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Introduction to max-linear programming

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Let $a \oplus b = \max(a, b)$ and $a \otimes b = a + b$ for $a, b \in \mathbb{R}$. Extend this pair of operations to matrices and vectors in the same way as in linear algebra. Being motivated by scheduling of multiprocessor interactive systems, we introduce max-linear programs of the form $f^T \otimes x \to \min$ (or max) subject to $A \otimes x \oplus c = B \otimes x \oplus d$ and develop solution methods for both of them. We prove that these methods are pseudo-polynomial if all entries are integers. This result is based on an existing pseudo-polynomial algorithm for solving the systems of the form $A \otimes x = B \otimes y$.

Keywords: max-linear programming; optimal solution; pseudo-polynomial algorithm.

1. Problem formulation

Consider the following 'multiprocessor interactive system' (MPIS).

Products $P_1, \ldots, P_m$ are prepared using n processors, every processor contributing to the completion of each product by producing a partial product. It is assumed that every processor can work on all products simultaneously and that all these actions on a processor start as soon as the processor starts to work. Let a_{ij} be the duration of the work of the jth processor needed to complete the partial product for P_i $(i = 1, \ldots, m;\ j = 1, \ldots, n)$. Let us denote by x_j the starting time of the jth processor $(j = 1, \ldots, n)$. Then, all partial products for P_i $(i = 1, \ldots, m)$ will be ready at time $\max(x_1 + a_{i1}, \ldots, x_n + a_{in})$. Now, suppose that independently k other processors prepare partial products for products $Q_1, \ldots, Q_m$ and the duration and starting times are b_{ij} and y_j, respectively. Then, the 'synchronization problem' is to find starting times of all $n + k$ processors so that each pair (P_i, Q_i) $(i = 1, \ldots, m)$ is completed at the same time. This task is equivalent to solving the system of equations

$$\max(x_1 + a_{i1}, \ldots, x_n + a_{in}) = \max(y_1 + b_{i1}, \ldots, y_k + b_{ik}) \quad (i = 1, \ldots, m).$$

It may also be required that P_i is not completed before a particular time c_i and similarly Q_i not before time d_i. Then, the equations are

$$\max(x_1 + a_{i1}, \ldots, x_n + a_{in}, c_i) = \max(y_1 + b_{i1}, \ldots, y_k + b_{ik}, d_i) \quad (i = 1, \ldots, m). \tag{1}$$

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If we denote $a \oplus b = \max(a, b)$ and $a \otimes b = a + b$ for $a, b \in \mathbb{R}$, then this system gets the form

$$\sum_{j=1,\ldots,n}^{\oplus} a_{ij} \otimes x_j \oplus c_i = \sum_{j=1,\ldots,k}^{\oplus} b_{ij} \otimes y_j \oplus d_i \quad (i = 1, \ldots, m). \tag{2}$$

Therefore, (1) (and also (2)) is called a 'two-sided system of max-linear equations' (or briefly a 'two-sided max-linear system' or just 'max-linear system').

LEMMA 1.1 (Cancellation law) Let $v, w, a, b \in \mathbb{R}$, $a > b$. Then, for any real x, we have

$$v \oplus a \otimes x = w \oplus b \otimes x \tag{3}$$

if and only if

$$v \oplus a \otimes x = w. \tag{4}$$

Proof. If x satisfies (3), then left-hand side $\geqslant a \otimes x > b \otimes x$. Hence, right-hand side $= w$ and (4) follows. If (4) holds, then $w \geqslant a \otimes x > b \otimes x$ and thus $w = w \oplus b \otimes x$. □

Lemma 1.1 shows that in a two-sided max-linear system, variables missing on one side of an equation may be artificially introduced using suitably taken small coefficients. We may therefore assume without loss of generality that (2) has the same variables on both sides, i.e. in the matrix–vector notation, it has the form

$$A \otimes x \oplus c = B \otimes x \oplus d,$$

where the pair of operations $(\oplus, \otimes)$ is extended to matrices and vectors in the same way as in linear algebra.

In applications, it may be required that the starting times are optimized with respect to a given criterion. In this paper, we consider the case when the objective function is also 'max-linear', i.e.

$$f(x) = f^T \otimes x = \max(f_1 + x_1, \ldots, f_n + x_n)$$

and it has to be either minimized or maximized. For instance, it may required that all processors in an MPIS are in motion as soon/as late as possible, i.e. the latest starting time of a processor is as small/big as possible. In this case, we would set $f(x) = \max(x_1, \ldots, x_n)$, i.e. all $f_j = 0$.

Thus, the problems we will study are

$$f^T \otimes x \to \min \text{ or } \max$$

s.t.

$$A \otimes x \oplus c = B \otimes x \oplus d.$$

Optimization problems of this type will be called 'max-linear programming problems' or, briefly, 'max-linear programs (MLPs)'.

Systems of max-linear equations were investigated already in the first publications dealing with the algebraic structure called max-algebra (sometimes also extremal algebra, path algebra or tropical algebra). In these publications, systems of equations with all variables on one side were considered (Cuninghame-Green, 1979; Vorobyov, 1967; Zimmermann, 1976; Butkovic, 2003). Other systems with a special structure were studied in the context of solving the eigenvalue problems in the corresponding algebraic structures or synchronization in discrete event systems (Baccelli *et al.*, 1992). Using the $(\oplus, \otimes)$-notation, the studied systems had one of the following forms: $A \otimes x = b$, $A \otimes x = x$ or $A \otimes x = x \oplus b$, where A is a given matrix and b is a given vector. Infinite-dimensional generalizations can be found, e.g. in Akian *et al.* (2005).

General two-sided max-linear systems have also been studied (Butkovic & Hegedus, 1984; Cuninghame-Green & Butkovic, 2003; Cuninghame-Green & Zimmermann, 2001; Walkup & Boriello, 1988). A general solution method was presented in Walkup & Boriello (1988), however, no complexity bound was given. In Cuninghame-Green & Butkovic (2003), a pseudo-polynomial algorithm, called the alternating method, has been developed. In Butkovic & Hegedus (1984), it was shown that the solution set is generated by a finite number of vectors and an elimination method was suggested. A general iterative approach suggested in Cuninghame-Green & Zimmermann (2001) assumes that finite upper and lower bounds for all variables are given. We make a substantial use of the alternating method for solving the two-sided max-linear systems in this paper and derive a bisection method for the MLP that repeatedly checks solvability of systems of the form $A \otimes x = B \otimes x$. To our knowledge, this problem has not been studied before. We prove that the number of calls of a subroutine for checking the feasibility is polynomial when applied to MLPs with integer entries, yielding a pseudo-polynomial computational complexity overall. Note that the problem of minimizing the function $2^{x_1} + 2^{x_2} + \cdots + 2^{x_n}$ subject to one-sided max-linear constraints is *NP*-complete. This result is motivated by a similar result presented in Cechlarova (2004) and details are presented at the end of the paper.

2. Max-algebraic prerequisites

Let $a \oplus b = \max(a, b)$ and $a \otimes b = a + b$ for $a, b \in \mathbb{R}$. If $a \in \mathbb{R}$, then the symbol a^{-1} stands in this paper for $-a$.

By 'max-algebra', we understand the analogue of linear algebra developed for the pair of operations $(\oplus, \otimes)$, extended to matrices and vectors in the same way as in linear algebra. That is, if $A = (a_{ij})$, $B = (b_{ij})$ and $C = (c_{ij})$ are matrices of compatible sizes with entries from $\mathbb{R}$, we write $C = A \oplus B$ if $c_{ij} = a_{ij} \oplus b_{ij}$ for all i, j and $C = A \otimes B$ if $c_{ij} = \sum_k^{\oplus} a_{ik} \otimes b_{kj} = \max_k(a_{ik} + b_{kj})$ for all i, j. If $\alpha \in \mathbb{R}$, then $\alpha \otimes A = A \otimes \alpha = (\alpha \otimes a_{ij})$. The main advantage of using max-algebra is the possibility of dealing with a class of non-linear problems in a linear-like way. This is due to the fact that basic rules (commutative, associative and distributive laws) hold in max-algebra to the same extent as in linear algebra.

Max-algebra has been studied by many authors and the reader is referred to Cuninghame-Green (1979, 1995), Heidergott *et al.* (2005), Baccelli *et al.* (1992) or Butkovic (2003) for more information, see also Cuninghame-Green (1962), Vorobyov (1967) and Zimmermann (1976). A chapter in Hogben *et al.* (2006) provides an excellent state of the art overview of the field.

We will now summarize some standard properties that will be used later on. The following holds for $a, b, c \in \mathbb{R}$:

$$a \oplus b \geqslant a,$$

$$a \geqslant b \Rightarrow a \oplus c \geqslant b \oplus c,$$

$$a \geqslant b \Leftrightarrow a \otimes c \geqslant b \otimes c.$$

For matrices (including vectors) A, B and C of compatible sizes over $\mathbb{R}$ and $a \in \mathbb{R}$, we have

$$A \oplus B \geqslant A,$$

$$A \geqslant B \Rightarrow A \oplus C \geqslant B \oplus C,$$

$$A \geqslant B \Rightarrow A \otimes C \geqslant B \otimes C,$$

$$A \geqslant B \Rightarrow C \otimes A \geqslant C \otimes B,$$

$$A \geqslant B \Rightarrow c \otimes A \geqslant c \otimes B,$$

$$(c \otimes A) \otimes B = A \otimes (c \otimes B).$$

The next statement readily follows from the above-mentioned relations.

LEMMA 2.1 Suppose $f \in \mathbb{R}^n$ and let $f(x) = f^T \otimes x$ be defined on $\mathbb{R}^n$. Then,

(a) $f(x)$ is max-linear, i.e. $f(\lambda \otimes x \oplus \mu \otimes y) = \lambda \otimes f(x) \oplus \mu \otimes f(y)$ for every $x, y \in \mathbb{R}^n$ and $\lambda, \mu \in \mathbb{R}$.

(b) $f(x)$ is isotone, i.e. $f(x) \leqslant f(y)$ for every $x, y \in \mathbb{R}^n$, $x \leqslant y$.

3. Max-linear programming problem and its basic properties

The aim of this paper is to develop methods for finding an $x \in \mathbb{R}^n$ that minimizes [maximizes] the function $f(x) = f^T \otimes x$ subject to

$$A \otimes x \oplus c = B \otimes x \oplus d, \tag{5}$$

where $f = (f_1, \ldots, f_n)^T \in \mathbb{R}^n$, $c = (c_1, \ldots, c_m)^T$, $d = (d_1, \ldots, d_m)^T \in \mathbb{R}^m$, $A = (a_{ij})$ and $B = (b_{ij}) \in \mathbb{R}^{m \times n}$ are given matrices and vectors. These problems will be denoted by $\text{MLP}^{\min}$ [$\text{MLP}^{\max}$] and we also denote everywhere $M = \{1, \ldots, m\}$ and $N = \{1, \ldots, n\}$. Note that it is not possible to convert $\text{MLP}^{\min}$ to $\text{MLP}^{\max}$ or vice versa.

Any system of the form (5) is called a ‘non-homogenous max-linear system’ and the set of solutions of this system will be denoted by S. The set of optimal solutions for $\text{MLP}^{\min}$ [$\text{MLP}^{\max}$] will be denoted by $S^{\min}$ [$S^{\max}$]. Any system of the form

$$E \otimes z = F \otimes z \tag{6}$$

is called a ‘homogenous max-linear system’ and the solution set to this system will be denoted by S_{h}. In the next proposition, we show that any non-homogenous max-linear system can easily be converted to a homogenous one. Here and elsewhere, the symbol 0 will be used to denote both the real number zero and the zero vector of an appropriate dimension.

PROPOSITION 3.1 Let $E = (A|0)$ and $F = (B|0)$ be matrices arising from A and B, respectively, by adding a zero column. If $x \in S$, then $(x|0) \in S_{\text{h}}$ and conversely, if $z = (z_1, \ldots, z_{n+1})^T \in S_{\text{h}}$, then $z_{n+1}^{-1} \otimes (z_1, \ldots, z_n)^T \in S$.

Proof. The statement follows straightforwardly from the definitions. □

Given $\text{MLP}^{\min}$ [$\text{MLP}^{\max}$], we denote

$$K = \max\{|a_{ij}|, |b_{ij}|, |c_i|, |d_j|, |f_j|; i \in M, j \in N\}. \tag{7}$$

THEOREM 3.1 (Cuninghame-Green & Butkovic, 2003) Let $E = (e_{ij})$, $F = (f_{ij}) \in \mathbb{Z}^{m\times n}$ and K' be the greatest of the values $|e_{ij}|, |f_{ij}|, i \in M, j \in N$. There is an algorithm of complexity $O(mn(m+n)K')$ that finds an x satisfying (6) or decides that no such x exists.

Proposition 3.1 and Theorem 3.1 show that the feasibility question for $\mathrm{MLP}^{\max}$ and $\mathrm{MLP}^{\min}$ can be solved in pseudo-polynomial time. We will use this result to develop bisection methods for solving $\mathrm{MLP}^{\min}$ and $\mathrm{MLP}^{\max}$. We will prove that these methods need a polynomial number of feasibility checks if all entries are integers and hence are also of pseudo-polynomial complexity.

The algorithm in Cuninghame-Green & Butkovic (2003) is an iterative procedure that starts with an arbitrary vector and then only uses the operations of $+, -$, max and min applied to the starting vector and the entries of E and F. Hence, using Proposition 3.1, we deduce the following theorem.

THEOREM 3.2 If all entries in a homogenous max-linear system are integers and the system has a solution, then this system has an integer solution. The same is true for non-homogenous max-linear systems.

As a corollary to Lemma 1.1, we have the following lemma.

LEMMA 3.1 Let $\alpha, \alpha' \in \mathbb{R}, \alpha' < \alpha$, and $f(x) = f^T \otimes x$, $f'(x) = f'^T \otimes x$, where $f'_j < f_j$ for every $j \in N$. Then, the following holds for every $x \in \mathbb{R}$: $f(x) = \alpha$ if and only if $f(x) \oplus \alpha' = f'(x) \oplus \alpha$.

The following proposition shows that the problem of attainment of a value for a MLP can be converted to a feasibility question.

PROPOSITION 3.2 $f(x) = \alpha$ for some $x \in S$ if and only if the following non-homogenous max-linear system has a solution:

$$A \otimes x \oplus c = B \otimes x \oplus d,$$

$$f(x) \oplus \alpha' = f'(x) \oplus \alpha,$$

where $\alpha' < \alpha$ and $f'(x) = f'^T \otimes x$, where $f'_j < f_j$ for every $j \in N$.

Proof. The statement follows from Lemmas 1.1 and 3.1. □

COROLLARY 3.1 If all entries in $\mathrm{MLP}^{\max}$ or $\mathrm{MLP}^{\min}$ are integers, then an integer objective function value is attained by a real feasible solution if and only if it is attained by an integer feasible solution.

Proof. It follows immediately from Theorem 3.2 and Proposition 3.2. □

COROLLARY 3.2 If all entries in $\mathrm{MLP}^{\max}$ or $\mathrm{MLP}^{\min}$ and α are integers, then the decision problem whether $f(x) = \alpha$ for some $x \in S \cap \mathbb{Z}^n$ can be solved by using $O(mn(m+n)K')$ operations where $K' = \max(K+1, |\alpha|)$.

Proof. For α' and f'_j in Proposition 3.2, we can take $\alpha - 1$ and $f_j - 1$, respectively. Using Theorem 3.1 and Proposition 3.2, the computational complexity then is

$$O((m+1)(n+1)(m+n+2)K') = O(mn(m+n)K').$$ □

A set $C \subseteq \mathbb{R}^n$ is said to be 'max-convex' if $\lambda \otimes x \oplus \mu \otimes y \in C$ for every $x, y \in C, \lambda, \mu \in \mathbb{R}$ with $\lambda \oplus \mu = 0$.

PROPOSITION 3.3 S and S_{h} are max-convex.

Proof.

$$\begin{aligned}
&A \otimes (\lambda \otimes x \oplus \mu \otimes y) \oplus c \\
&\quad = A \otimes (\lambda \otimes x \oplus \mu \otimes y) \oplus \lambda \otimes c \oplus \mu \otimes c \\
&\quad = \lambda \otimes (A \otimes x \oplus c) \oplus \mu \otimes (A \otimes y \oplus c) \\
&\quad = \lambda \otimes (B \otimes x \oplus d) \oplus \mu \otimes (B \otimes y \oplus d) \\
&\quad = B \otimes (\lambda \otimes x \oplus \mu \otimes y) \oplus \lambda \otimes d \oplus \mu \otimes d \\
&\quad = B \otimes (\lambda \otimes x \oplus \mu \otimes y) \oplus d.
\end{aligned}$$

Hence, S is max-convex and S_{h} is max-convex for similar reasons. □

PROPOSITION 3.4 If $x, y \in S$, $f(x) = \alpha < \beta = f(y)$, then for every $\gamma \in (\alpha, \beta)$, there is a $z \in S$ satisfying $f(z) = \gamma$.

Proof. Let $\lambda = 0$, $\mu = \beta^{-1} \otimes \gamma$ and $z = \lambda \otimes x \oplus \mu \otimes y$. Then, $\lambda \oplus \mu = 0$ and $z \in S$ by Proposition 3.3 and by Lemma 2.1, we have

$$f(z) = \lambda \otimes f(x) \oplus \mu \otimes f(y) = \alpha \oplus \beta^{-1} \otimes \gamma \otimes \beta = \gamma. \qquad \square$$

Before we develop solutions methods for solving the optimization problems $\mathrm{MLP}^{\min}$ and $\mathrm{MLP}^{\max}$, we need to find and prove criteria for the existence of optimal solutions. For simplicity, we denote $\inf_{x \in S} f(x)$ by $f^{\min}$ and similarly $\sup_{x \in S} f(x)$ by $f^{\max}$.

We start with the lower bound. We may assume without loss of generality that in (5) we have $c \geqslant d$. Let $M^{>} = \{i \in M; c_i > d_i\}$. For $r \in M^{>}$, we denote

$$L_r = \min_{k \in N} f_k \otimes c_r \otimes b_{rk}^{-1}$$

and

$$L = \max_{r \in M^{>}} L_r.$$

As usual $\max \emptyset = -\infty$ by definition.

LEMMA 3.2 If $c \geqslant d$, then $f(x) \geqslant L$ for every $x \in S$.

Proof. If $M^{>} = \emptyset$, then the statement follows trivially since $L = -\infty$. Let $x \in S$ and $r \in M^{>}$. Then,

$$(B \otimes x)_r \geqslant c_r$$

and so

$$x_k \geqslant c_r \otimes b_{rk}^{-1}$$

for some $k \in N$. Hence, $f(x) \geqslant f_k \otimes x_k \geqslant f_k \otimes c_r \otimes b_{rk}^{-1} \geqslant L_r$ and the statement now follows. □

THEOREM 3.3 $f^{\min} = -\infty$ if and only if $c = d$.

Proof. If $c = d$, then $\alpha \otimes x \in S$ for any $x \in \mathbb{R}^n$ and every $\alpha \in \mathbb{R}$ small enough. Hence, by letting $\alpha \to -\infty$, we have $f(\alpha \otimes x) = \alpha \otimes f(x) \to -\infty$.

If $c \neq d$, then without loss of generality $c \geqslant d$ and the statement now follows by Lemma 3.2 since $L > -\infty$. □

Now, we discuss the upper bound.

LEMMA 3.3 Let $c \geqslant d$. If $x \in S$ and $(A \otimes x)_i > c_i$ for all $i \in M$, then $x' = \alpha \otimes x \in S$ and $(A \otimes x')_i = c_i$ for some $i \in M$, where

$$\alpha = \max_{i \in M}(c_i \otimes (A \otimes x)_i^{-1}). \tag{8}$$

Proof. Let $x \in S$. If

$$(A \otimes x)_i > c_i$$

for every $i \in M$, then $A \otimes x = B \otimes x$. For every $\alpha \in \mathbb{R}$, we also have

$$A \otimes (\alpha \otimes x) = B \otimes (\alpha \otimes x).$$

It follows from the choice of α that also

$$(A \otimes (\alpha \otimes x))_i = \alpha \otimes (A \otimes x)_i \geqslant c_i$$

for every $i \in M$ with equality for at least one $i \in M$. Hence, $x' \in S$ and the lemma follows. □

Let us denote

$$U = \max_{r \in M} \max_{j \in N} f_j \otimes a_{rj}^{-1} \otimes c_r.$$

LEMMA 3.4 If $c \geqslant d$, then the following holds:

(a) if $x \in S$ and $(A \otimes x)_r \leqslant c_r$ for some $r \in M$, then $f(x) \leqslant U$;

(b) if $A \otimes x = B \otimes x$ has no solution, then $f(x) \leqslant U$ for every $x \in S$.

Proof. (a) Since

$$a_{rj} \otimes x_j \leqslant c_r$$

for all $j \in N$, we have

$$f(x) \leqslant \max_{j \in N} f_j \otimes a_{rj}^{-1} \otimes c_r \leqslant U.$$

(b) If $S = \emptyset$, then the statement holds trivially. Let $x \in S$. Then,

$$(A \otimes x)_r \leqslant c_r$$

for some $r \in M$ since otherwise $A \otimes x = B \otimes x$, and the statement now follows from (a). □

THEOREM 3.4 $f^{\max} = +\infty$ if and only if $A \otimes x = B \otimes x$ has a solution.

Proof. We may assume without loss of generality that $c \geqslant d$. If $A \otimes x = B \otimes x$ has no solution, then the statement follows from Lemma 3.4. If it has a solution, say z, then for all sufficiently big $\alpha \in \mathbb{R}$, we have

$$A \otimes (\alpha \otimes z) = B \otimes (\alpha \otimes z) \geqslant c \oplus d$$

and hence $\alpha \otimes z \in S$. The statement now follows by letting $\alpha \longrightarrow +\infty$. □

We also need to show that the maximal [minimal] value is attained if $S \neq \emptyset$ and $f^{\max} < +\infty$ [$f^{\min} > -\infty$]. Due to continuity of f, this will be proved by showing that both for minimization and maximization the set S can be reduced to a compact subset. To achieve this, we denote the following for $j \in N$:

$$h_j = \min\left(\min_{r \in M} a_{rj}^{-1} \otimes c_j, \min_{r \in M} b_{rj}^{-1} \otimes d_j,\ f_j^{-1} \otimes L\right), \tag{9}$$

$$h'_j = \min\left(\min_{r \in M} a_{rj}^{-1} \otimes c_j, \min_{r \in M} b_{rj}^{-1} \otimes d_j\right) \tag{10}$$

and $h = (h_1, \ldots, h_n)^T$, $h' = (h'_1, \ldots, h'_n)^T$. Note that h is finite if and only if $f^{\min} > -\infty$.

PROPOSITION 3.5 For any $x \in S$, there is an $x' \in S$ such that $x' \geqslant h$ and $f(x) = f(x')$.

Proof. Let $x \in S$. It is sufficient to set $x' = x \oplus h$ since if $x_j < h_j$, $j \in N$, then x_j is not active on any side of any equation or in the objective function and therefore, changing x_j to h_j will not affect any of the equations or the objective function value. □

COROLLARY 3.3 If $f^{\min} > -\infty$ and $S \neq \emptyset$, then there is a compact set $\overline{S}$ such that

$$f^{\min} = \min_{x \in \overline{S}} f(x).$$

Proof. Note that h is finite since $f^{\min} > -\infty$. Let $\tilde{x} \in S$, $\tilde{x} \geqslant h$, then

$$\overline{S} = S \cap \{x \in \mathbb{R}^n; h_j \leqslant x_j \leqslant f_j^{-1} \otimes f(\tilde{x}), j \in N\}$$

is a compact subset of S and $\tilde{x} \in S$. If there was a $y \in S$, $f(y) < \min_{x \in \overline{S}} f(x) \leqslant f(\tilde{x})$, then by Proposition 3.5, there is a $y' \geqslant h$, $y' \in S$, $f(y') = f(y)$. Hence,

$$f_j \otimes y'_j \leqslant f(y') = f(y) \leqslant f(\tilde{x})$$

for every $j \in N$ and thus $y' \in \overline{S}$, $f(y') < \min_{x \in \overline{S}} f(x)$, a contradiction. □

PROPOSITION 3.6 For any $x \in S$, there is an $x' \in S$ such that $x' \geqslant h'$ and $f(x) \leqslant f(x')$.

Proof. Let $x \in S$ and $j \in N$. It is sufficient to set $x' = x \oplus h'$ since if $x_j < h'_j$, then x_j is not active on any side of any equation and therefore changing x_j to h'_j does not violate any of the equations. The rest follows from isotonicity of $f(x)$. □

Let $\overline{S}' = S \cap \{x \in \mathbb{R}^n; h'_j \leqslant x_j \leqslant f_j^{-1} \otimes U, j \in N\}$.

COROLLARY 3.4 If $f^{\max} < +\infty$, then

$$f^{\max} = \max_{x \in \overline{S}'} f(x).$$

Proof. The statement follows immediately from Lemma 3.4, Theorem 3.4 and Proposition 3.6. □

COROLLARY 3.5 If $S \neq \emptyset$ and $f^{\min} > -\infty$ [$f^{\max} < +\infty$], then $S^{\min} \neq \emptyset$ [$S^{\max} \neq \emptyset$].

It follows from Lemma 3.2 that $f^{\max} > L$. However, this information is not useful if $c = d$ since then $L = -\infty$. Since we will need a lower bound for $f^{\max}$ even when $c = d$, we define $L' = f(h')$ and formulate the following.

COROLLARY 3.6 If $x \in S$, then $x' = x \oplus h'$ satisfies $f(x') \geqslant L'$ and thus $f^{\max} \geqslant L'$.

4. The algorithms

It follows from Proposition 3.1 and Theorem 3.1 that in pseudo-polynomial time either a feasible solution to (5) can be found or it can be decided that no such solution exists. Due to Theorems 3.3 and 3.4, we can also recognize the cases when the objective function is unbounded. We may therefore assume that a feasible solution exists, the objective function is bounded (from below or above depending on whether we wish to minimize or maximize) and hence an optimal solution exists (Corollary 3.5). If $x^0 \in S$ is found, then using the scaling (if necessary) proposed in Lemma 3.3 or Corollary 3.6, we find (another) x^0 satisfying $L \leqslant f(x^0) \leqslant U$ or $L' \leqslant f(x^0) \leqslant U$ (see Lemmas 3.2 and 3.4). The use of the bisection method applied to either $(L, f(x^0))$ or $(f(x^0), U)$ for finding a minimizer or maximizer of $f(x)$ is then justified by Proposition 3.4. The algorithms are based on the fact that (see Proposition 3.2) checking the existence of an $x \in S$ satisfying $f(x) = \alpha$ for a given $\alpha \in \mathbb{R}$ can be converted to a feasibility problem. They stop when the interval of uncertainty is shorter than a given precision $\varepsilon > 0$.

ALGORITHM 4.1 MAXLINMIN (max-linear minimization)

Input: $f = (f_1, \ldots, f_n)^T \in \mathbb{R}^n$, $c = (c_1, \ldots, c_m)^T$, $d = (d_1, \ldots, d_m)^T \in \mathbb{R}^m$, $c \geqslant d$, $c \neq d$, $A = (a_{ij})$, $B = (b_{ij}) \in \mathbb{R}^{m\times n}$, $\varepsilon > 0$.
Output: $x \in S$ such that $f(x) - f^{\min} \leqslant \varepsilon$.

1. If $L = f(x)$ for some $x \in S$, then stop ($f^{\min} = L$).
2. Find an $x^0 \in S$. If $(A \otimes x^0)_i > c_i$ for all $i \in M$, then scale x^0 by α defined in (8).
3. $L(0) := L$, $U(0) := f(x^0)$, $r := 0$.
4. $\alpha := \frac{1}{2}(L(r) + U(r))$.
5. Check whether $f(x) = \alpha$ is satisfied by some $x \in S$ and in the positive case find one.
   If yes, then $U(r+1) := \alpha$, $L(r+1) := L(r)$.
   If not, then $U(r+1) := U(r)$, $L(r+1) := \alpha$.
6. $r := r + 1$.
7. If $U(r) - L(r) \leqslant \varepsilon$, then stop else go to 4.

THEOREM 4.1 Algorithm MAXLINMIN is correct and the number of iterations before termination is

$$O\left(\log_2 \frac{U - L}{\varepsilon}\right).$$

Proof. Correctness follows from Proposition 3.4 and Lemma 3.2. Since $c \neq d$, we have the following at the end of Step 2: $f(x^0) \geqslant L > -\infty$ (Lemma 3.2) and $U(0) := f(x^0) \leqslant U$ by Lemma 3.4. Thus, the number of iterations is $O\left(\log_2 \frac{U-L}{\varepsilon}\right)$ since after every iteration the interval of uncertainty is halved. □

ALGORITHM 4.2 MAXLINMAX (max-linear maximization)

Input: $f = (f_1, \ldots, f_n)^T \in \mathbb{R}^n$, $c = (c_1, \ldots, c_m)^T$, $d = (d_1, \ldots, d_m)^T \in \mathbb{R}^m$,
$A = (a_{ij})$, $B = (b_{ij}) \in \mathbb{R}^{m \times n}$, $\varepsilon > 0$.
Output: $x \in S$ such that $f^{\max} - f(x) \leqslant \varepsilon$ or an indication that $f^{\max} = +\infty$.

1. If $U = f(x)$ for some $x \in S$, then stop ($f^{\max} = U$).
2. Check whether $A \otimes x = B \otimes x$ has a solution. If yes, stop ($f^{\max} = +\infty$).
3. Find an $x^0 \in S$ and set $x^0 := x^0 \oplus h'$, where h' is as defined in (10).
4. $L(0) := f(x^0)$, $U(0) := U$, $r := 0$.
5. $\alpha := \frac{1}{2}(L(r) + U(r))$.
6. Check whether $f(x) = \alpha$ is satisfied by some $x \in S$ and in the positive case find one.
   If yes, then $U(r+1) := U(r)$, $L(r+1) := \alpha$.
   If not, then $U(r+1) := \alpha$, $L(r+1) := L(r)$.
7. $r := r + 1$.
8. If $U(r) - L(r) \leqslant \varepsilon$, then stop else go to 5.

THEOREM 4.2 Algorithm MAXLINMAX is correct and the number of iterations before termination is

$$O\left(\log_2 \frac{U - L'}{\varepsilon}\right).$$

Proof. Correctness follows from Proposition 3.4 and Lemma 3.4. By Lemma 3.4 and Corollary 3.6, $U \geqslant f(x^0) \geqslant L'$ and thus the number of iterations is $O\left(\log_2 \frac{U-L'}{\varepsilon}\right)$ since after every iteration the interval of uncertainty is halved. □

5. The integer case

The algorithms of Section 4 may immediately be applied to $\mathrm{MLP}^{\min}$ or $\mathrm{MLP}^{\max}$ when all input data are integers. However, we show that in such a case $f^{\min}$ and $f^{\max}$ are integers and therefore, the algorithms find an ‘exact’ solution once the interval of uncertainty is of length 1 since then either $L(r)$ or $U(r)$ is the optimal value. Note that L and U are now integers and we will show how integrality of $L(r)$ and $U(r)$ can be maintained during the run of the algorithms. This implies that the algorithms will find exact optimal solutions in a finite number of steps and we will prove that their computational complexity is pseudo-polynomial.

THEOREM 5.1 If A, B, c, d and f are integers, $S \neq \emptyset$ and $f^{\min} > -\infty$, then $f^{\min} \in \mathbb{Z}$ (and therefore, $S^{\min} \cap \mathbb{Z}^n \neq \emptyset$).

Proof. Suppose $f^{\min} \notin \mathbb{Z}$ and let $z = (z_1, \ldots, z_n)^T \in S^{\min}$. We assume again without loss of generality that $c \geqslant d$. For any $x \in \mathbb{R}^n$, denote

$$F(x) = \{j \in N;\ f_j \otimes x_j = f(x)\}.$$

Hence, we have

$$z_j \notin \mathbb{Z} \quad \text{for every } j \in F(z). \tag{11}$$

We will now show that all z_j, $j \in F(z)$, can be reduced while maintaining feasibility which will be a contradiction with optimality of z. To prove this, we develop a special procedure called the reduction algorithm. Let us first denote the following for $x \in \mathbb{R}^n$:

$$Q(x) = \{i \in M; (A \otimes x)_i > c_i\}$$

and for $i \in M$ and $x \in \mathbb{R}^n$,

$$T_i(x) = \{j \in N; a_{ij} \otimes x_j = (A \otimes x)_i\},$$

$$R_i(x) = \{j \in N; b_{ij} \otimes x_j = (B \otimes x)_i\}.$$

Since all entries are integers, $a_{ij} \otimes z_j = c_i$ cannot hold for any $i \in M$ and $j \in F(z)$ and if $a_{ij} \otimes z_j < c_i$ for every $i \in M$ and $j \in F(z)$, then all z_j, $j \in F(x)$, could be reduced without violating any equation which contradicts the optimality of z. Hence, $Q(z) \neq \emptyset$.

Reduction algorithm

1. $P(z) := F(z)$.
2. $E_1 := \{i \in Q(z); T_i(z) \subseteq P(z) \text{ and } R_i(z) \nsubseteq P(z)\}$,
   $E_2 := \{i \in Q(z); T_i(z) \nsubseteq P(z) \text{ and } R_i(z) \subseteq P(z)\}$.
3. If $E_1 \cup E_2 = \emptyset$, then $P(z)$ is the set of indices of variables to be reduced, STOP.
4. $P(z) := P(z) \cup \cup_{i \in E_1} (R_i(z) \setminus P(z)) \cup \cup_{i \in E_2} (T_i(z) \setminus P(z))$.
5. Go to 2.

Claim: Reduction algorithm terminates after a finite number of steps and at termination,

$$z_j \notin \mathbb{Z} \quad \text{for } j \in P(z). \tag{12}$$

Proof of claim: Finiteness follows from the fact that the set $P(z)$ strictly increases in size at every iteration and $P(z) \subseteq N$. For the remaining part of the claim, it is sufficient to prove the following for any iteration of this algorithm: if (12) holds at Step 2, then it is also true at Step 5. The statement then follows from the fact that (12) is true when Step 2 is reached for the first time due to Step 1 and assumption (11). Consider therefore a fixed iteration at the beginning of which (12) holds. Suppose without loss of generality that $E_1 \cup E_2 \neq \emptyset$ and take any $i \in E_1$. Hence, $z_j \notin \mathbb{Z}$ for $j \in T_i(z)$, thus $(A \otimes z)_i \notin \mathbb{Z}$. But $i \in Q(z)$, implying $(B \otimes z)_i = (A \otimes z)_i$ and so $(B \otimes z)_i \notin \mathbb{Z}$ too. Since b_{ij} are also integers, this yields that $z_j \notin \mathbb{Z}$ for $j \in R_i(z)$. Therefore, $z_j \notin \mathbb{Z}$ for $j \in \cup_{i \in E_1} (R_i(z) \setminus P(z))$. Similarly, $z_j \notin \mathbb{Z}$ for $j \in \cup_{i \in E_2} (T_i(z) \setminus P(z))$ and the claim follows.

If $i \in M \setminus Q(z)$, then by integrality of the entries, both $a_{ij} \otimes z_j < c_i$ and $b_{ij} \otimes z_j < c_i$ for $j \in P(z)$. We conclude that all z_j for $j \in P(z)$ can be reduced without violating any of the equations, a contradiction with optimality of z.

Hence, $f^{\min} \in \mathbb{Z}$. The existence of an integer optimal solution now follows from Corollary 3.1. □

THEOREM 5.2 If A, B, c, d and f are integers, $S \neq \emptyset$ and $f^{\max} < +\infty$, then $f^{\max} \in \mathbb{Z}$ (and therefore, $S^{\max} \cap \mathbb{Z}^n \neq \emptyset$).

Proof. (Sketch) The proof follows the ideas of the proof of Theorem 5.1. We suppose $c \geqslant d$, $f^{\max} \notin \mathbb{Z}$ and let $z = (z_1, \ldots, z_n)^T \in S^{\max}$. We take one fixed $j \in F(z)$ (hence $z_j \notin \mathbb{Z}$) and show that it is possible to increase z_j without violating equality in any of the equations. Similarly as in the proof of Theorem 5.1, it is shown that the increase of z_j only forces the non-integer components of z to increase. Due to integrality of all entries, it is not possible that the equality in an equation is achieved by both integer and non-integer components of z. At the same time, an equality of the form $(A \otimes z)_i = c_i$ (if any) cannot be attained by non-integer components, thus $a_{ij} \otimes z_j < c_i$ and $b_{ij} \otimes z_j < c_i$ whenever $z_j \notin \mathbb{Z}$ and hence there is always scope for an increase of $z_j \notin \mathbb{Z}$. The rest of the argument is the same as in the proof of Theorem 5.1. □

Integer modifications of the algorithms are now straightforward since L, L' and U are also integers: we only need to ensure that the algorithms start from an integer vector (see Theorem 3.2) and that the integrality of both ends of the intervals of uncertainty is maintained, for instance, by taking one of the integer parts of the middle of the interval.

We start with the minimization. Note that

$$L, L', U \in [-3K, 3K], \tag{13}$$

where K is defined by (7).

ALGORITHM 5.1 INTEGER MAXLINMIN (integer max-linear minimization)

Input: $f = (f_1, \ldots, f_n)^T \in \mathbb{Z}^n, c = (c_1, \ldots, c_m)^T, d = (d_1, \ldots, d_m)^T \in \mathbb{Z}^m, c \geqslant d, c \neq d,$
$A = (a_{ij}), B = (b_{ij}) \in \mathbb{Z}^{m \times n}$.
Output: $x \in S^{\min} \cap \mathbb{Z}^n$.

1. If $L = f(x)$ for some $x \in S \cap \mathbb{Z}^n$, then stop ($f^{\min} = L$).
2. Find $x^0 \in S \cap \mathbb{Z}^n$. If $(A \otimes x^0)_i > c_i$ for all $i \in M$, then scale x^0 by α defined in (8).
3. $L(0) := L, U(0) := f(x^0), r := 0$.
4. $\alpha := \left\lceil \frac{1}{2}(L(r) + U(r)) \right\rceil$.
5. Check whether $f(x) = \alpha$ is satisfied by some $x \in S \cap \mathbb{Z}^n$ and in the positive case find one.
   If x exists, then $U(r+1) := \alpha$, $L(r+1) := L(r)$.
   If it does not, then $U(r+1) := U(r)$, $L(r+1) := \alpha$.
6. $r := r + 1$.
7. If $U(r) - L(r) = 1$, then stop ($U(r) = f^{\min}$) else go to 4.

THEOREM 5.3 Algorithm INTEGER MAXLINMIN is correct and terminates after using $O(mn(m + n)K \log K)$ operations.

Proof. Correctness follows from the correctness of MAXLINMIN and Theorem 5.1. For computational complexity, first note that the number of iterations is $O(\log(U - L)) \leqslant O(\log 6K) = O(\log K)$. The computationally prevailing part of the algorithm is the checking whether $f(x) = \alpha$ for some $x \in S \cap \mathbb{Z}^n$ when α is given. By Corollary 3.2, this can be done using $O(mn(m + n)K')$ operations, where $K' = \max(K + 1, |\alpha|)$. Since $\alpha \in [L, U]$, using (13), we have $K' = O(K)$. Hence, the computational complexity of checking whether $f(x) = \alpha$ for some $x \in S \cap \mathbb{Z}^n$ is $O(mn(m + n)K)$ and the statement follows. □

ALGORITHM 5.2 INTEGER MAXLINMAX (integer max-linear maximization)

Input: $f = (f_1, \ldots, f_n)^T \in \mathbb{Z}^n, c = (c_1, \ldots, c_m)^T, d = (d_1, \ldots, d_m)^T \in \mathbb{Z}^m,$

$A = (a_{ij}), B = (b_{ij}) \in \mathbb{Z}^{m\times n}$.
Output: $x \in S^{\max} \cap \mathbb{Z}^n$ or an indication that $f^{\max} = +\infty$.

1. If $U = f(x)$ for some $x \in S \cap \mathbb{Z}^n$, then stop ($f^{\max} = U$).
2. Check whether $A \otimes x = B \otimes x$ has a solution. If yes, stop ($f^{\max} = +\infty$).
3. Find an $x^0 \in S \cap \mathbb{Z}^n$ and set $x^0 := x^0 \oplus h'$, where h' is as defined in (10).
4. $L(0) := f(x^0), U(0) := U, r := 0$.
5. $\alpha := \left\lfloor \frac{1}{2}(L(r) + U(r)) \right\rfloor$.
6. Check whether $f(x) = \alpha$ is satisfied by some $x \in S \cap \mathbb{Z}^n$ and in the positive case find one.
   If x exists, then $U(r+1) := U(r), L(r+1) := \alpha$.
   If not, then $U(r+1) := \alpha, L(r+1) := L(r)$.
7. $r := r + 1$.
8. If $U(r) - L(r) = 1$, then stop ($L(r) = f^{\max}$) else go to 5.

THEOREM 5.4 Algorithm INTEGER MAXLINMAX is correct and terminates after using $O(mn(m + n)K \log K)$ operations.

Proof. Correctness follows from the correctness of MAXLINMAX and Theorem 5.2. The computational complexity part follows the lines of the proof of Theorem 5.3 after replacing L by L'. □

6. An example

Let us consider the MLP (minimization) in which

$$f = (3, 1, 4, -2, 0)^T,$$

$$A = \begin{pmatrix} 17 & 12 & 9 & 4 & 9 \\ 9 & 0 & 7 & 9 & 10 \\ 19 & 4 & 3 & 7 & 11 \end{pmatrix},$$

$$B = \begin{pmatrix} 2 & 11 & 8 & 10 & 9 \\ 11 & 0 & 12 & 20 & 3 \\ 2 & 13 & 5 & 16 & 4 \end{pmatrix}$$

$$c = \begin{pmatrix} 12 \\ 15 \\ 13 \end{pmatrix}, \quad d = \begin{pmatrix} 12 \\ 12 \\ 3 \end{pmatrix}$$

and the starting vector is

$$x^0 = (-6, 0, 3, -5, 2)^T.$$

Clearly, $f(x^0) = 7$, $M^> = \{2, 3\}$ and the lower bound is

$$\begin{aligned} L &= \max_{r\in M^>} \min_{k\in N} f_k \otimes c_r \otimes b_{rk}^{-1} \\ &= \max(\min(7, 16, 7, -7, 12), \min(14, 1, 12, -5, 9)) = -5. \end{aligned}$$

We now make a record of the run of INTEGER MAXLINMIN for this problem.

Iteration 1: Check whether $L = -5$ is attained by $f(x)$ for some $x \in S$ by solving the system

$$\begin{pmatrix} 17 & 12 & 9 & 4 & 9 & 12 \\ 9 & 0 & 7 & 9 & 10 & 15 \\ 19 & 4 & 3 & 7 & 11 & 13 \\ 3 & 1 & 4 & -2 & 0 & -6 \end{pmatrix} \otimes w = \begin{pmatrix} 2 & 11 & 8 & 10 & 9 & 12 \\ 11 & 0 & 12 & 20 & 3 & 12 \\ 2 & 13 & 5 & 16 & 4 & 3 \\ 2 & 0 & 3 & -3 & -1 & -5 \end{pmatrix} \otimes w.$$

There is no solution, hence $L(0) := -5, U(0) := 7, r := 0$ and $\alpha := 1$.

Check whether $f(x) = 1$ is satisfied by some $x \in S$ by solving

$$\begin{pmatrix} 17 & 12 & 9 & 4 & 9 & 12 \\ 9 & 0 & 7 & 9 & 10 & 15 \\ 19 & 4 & 3 & 7 & 11 & 13 \\ 3 & 1 & 4 & -2 & 0 & 0 \end{pmatrix} \otimes w = \begin{pmatrix} 2 & 11 & 8 & 10 & 9 & 12 \\ 11 & 0 & 12 & 20 & 3 & 12 \\ 2 & 13 & 5 & 16 & 4 & 3 \\ 2 & 0 & 3 & -3 & -1 & 1 \end{pmatrix} \otimes w.$$

There is a solution $x = (-6, 0, -3, -5, 1)^T$. Hence, $U(1) := 1, L(1) := -5, r := 1$ and $U(1) - L(1) > 1$.

Iteration 2: Check whether $f(x) = -2$ is satisfied by some $x \in S$ by solving

$$\begin{pmatrix} 17 & 12 & 9 & 4 & 9 & 12 \\ 9 & 0 & 7 & 9 & 10 & 15 \\ 19 & 4 & 3 & 7 & 11 & 13 \\ 3 & 1 & 4 & -2 & 0 & -3 \end{pmatrix} \otimes w = \begin{pmatrix} 2 & 11 & 8 & 10 & 9 & 12 \\ 11 & 0 & 12 & 20 & 3 & 12 \\ 2 & 13 & 5 & 16 & 4 & 3 \\ 2 & 0 & 3 & -3 & -1 & -2 \end{pmatrix} \otimes w.$$

There is no solution. Hence, $U(2) := 1, L(2) := -2, r := 2$ and $U(2) - L(2) > 1$.

Iteration 3: Check whether $f(x) = 0$ is satisfied by some $x \in S$ by solving

$$\begin{pmatrix} 17 & 12 & 9 & 4 & 9 & 12 \\ 9 & 0 & 7 & 9 & 10 & 15 \\ 19 & 4 & 3 & 7 & 11 & 13 \\ 3 & 1 & 4 & -2 & 0 & -1 \end{pmatrix} \otimes w = \begin{pmatrix} 2 & 11 & 8 & 10 & 9 & 12 \\ 11 & 0 & 12 & 20 & 3 & 12 \\ 2 & 13 & 5 & 16 & 4 & 3 \\ 2 & 0 & 3 & -3 & -1 & 0 \end{pmatrix} \otimes w.$$

There is no solution. Hence, $U(3) := 1, L(3) := 0, U(1) - L(1) = 1$, stop, $f^{\min} = 1$, an optimal solution is $x = (-6, 0, -3, -5, 1)^T$.

7. An easily solvable special case

One-sided systems of max-linear equations have been studied for many years and they are very well understood (Cuninghame-Green, 1979, 1995; Zimmermann, 1976; Butkovic, 2003). Note that a one-sided system is a special case of a two-sided system (5) where $a_{ij} > b_{ij}$ and $c_i < d_i$ for every i and j. Not surprisingly, MLPs with one-sided constraints have also been known for some time (Zimmermann, 1976). Here, we present this special case for the sake of completeness.

Let us consider one-sided systems of the form

$$A \otimes x = b, \tag{14}$$

where $A = (a_{ij}) \in \mathbb{R}^{m\times n}$ and $b = (b_1, \ldots, b_m)^T \in \mathbb{R}^m$. These systems can be solved more easily than their linear algebraic counterparts. One of the methods follows from the next theorem in which $S = \{x \in \mathbb{R}^n; A \otimes x = b\}$.

THEOREM 7.1 Let $\overline{x} = (\overline{x}_1, \ldots, \overline{x}_n)^\top$, where $\overline{x}_j = \min_{i\in M} b_i \otimes a_{ij}^{-1}$ for $j \in N$. Then,

(a) $x \leqslant \overline{x}$ for every $x \in S$ and

(b) $x \in S$ if and only if $x \leqslant \overline{x}$ and

$$\bigcup_{j:\, x_j = \overline{x}_j} M_j = M,$$

where for $j \in N$,

$$M_j = \{i \in M; \overline{x}_j = b_i \otimes a_{ij}^{-1}\}.$$

Proof. Can be found in standard texts on max-algebra (Cuninghame-Green, 1979; Heidergott *et al.*, 2005; Zimmermann, 1976). □

Suppose that $f = (f_1, \ldots, f_n)^T \in \mathbb{R}^n$ is given. The task of minimizing [maximizing] $f(x) = f^T \otimes x$ subject to (14) will be denoted by $\mathrm{MLP}_1^{\min}$ [$\mathrm{MLP}_1^{\max}$]. The sets of optimal solutions will be denoted $S_1^{\min}$ and $S_1^{\max}$, respectively. It follows from Theorem 7.1 and the isotonicity of $f(x)$ that $\overline{x} \in S_1^{\max}$. We now present a simple algorithm which solves $\mathrm{MLP}_1^{\min}$.

ALGORITHM 7.1 ONEMAXLINMIN (one-sided max-linear minimization)

Input: $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$.

Output: $x \in S_1^{\min}$.

1. Find $\overline{x}$ and M_j, $j \in N$.
2. Sort $(f_j \otimes \overline{x}_j;\ j \in N)$, without loss of generality let
$$f_1 \otimes \overline{x}_1 \leqslant f_2 \otimes \overline{x}_2 \leqslant \cdots \leqslant f_n \otimes \overline{x}_n.$$
3. $J := \{1\}, r = 1$.
4. If
$$\bigcup_{j\in J} M_j = M,$$
then stop ($x_j = \overline{x}_j$ for $j \in J$ and x_j small enough for $j \notin J$).
5. $r := r + 1$, $J := J \cup \{r\}$.
6. Go to 4.

THEOREM 7.2 Algorithm ONEMAXLINMIN is correct and its computational complexity is $O(mn^2)$.

Proof. Correctness is obvious and computational complexity follows from the fact that the loop 4–6 is repeated at most n times and each run is $O(mn)$. Step 1 is $O(mn)$ and Step 2 is $O(n \log n)$. □

Note that the problem of minimizing the function $2^{x_1} + 2^{x_2} + \cdots + 2^{x_n}$ subject to one-sided max-linear constraints is *NP*-complete since the classical minimum set-covering problem (MSCP) can be formulated as a special case of this problem with matrix A over $\{0, -1\}$ and $b = 0$. Indeed, given a finite set $M = \{v_1, \ldots, v_m\}$ and a collection $M_1, \ldots, M_n$ of its subsets, consider the 'MSCP' for this system, i.e. the task of finding the smallest k such that

$$M_{i_1} \cup \cdots \cup M_{i_k} = M$$

for some $i_1, \ldots, i_k \in \{1, \ldots, n\}$. MSCP is known to be *NP*-complete (Rosen *et al.*, 2000). Let Q be the minimization problem

$$f(x) = 2^{x_1} + \cdots + 2^{x_n} \to \min$$

subject to

$$A \otimes x = b,$$

where $A = (a_{ij}) \in \mathbb{R}^{m \times n}$, $b = 0 \in \mathbb{R}^m$ and

$$a_{ij} = \begin{cases} 0, & \text{if } i \in M_j, \\ -1, & \text{otherwise.} \end{cases}$$

It follows from Theorem 7.1 that at every local minimum $x = (x_1, \ldots, x_n)^\top$, every x_j is either 0 or $-\infty$ and

$$\bigcup_{x_j = 0} M_j = M.$$

Thus, every local minimum x corresponds to a covering of M and the value $f(x)$ is the number of subsets used in this covering. Therefore, Q is polynomially equivalent to MSCP.

Note also that some results of this paper may be extended to the case when the objective function is 'isotone', i.e. $f(x) \leqslant f(y)$ whenever $x \leqslant y$. This generalization is beyond the scope of the present paper.

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