimal matching. We conclude the proof by showing that $G \setminus G' \in \mathcal{H}_{n,p}^{\varepsilon, \lceil (3p)^{-1} \rceil/n}$.

Proof. Let $1/n \ll \eta \ll \varepsilon < 10^{-6}$. Let $Y \subseteq V$ be any set of vertices of size $\lfloor ((1+\eta)2p)^{-1} \rfloor$. Now expose all edges of G incident to Y. Let $\mathcal{E}_1$ be the event that, for each vertex $y \in Y$, we have

$$d_G(y) = (1 \pm \eta)np. \tag{3.5}$$

Note that Lemma 2.3 implies that $\mathcal{E}_1$ happens a.a.s. We condition on the event $\mathcal{E}_1$. Thus we have

$$|N_G(Y)| \leq \sum_{y \in Y} d_G(y) \leq |Y|(1+\eta)np \leq n/2.$$

Fix pairwise disjoint sets $X, U_1, U_2 \subseteq V \setminus (Y \cup N_G(Y))$ with $|X| = 100\varepsilon n$ and $|X|$ even, $|U_1| = |X|/2$ and $|U_2| = |X|/2 - 2$. Let $U := U_1 \cup U_2$.

Now expose all remaining edges of G (i.e. those not incident to Y). Let $\mathcal{E}_2$ be the event that the following hold for all $v \in V \setminus Y$ and $Z \in \{X, U_1, U_2\}$:

$$e_G(v, Z) = (1 \pm \eta)|Z|p, \tag{3.6}$$

$$d_G(v) = (1 \pm \eta)np. \tag{3.7}$$

By Lemma 2.3 and Lemma 2.4, the event $\mathcal{E}_2$ happens a.a.s. under conditioning on $\mathcal{E}_1$. We condition on the event that both $\mathcal{E}_1$ and $\mathcal{E}_2$ hold, i.e. that G satisfies (3.5)–(3.7). We will show that every such G is not $(\varepsilon, \lceil (3p)^{-1} \rceil/n)$-Chvátal-resilient with respect to containing an optimal matching.

Choose an arbitrary ordering $u_1, \dots, u_{100\varepsilon n-2}$ of the vertices in U in such a way that all vertices in U_1 come before the vertices in U_2. We construct a spanning subgraph G' of G as follows.

For each $i \in [100\varepsilon n - 2]$, we choose $\max\{0, \lfloor (i-\varepsilon n)p \rfloor\}$ edges in $E_G(u_i, X)$ uniformly at random and delete them. We further delete all edges in $G[X]$ and all edges in $G[X, V \setminus (X \cup U)]$ to obtain G'.

We now show that, with probability $1 - o(1)$, for each $x \in X$ we have

$$20\varepsilon np \leq d_{G'}(x) \leq (100\varepsilon + \eta)np. \tag{3.8}$$

As x is only adjacent to vertices in U, the upper bound follows from (3.6). To show that the lower bound holds with probability $1 - o(1)$, note that, for all $i \in [50\varepsilon n]$ and $x \in X$ with $xu_i \in E(G)$, we have

$$\mathbb{P}[xu_i \in E(G')] = 1 - \frac{\max\{0, \lfloor (i-\varepsilon n)p \rfloor\}}{e_G(u_i, X)} \overset{(3.6)}{\geq} 1 - \frac{(|X|/2 - \varepsilon n)p}{(1-\eta)|X|p} \geq \frac{1}{2}.$$

For a fixed $x \in X$, the events that $u_i x \in E(G')$ for some $u_i \in N_G(x) \cap U_1$ are independent and satisfy

$$\mathbb{E}\left[|\{u_i x \in E(G') : u_i \in N_G(x) \cap U_1\}| \right] \geq \frac{1}{2} e_G(x, U_1) \overset{(3.6)}{\geq} 24\varepsilon np.$$

Hence, by Lemma 2.1, (3.8) holds with probability $1 - o(1)$. Fix a choice of G' which satisfies (3.8).

From the construction, X is an independent set of G' and $N_{G'}(X) \subseteq U$. Thus $|N_{G'}(X)| \leq |U| < |X| - 1$, hence G' does not contain an optimal matching.

Now it suffices to show that $H := G \setminus G' \in \mathcal{H}_{n,p}^{\varepsilon, \lceil (3p)^{-1} \rceil / n}$. From the construction, it is easy to see that, for all $y \in Y$, $i \in [100\varepsilon n - 2]$ and $v \in V \setminus (X \cup U \cup Y)$, we have

$$d_H(y) = 0, \quad d_H(u_i) = \max\{0, \lfloor (i-\varepsilon n)p \rfloor\} \quad \text{and} \quad d_H(v) \overset{(3.6)}{\leq} (100\varepsilon + \eta)np. \tag{3.9}$$

Furthermore, (3.8) with (3.7) implies that, for each $x \in X$, we have

$$(1 - 100\varepsilon - 2\eta)np \leq d_H(x) \leq (1 - 20\varepsilon + \eta)np. \tag{3.10}$$

Let $v_1, \dots, v_n$ be an ordering of V with $d_H(v_1) \geq \dots \geq d_H(v_n)$. Let $w_1, \dots, w_n$ be an ordering of V in such a way that all vertices of X come first, then the vertices in $V \setminus (X \cup U \cup Y)$ come next, then the vertices $u_{100\varepsilon n-2}, \dots, u_1$ come in this order and, finally, the vertices in Y. For each $0 < \alpha \in \mathbb{R}$, let $W_\alpha := \{w_j : j \geq n - \lceil \alpha \rceil\}$.

Let $\gamma := \lceil (3p)^{-1} \rceil / n$. We now show that $H \in \mathcal{H}_{n,p}^{\varepsilon,\gamma}$. As (3.9) and (3.10) imply $\Delta(H) \leq (1 - 19\varepsilon)np$, H satisfies (1.2) for all $i \in [18\varepsilon n]$. Note that for each $i \in [100\varepsilon n] \setminus [18\varepsilon n]$, (3.9) implies that for each $w \in W_{i+(3p)^{-1}}$ we have $d_H(w) \leq (i - \varepsilon n)p$. As this provides at least $|W_{i+(3p)^{-1}}| = i + \gamma n + 1$ vertices satisfying this, we have $d_H(v_{n-i-\gamma n}) \leq (i - \varepsilon n)p$. Thus H satisfies (1.3) for all $i \in [100\varepsilon n] \setminus [18\varepsilon n]$.

Finally, for $i \in [n/2 - 1] \setminus [100\varepsilon n]$, we consider W_{n-i}. As $W_{n-i} \subseteq V \setminus X$, for each $w \in W_{n-i}$, (3.9) implies that $d_H(w) \leq 101\varepsilon np$. This provides at least $n-i+1$ vertices with $d_H(w) \leq 101\varepsilon np$. Hence, we have $d_H(v_i) \leq 101\varepsilon np \leq (n-i)p - \varepsilon np$, where the final inequality holds with room to spare. Thus H satisfies (1.2) for all $i \in [n/2 - 1] \setminus [100\varepsilon n]$. Hence, $H \in \mathcal{H}_{n,p}^{\varepsilon,\gamma}$. Therefore, $G_{n,p}$ a.a.s. contains a subgraph $H \in \mathcal{H}_{n,p}^{\varepsilon,\gamma}$ such that $G_{n,p} \setminus H$ does not contain an optimal matching. □

4. Pósa's theorem for Hamilton cycles in random graphs

Our approach for the proof of Theorem 1.2 builds on the ideas of Lee and Sudakov [23], with some modifications and additional steps to account for the increased flexibility in the choice of the graph H that we remove. Thus we only describe the necessary tools as well as the main steps. The corresponding proofs that we omit here can be found in the appendix. For $H \in \mathcal{H}_{n,p}^{\varepsilon}$, we rely heavily on the fact that graphs of the form $G_{n,p} \setminus H$ have good expansion properties; namely, they satisfy Proposition 2.5.

Whenever we consider a path P on a vertex set W we mean that $V(P) \subseteq W$. Let G be a graph and let $P = v_1 \dots v_\ell$ be a path on $V(G)$. Let $v \coloneqq v_1$ and $u \coloneqq v_\ell$ be the endpoints of P. Suppose $v_i \in N_G(v)$ for some $i \neq \ell$. Then, we can also consider the path $P' = v_{i-1}v_{i-2} \dots vv_iv_{i+1} \dots u$ in $G \cup P$. We refer to the path P' as a *rotation* of P within G with *fixed endpoint* u and *pivot* v_i. We call $v_{i-1}v_i$ the *broken edge* of the rotation.

Starting from P, we will consider successive rotations of P to obtain new paths, always leaving one of the endpoints of P fixed. We only consider rotations whose broken edges are edges in the original path P.

For any vertex $x \in V(P)$, let $x^-_{P,u}$ and $x^+_{P,u}$ denote the predecessor and successor of x along P, respectively (where P is oriented towards the fixed endpoint u). Similarly, given any set $X \subseteq V(P)$, we denote $X^+_{P,u} \coloneqq \{x^+_{P,u} : x \in X\}$ and $X^-_{P,u} \coloneqq \{x^-_{P,u} : x \in X\}$.

Let $\mathcal{R}_{G,P,u} \subseteq V(P)$ be the set of all vertices $x \in V(P)$ such that there exists a path P_x in $G \cup P$ with endpoints u and x which can be obtained by taking successive rotations of P within G with fixed endpoint u. (As mentioned before, we only consider rotations whose broken edges are in P.) Whenever we consider a vertex $x \in \mathcal{R}_{G,P,u}$, the notation P_x will be used to denote a path with endpoints x and u which can be obtained by the minimum number of rotations of P (whenever there is more than one choice for P_x, we fix such a choice arbitrarily among all the possibilities). Let $R^0_{G,P,u} \coloneqq \{v\}$ and $R^t_{G,P,u}$ be the set of vertices $x \in \mathcal{R}_{G,P,u}$ such that P_x is obtained by at most t rotations.

Given any set $A \subseteq \mathcal{R}_{G,P,u}$, we denote by $R_{G,P,u}(A)$ the union of A and the set of endpoints of all paths which are obtained via a single rotation of P_a with u as a fixed endpoint, for any $a \in A$.

The following observation is well-known. We include the short proof in the appendix.

Lemma 4.1. *Let G be a graph. Let P' be a path on $V(G)$ and let $P = v_1 \dots v_\ell$ be a longest path in $G \cup P'$. Then, for all $t \geq 0$ we have*

$$|R^{t+1}_{G,P,v_\ell}| \geq \frac{1}{2}(|N_G(R^t_{G,P,v_\ell})| - 3|R^t_{G,P,v_\ell}|).$$

Next, we restrict ourselves to the random graph $G_{n,p}$. Given a 'large' set A of endpoints obtainable via a 'small' number of successive rotations of a longest path P, we prove a lower bound on the number of endpoints obtainable from A via one further rotation.

Lemma 4.2. *Let $0 < 1/C \ll \eta \ll \varepsilon < 1$. For $p \geq C \log n/n$, the random graph $G = G_{n,p}$ a.a.s. satisfies the following. Let G' be a subgraph of G and P' be a path on V. Let $P = v_1 \dots v_\ell$ be a longest path in $G' \cup P'$. Then, for all $A \subseteq R^{\eta \log n}_{G',P,v_\ell}$ with $|A| \geq \varepsilon n/100$, we have that $|R_{G',P,v_\ell}(A)| \geq p^{-1}\delta_{G'}(A) - \varepsilon n/10$.*

The proof of Lemma 4.2 is similar to (part of) the proof of Lemma 3.2 in [23]. For completeness, we include the details in the appendix.

We now combine the two previous results to give a lower bound on the number of endpoints which can be generated via successive rotations of a path P with one fixed endpoint.

Lemma 4.3. *Let $0 < 1/C \ll \varepsilon < 1$. For $p \geq C \log n/n$, the random graph $G = G_{n,p}$ a.a.s. satisfies the following. Let $H \in \mathcal{H}^\varepsilon_{n,p}(G)$ and $G' \coloneqq G \setminus H$. Let P' be a path on V. For any longest path $P = v_1 \dots v_\ell$ in $G' \cup P'$, there exists $U \subseteq V$ with $|U| \geq (1/2 + \varepsilon/4)n$ such that, for every $v \in U$, there exists a longest path Q_v in $G' \cup P'$ with endpoints $u \coloneqq v_\ell$ and v, where $V(Q_v) = V(P)$.*

Proof. Throughout this proof we write R^t for $R^t_{G',P,u}$ and $R(A) \coloneqq R_{G',P,u}(A)$ for any $A \subseteq \mathcal{R}_{G',P,u}$. Let η be a number such that $1/C \ll \eta \ll \varepsilon$. Condition on the event that the following holds for all $v \in V$:

$$d_G(v) = (1 \pm \eta)np. \tag{4.1}$$

We also condition on the event that the assertions of Proposition 2.5 and Lemma 4.2 hold for G. By Lemmas 2.3 and 4.2 and Proposition 2.5, each of these events holds a.a.s.

Note that (4.1) and the fact that $H \in \mathcal{H}^{\varepsilon}_{n,p}(G)$ imply that, for any set $X \subseteq V$ with $|X| \geq \varepsilon n/10$,

$$\text{there exists a set } X' \subseteq X \text{ with } |X'| \geq \varepsilon n/20 \text{ and } \delta_{G'}(X') \geq \min\{|X|, n/2\}p + \varepsilon np/2. \tag{4.2}$$

Note that, since P is a longest path in $G' \cup P'$, we have that $N_{G'}(x) \subseteq V(P)$ for all $x \in \mathcal{R}_{G',P,u}$. We will consider successive rotations of P, keeping u fixed, to derive a lower bound on the number of distinct endpoints of different longest paths in $G' \cup P'$ with an endpoint u.

By Lemma 4.1 together with the assertion of Proposition 2.5(i), for each $t \geq 0$, we have

$$|R^{t+1}| \geq \frac{1}{2}\left(\min\left\{\frac{1}{2}\varepsilon|R^t|np, \frac{1}{2}\varepsilon n(\log n)^{-1/4}\right\} - 3|R^t|\right).$$

As $R^0 = \{v_1\}$ and $\varepsilon np/2 > \log n$, the above inequality implies that there exists $s \in \mathbb{N}$ with $s \leq \frac{1}{2}\eta \log n$ such that

$$|R^s| \geq \frac{\varepsilon n}{5(\log n)^{1/4}}. \tag{4.3}$$

Again, by applying Lemma 4.1 together with the assertion of Proposition 2.5(ii), we obtain that $|R^{s+1}| \geq \varepsilon n/10$.

Now, in order to show that $|R^{s+5\varepsilon^{-1}+1}| \geq (1/2 + \varepsilon/4)n$, we will iteratively construct sets $Y_0, \ldots, Y_{5\varepsilon^{-1}}$ as follows.

Let $Y_0 := R^{s+1}$. Suppose that for some $0 \leq j < 5\varepsilon^{-1}$ we have already constructed Y_j with $|Y_j| \geq (j+1)\varepsilon n/10$. We use (4.2) to obtain a subset $Y_j' \subseteq Y_j$ with $|Y_j'| \geq \varepsilon n/20$ and $\delta_{G'}(Y_j') \geq (j+1)\varepsilon np/10 + \varepsilon np/2$. Let $Y_{j+1} := R(Y_j')$. By Lemma 4.2, we have

$$|Y_{j+1}| = |R(Y_j')| \geq p^{-1}\delta_{G'}(Y_j') - \varepsilon n/10 \geq (j+1)\varepsilon n/10 + \varepsilon n/2 - \varepsilon n/10 \geq (j+2)\varepsilon n/10.$$

Note that we can apply Lemma 4.2 as $s + 1 + j \leq s + 5\varepsilon^{-1} \leq \frac{1}{2}\eta \log n + 5\varepsilon^{-1} \leq \eta \log n$. By repeating this for $0 \leq j < 5\varepsilon^{-1}$, we have $|Y_{5\varepsilon^{-1}}| \geq (1/2 + \varepsilon/4)n$.

By the construction, $Y_{5\varepsilon^{-1}} \subseteq R^{s+5\varepsilon^{-1}+1} \subseteq R^{\eta \log n}$. Letting $U := R^{\eta \log n}$ concludes the proof. □

Definition 4.4. *Let $\delta > 0$. We say that a connected n-vertex graph G has property $RE(\delta)$ if one of the following holds for every path P on $V(G)$:*

(i) there exists a path longer than P in the graph $G \cup P$,

(ii) there exists $S_P \subseteq V(G)$ with $|S_P| \geq \delta n$ and a collection $\{T_v : v \in S_P\}$ of subsets of $V(G)$ with $|T_v| \geq \delta n$ for all $v \in S_P$ satisfying the following: for all $v \in S_P$ and $w \in T_v$, the graph $G \cup P$ contains a path Q between v and w with $V(Q) = V(P)$.

Lemma 4.5. *For every $0 < \varepsilon < 1$ there exists $C > 0$ such that, for $p \geq C \log n/n$, the random graph $G = G_{n,p}$ a.a.s. satisfies the following. Let $H \in \mathcal{H}^{\varepsilon}_{n,p}(G)$ and $G' := G \setminus H$. Then, G' satisfies $RE(1/2 + \varepsilon/4)$.*

Proof. Recall that G a.a.s. satisfies the assertions of Proposition 2.5 and Lemma 4.3. We prove that G' satisfies $RE(1/2 + \varepsilon/4)$ conditioned on this.

By Proposition 2.5(iii), G' is connected. Let P be any path on V. We may assume that $G' \cup P$ does not contain a path which is longer than P. Let one of the endpoints of P be u. By Lemma 4.3, there exists $S_P \subseteq V$ with $|S_P| \geq (1/2+\varepsilon/4)n$ and such that, for every $v \in S_P$, there exists a path $Q_v \subseteq G' \cup P$ with endpoints u and v such that $V(Q_v) = V(P)$. For each path Q_v we can fix v and apply Lemma 4.3 again to obtain a set $T_v \subseteq V$ such that $|T_v| \geq (1/2 + \varepsilon/4)n$ and for every $x \in T_v$ there is a path $Q_{xv} \subseteq G' \cup P$ from x to v with $V(Q_{xv}) = V(P)$. The result follows. □

Definition 4.6. *Let $\delta > 0$ and let G_1 be a graph on n vertices with property $RE(\delta)$. We say that a graph G_2 with $V(G_2) = V(G_1)$ complements G_1 if, for every path P on $V(G_1)$, one of the following holds:*

- *(i) there exists a path longer than P in $G_1 \cup P$,*
- *(ii) there exist sets S_P and T_v as in Definition 4.4 and vertices $v \in S_P$ and $w \in T_v$ such that vw is an edge of $G_1 \cup G_2$.*

Proposition 4.7 ([23])**.** *Let $\delta > 0$. For every $G_1 \in RE(\delta)$ and G_2 complementing G_1, the union $G_1 \cup G_2$ is Hamiltonian.*

Finally, we state two lemmas which are used to complete the proof of Theorem 1.2. The first says that, given $G = G_{n,p}$ and $H \in \mathcal{H}^{\varepsilon}_{n,p}(G)$, the graph $G \setminus H$ complements every 'small' subgraph of G which has property $RE(1/2 + \varepsilon/4)$. The final lemma then says that G' actually contains some such 'small' graph as a subgraph. We include the details in the appendix.

Lemma 4.8. *For every $0 < \varepsilon < 1$, there exist $C, \delta > 0$ such that for $p \geq C \log n/n$ we have that $G = G_{n,p}$ a.a.s. satisfies the following property: for any $H \in \mathcal{H}^{\varepsilon}_{n,p}(G)$, the graph $G \setminus H$ complements all graphs $R \subseteq G$ which satisfy $RE(1/2 + \varepsilon/4)$ and have at most $\delta n^2 p$ edges.*

Lemma 4.9. *For all $0 < \varepsilon, \delta \leq 1$, there exists $C > 0$ such that, for $p \geq C \log n/n$, the graph $G = G_{n,p}$ a.a.s. satisfies the following property. Let $H \in \mathcal{H}^{2\varepsilon}_{n,p}(G)$. Then, $G \setminus H$ contains a subgraph with at most $\delta n^2 p$ edges satisfying $RE(1/2 + \varepsilon/4)$.*

The proof of Theorem 1.2 now follows from the previous results.

Proof of Theorem 1.2. Let $1/n \ll 1/C \ll \delta \ll \varepsilon$. Condition on the assertions of Lemmas 4.8 and 4.9 holding with $\varepsilon/2$ instead of ε, which happens a.a.s. We will show that for any $H \in \mathcal{H}^{\varepsilon}_{n,p}(G)$, the graph $G \setminus H$ is Hamiltonian.

Let H be a graph as above. By Lemma 4.9, there exists a subgraph G^* of $G \setminus H$ which has at most $\delta n^2 p$ edges and satisfies property $RE(1/2 + \varepsilon/8)$. By Lemma 4.8 we have that $G \setminus H$ complements G^*. Therefore, Proposition 4.7 implies that $G \setminus H$ is Hamiltonian. □

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Appendix A. Proofs of Lemmas 4.1, 4.2, 4.8 and 4.9

Proof of Lemma 4.1. Throughout the proof we write $R^t := R^t_{G,P,v_\ell}$ and, for all $x \in V(P)$, $x^+ := x^+_{P,v_\ell}$ and $x^- := x^-_{P,v_\ell}$. Since P is a longest path, we must have that $N_G(x) \subseteq V(P)$ for all $x \in \mathcal{R}_{G,P,v_\ell}$. Let $T := \{x \in N_G(R^t) \setminus R^t \mid x^-, x^+ \notin R^t\}$. It follows that if $x \in T$, then the segment of P formed by x^-, x and x^+ is preserved under any sequence of t rotations of P. Since $x \in N_G(R^t) \setminus R^t$, it follows that one of x^-, x^+ must be in $R_{G,P,v_\ell}(R^t) = R^{t+1}$. Now let $T_+ := \{x^+ \mid x \in T, x^+ \in R^{t+1}\}$ and $T_- := \{x^- \mid x \in T, x^- \in R^{t+1}\}$. We have that either $|T_+| \geq |T|/2$ or $|T_-| \geq |T|/2$. It follows that

$$|R^{t+1}| \geq \frac{1}{2}|T| \geq \frac{1}{2}|N_G(R^t) \setminus R^t| - |R^t|. \qquad \square$$

Proof of Lemma 4.2. Choose a constant c satisfying $\eta \ll 1/c \ll 1$. We condition on the event that the following holds for all $X, Y \subseteq V$:

$$e'_G(X,Y) = |X||Y|p \pm c\sqrt{|X||Y|np}. \tag{A.1}$$

Indeed, Lemma 2.2 implies this event a.a.s. occurs.

Let $H := G \setminus G'$. We partition P into $k := \eta^{1/2} \log n$ vertex-disjoint intervals $P_1, \dots, P_k$ with $V(P) = \bigcup_{i \in [k]} V(P_i)$, whose lengths are as equal as possible. By abusing notation, we will also view P_i and P as vertex sets. Consider any $A \subseteq R^{\eta \log n}_{G',P,v_\ell}$ with $|A| \geq \varepsilon n/100$. Throughout this proof, we write $R(A) := R_{G,P,v_\ell}(A)$. For each $i \in [k]$, let $\hat{X}_i \subseteq A$ be the collection of all those vertices $x \in A$ for which some edge in P_i is broken in the sequence of rotations resulting in P_x. Let $X_{i,+}$ and $X_{i,-}$ be the collections of all those vertices $x \in A$ such that P_i is unbroken (i.e. it contains no broken edges) in the sequence of rotations resulting in P_x, and where P_x (when directed from x to v_ℓ) traverses P_i in the original and reverse order, respectively. Note that $A = \hat{X}_i \cup X_{i,+} \cup X_{i,-}$ for every $i \in [k]$. Let $I := \{i \in [k] : |\hat{X}_i| \geq \eta^{1/4}|A|\}$.

We claim that

$$|I| \leq \eta^{3/4} \log n. \tag{A.2}$$

Indeed, recall that each vertex in A is obtained by at most $\eta \log n$ rotations of P. By considering the total sum of the number of rotations performed to obtain each different endpoint in A we observe that

$$\eta^{1/4}|A| \cdot |I| \leq |A| \cdot \eta \log n,$$

which implies (A.2).

Claim 4. *We have* $e'_H(A, V) \geq |A||V \setminus R(A)|p - \eta^{1/5}n^2p$.

Proof. To prove this, note that, since P is a longest path, we have $e'_{G'}(A, V \setminus P) = 0$. Hence,

$$e'_H(A, V \setminus P) = e'_G(A, V \setminus P). \tag{A.3}$$

Throughout this proof, for any $X \subseteq V(P)$ we write $X^+ := X^+_{P,v_\ell}$ and $X^- := X^-_{P,v_\ell}$. For vertices $v_j \in P_i \cap P_i^-$ and $x \in X_{i,+}$, if xv_{j+1} is an edge in G', then we have $v_j \in R(A)$. In other words, x has no edges to $(P_i \cap P_i^+) \setminus R(A)^+$ in the graph G'. By a similar argument, a vertex $x \in X_{i,-}$ has no edges to $(P_i \cap P_i^-) \setminus R(A)^-$ in G'. Thus, we have

$$e'_{G'}(X_{i,+}, (P_i \cap P_i^+) \setminus R(A)^+) = 0 \quad \text{and} \quad e'_{G'}(X_{i,-}, (P_i \cap P_i^-) \setminus R(A)^-) = 0. \tag{A.4}$$

As $G' = G \setminus H$, this implies that all edges of G between $X_{i,*}$ and $(P_i \cap P_i^*) \setminus R(A)^*$ belong to H, for $* \in \{+, -\}$. As $P_i \cap P_i^*$ and P_i^* differ by exactly one vertex, by (A.3) and (A.4) we have

$$\begin{aligned}
e'_H(A, V) &\geq e'_G(A, V \setminus P) + \sum_{* \in \{+,-\}} \sum_{i=1}^{k} \left(e'_G\left(X_{i,*}, \left(P_i \cap P_i^*\right) \setminus R(A)^*\right)\right.\\
&\geq e'_G(A, V \setminus P) + \sum_{* \in \{+,-\}} \sum_{i=1}^{k} \left(e'_G\left(X_{i,*}, \left(P_i \setminus R(A)\right)^*\right) - 4kn\right.\\
&\overset{\text{(A.1)}}{\geq} |A||V \setminus P|p - c\sqrt{|A|n^2p} + \sum_{* \in \{+,-\}} \sum_{i=1}^{k} \left(|X_{i,*}||P_i \setminus R(A)|p - c\sqrt{|X_{i,*}||P_i|np}\right) - 4kn.\\
&\geq |A||V \setminus P|p + \sum_{* \in \{+,-\}} \sum_{i=1}^{k} |X_{i,*}||P_i \setminus R(A)|p - 4c\sqrt{kn^3p}\\
&\geq |A||V \setminus P|p + \sum_{i=1}^{k} |A \setminus \hat{X}_i||P_i \setminus R(A)|p - 4c\sqrt{kn^3p},
\end{aligned}$$

where we used that $|V(P_i)| \leq |P|/k + 1$ in the penultimate inequality, and the fact that $A = \hat{X}_i \cup X_{i,+} \cup X_{i,-}$ in the final inequality. By the definition of I, we have $|A \setminus \hat{X}_i| \geq (1 - \eta^{1/4})|A|$ for all $i \in [k] \setminus I$. Therefore, we have

$$\begin{aligned}
e'_H(A, V) &\geq |A||V \setminus P|p + (1 - \eta^{1/4})|A|p \sum_{i \in [k] \setminus I} |P_i \setminus R(A)| - 4c\sqrt{kn^3p}\\
&\overset{\text{(A.2)}}{\geq} |A||V \setminus R(A)|p - 2\eta^{1/4}|A|np - 4c\sqrt{kn^3p}\\
&\geq |A||V \setminus R(A)|p - \eta^{1/5}n^2p.
\end{aligned}$$

We obtain the final inequality as $p \geq \log n/n$ implies $\sqrt{kn^3p} \leq \eta^{1/4}n^2p$. This proves the claim. □

On the other hand, we have $e'_{G'}(A,V) \geq |A|\delta_{G'}(A)$ and, by (A.1), we have

$$e'_G(A,V) \leq |A|np + c\sqrt{n^3p} \leq (1+\eta)|A|np.$$

Therefore,

$$e'_H(A,V) = e'_G(A,V) - e'_{G'}(A,V) \leq (1+\eta)|A|np - |A|\delta_{G'}(A).$$

Combining this with Claim 4 gives the desired inequality,

$$|R(A)| \geq \delta_{G'}(A)p^{-1} - \eta n - \frac{\eta^{1/5}n^2}{|A|} \geq \delta_{G'}(A)p^{-1} - \varepsilon n/10. \qquad \square$$

Proof of Lemma 4.8. Let $1/C \ll \delta \ll \varepsilon$. Let $\mathcal{G}$ be the family of all subgraphs of the form $G \setminus H$, for all $H \in \mathcal{H}^{\varepsilon}_{n,p}$. (Note that we have $H \in \mathcal{H}^{\varepsilon}_{n,p}$ here instead of $H \in \mathcal{H}^{\varepsilon}_{n,p}(G)$, because this is more convenient for the argument below. But this results in the same family $\mathcal{G}$.)

The probability that the assertion of the lemma fails is

$$\begin{aligned} p^* &:= \mathbb{P}\Big[\bigcup_{R \in RE(1/2+\varepsilon/4),\ e(R) \leq \delta n^2 p} (\{R \subseteq G\} \cap \{\text{some } G' \in \mathcal{G} \text{ does not complement } R\})\Big] \\ &\leq \sum_{R \in RE(1/2+\varepsilon/4),\ e(R) \leq \delta n^2 p} \mathbb{P}[\text{some } G' \in \mathcal{G} \text{ does not complement } R \mid R \subseteq G]\,\mathbb{P}[R \subseteq G], \end{aligned} \tag{A.5}$$

where the union and sum are taken over all labelled graphs R on V which have property $RE(1/2+\varepsilon/4)$ and at most $\delta n^2 p$ edges.

Let R be a fixed graph on V with property $RE(1/2+\varepsilon/4)$ and at most $\delta n^2 p$ edges. Let P be a fixed path on V. If in $R \cup P$ there is a path longer than P, then condition (i) of Definition 4.6 is already satisfied, so we can assume that there is no such path in $R \cup P$. Then, by the definition of property $RE(1/2+\varepsilon/4)$, we can find a set $S_P \subseteq V$ and, for every $v \in S_P$, a corresponding set $T_{v,P} \subseteq V$, as in Definition 4.4. We can assume that for each $v \in S_P$ we have that $e_R(v, T_{v,P}) = 0$, as otherwise R complements itself and there is nothing more to prove. For each $S \subseteq S_P$ with $|S| = \varepsilon n/8$ let $\mathcal{H}_S \subseteq \mathcal{H}^{\varepsilon}_{n,p}$ be the collection of graphs $H \in \mathcal{H}^{\varepsilon}_{n,p}$ for which every $v \in S$ is of the form v_i for $i \geq n/2$ with respect to the ordering of $V(H)$ given in Definition 1.1. Note that $\bigcup_{S \subseteq S_P : |S| = \varepsilon n/8} \mathcal{H}_S = \mathcal{H}^{\varepsilon}_{n,p}$. Thus, given any such $S \subseteq S_P$ and $H \in \mathcal{H}_S$, we have $d_H(v) \leq (1/2-\varepsilon)np$ for all $v \in S$. For each such $S \subseteq S_P$ and all $v \in S$, define $T_{v,P,S} := T_{v,P} \setminus S$. Note that $|T_{v,P,S}| \geq (1/2+\varepsilon/8)n$.

Fix $S \subseteq S_P$ and $v \in S$. Since $|T_{v,P,S}| \geq (1/2+\varepsilon/8)n$, by Lemma 2.1 we have

$$\mathbb{P}[e_G(v, T_{v,P,S}) \leq np/2 \mid R \subseteq G] \leq e^{-\Omega_\varepsilon(np)}.$$

Since S is disjoint from all sets of the form $T_{v,P,S}$, these events are independent for different vertices. Thus, using that $|S| = \varepsilon n/8$, we can see that

$$\mathbb{P}[e_G(v, T_{v,P,S}) \leq np/2 \ \text{ for all } v \in S \mid R \subseteq G] \leq e^{-\Omega_\varepsilon(n^2p)}. \tag{A.6}$$

Note that if there exists $v \in S$ such that $e_G(v, T_{v,P,S}) > np/2$, then for each $H \in \mathcal{H}_S$ we have $e_{G \setminus H}(v, T_{v,P,S}) > np/2 - (1/2-\varepsilon)np > 0$. Therefore, if some $G' \in \mathcal{G}$ does not complement R, there must exist some path P on V and some $S \subseteq S_P$ with $|S| = \varepsilon n/8$ such that all of the vertices of S have fewer than $np/2$ neighbours in $T_{v,P,S} = T_{v,P} \setminus S$. Note that there are at most $n \cdot n!$ choices for the path P and 2^n choices for the set S. Taking the union bound over all choices of the path P and the set S, by (A.6) we have

$$\mathbb{P}[\text{some } G' \in \mathcal{G} \text{ does not complement } R \mid R \subseteq G] \leq n2^n\, n!\, e^{-\Omega_\varepsilon(n^2p)} = e^{-\Omega_\varepsilon(n^2p)}.$$

Combining this with (A.5), we have

$$p^* \leq e^{-\Omega_\varepsilon(n^2p)} \sum_{R \in RE(1/2+\varepsilon/4),\ e(R) \leq \delta n^2 p} \mathbb{P}(R \subseteq G)$$

$$\begin{aligned}
&\leq e^{-\Omega_\varepsilon(n^2p)} \sum_{k=1}^{\delta n^2 p} \binom{\binom{n}{2}}{k} p^k \leq e^{-\Omega_\varepsilon(n^2p)} \sum_{k=1}^{\delta n^2 p} \left(\frac{en^2p}{k}\right)^k \\
&\leq e^{-\Omega_\varepsilon(n^2p)} (\delta n^2 p) \left(\frac{e}{\delta}\right)^{\delta n^2 p} \leq e^{-\Omega_\varepsilon(n^2p)} e^{O(\delta n^2 p \log(1/\delta))} \\
&= o(1),
\end{aligned}$$

where the penultimate inequality holds since $(en^2p/k)^k$ is monotone increasing in the range $1 \leq k \leq \delta n^2 p$. □

Proof of Lemma 4.9. Let $1/n \ll 1/C \ll \varepsilon, \delta$ and $1/c < 1$. Let $p' := \delta p$. We say that a graph F on V is *good* if it has at most $n^2 p' = \delta n^2 p$ edges and, for all $H \in \mathcal{H}^{\varepsilon}_{n,p'}$, the graph $F \setminus H$ satisfies $RE(1/2 + \varepsilon/4)$. Otherwise, we call it *bad*. Given any graph F on V, let $\hat{F}$ be the graph obtained from F by taking every edge of F independently with probability δ.

Let $\hat{\mathbb{P}}$ be the measure associated with the experiment $\hat{F}$. Let $\mathbb{P}_{\text{total}}$ be the product measure obtained from considering the experiments yielding $G_{n,p}$ and $\hat{G}_{n,p}$ (i.e. with respective measures $\mathbb{P}$ and $\hat{\mathbb{P}}$). Note that, by definition, the edge distribution of $\hat{G}_{n,p}$ is identical to that of $G_{n,p'}$. It follows by Lemmas 2.2 and 4.5 that $\mathbb{P}_{\text{total}}[\hat{G}_{n,p} \text{ is good}] = \mathbb{P}[G_{n,p'} \text{ is good}] = 1 - o(1)$.

Let $\mathcal{F}$ be the collection of all graphs F on V for which $\hat{\mathbb{P}}[\hat{F} \text{ is good}] \geq 3/4$. Since

$$o(1) = \mathbb{P}_{\text{total}}[\hat{G}_{n,p} \text{ is bad}] \geq \mathbb{P}[G_{n,p} \notin \mathcal{F}]\, \mathbb{P}_{\text{total}}[\hat{G}_{n,p} \text{ is bad} \mid G_{n,p} \notin \mathcal{F}] \geq \mathbb{P}[G_{n,p} \notin \mathcal{F}]/4,$$

we know that $\mathbb{P}[G_{n,p} \notin \mathcal{F}] = o(1)$ or, in other words, $\mathbb{P}[G_{n,p} \in \mathcal{F}] = 1 - o(1)$. Thus, from now on, we consider $G = G_{n,p}$ and condition on the event that $G \in \mathcal{F}$.

Let $H \in \mathcal{H}^{2\varepsilon}_{n,p}(G)$. Using Lemma 2.1 and taking a union bound over all vertices in V, we have that $\hat{\mathbb{P}}[\hat{G} \cap H \in \mathcal{H}^{\varepsilon}_{n,p'}] = 1 - o(1)$. Since $\hat{G}$ is good with probability at least $3/4$, and $\hat{G} \cap H \in \mathcal{H}^{\varepsilon}_{n,p'}$ with probability $1 - o(1)$, there exists a choice of $\hat{G}$ which satisfies these two properties. For such $\hat{G}$, by the definition of good, the graph $\hat{G} \setminus H$ satisfies $RE(1/2 + \varepsilon/4)$. Moreover, $\hat{G}$ has at most $\delta n^2 p$ edges and, hence, so does $\hat{G} \setminus H$. Since $\hat{G} \setminus H \subseteq G \setminus H$, the result follows. □

Padraig Condon, Alberto Espuny Díaz, Daniela Kühn and Deryk Osthus
School of Mathematics
University of Birmingham
Birmingham
B15 2TT
UK

Jaehoon Kim
Mathematics Institute
University of Warwick
Coventry
CV4 7AL
UK

E-mail addresses: `{pxc644, axe673, d.kuhn, d.osthus}@bham.ac.uk`, `Jaehoon.Kim.1@warwick.ac.uk`.