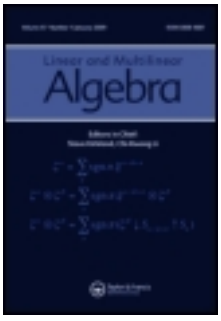

Linear and Multilinear Algebra
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/glma20

Z-matrix equations in max-algebra, nonnegative linear algebra and other semirings

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Available online: 06 Feb 2012

To cite this article: Peter Butkovič, Hans Schneider & Sergeĭ Sergeev (2012): Z-matrix equations in max-algebra, nonnegative linear algebra and other semirings, Linear and Multilinear Algebra, DOI:10.1080/03081087.2012.656107

To link to this article: http://dx.doi.org/10.1080/03081087.2012.656107

Linear and Multilinear Algebra
2012, 1–20, iFirst

Z-matrix equations in max-algebra, nonnegative linear algebra and other semirings

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Communicated by C.-K. Li

(*Received 30 September 2011; final version received 6 January 2012*)

We study the max-algebraic analogue of equations involving Z-matrices and M-matrices, with an outlook to a more general algebraic setting. We show that these equations can be solved using the Frobenius trace-down method in a way similar to that in nonnegative linear algebra [G.F. Frobenius, Über Matrizen aus nicht negativen Elementen. *Sitzungsber. Kön. Preuss. Akad. Wiss.*, 1912, in Ges. Abh., Vol. 3, Springer, 1968, pp. 546–557; D. Hershkowitz and H. Schneider, *Solutions of Z-matrix equations*, Linear Algebra Appl. 106 (1988), pp. 25–38; H. Schneider, *The influence of the marked reduced graph of a nonnegative matrix on the Jordan form and on related properties: A survey*, Linear Algebra Appl. 84 (1986), pp. 161–189], characterizing the solvability in terms of supports and access relations. We give a description of the solution set as combination of the least solution and the eigenspace of the matrix, and provide a general algebraic setting in which this result holds.

Keywords: max-algebra; nonnegative linear algebra; idempotent semiring; Z-matrix equations; Kleene star

AMS Subject Classifications: 15A80; 15A06; 15B48

1. Introduction

A Z-matrix is a square matrix of the form $\lambda I - A$ where λ is real and A is an (elementwise) nonnegative matrix. It is called an M-matrix if $\lambda \geq \rho(A)$, where $\rho(A)$ is the Perron root (spectral radius) of A and it is nonsingular if and only if $\lambda > \rho(A)$. Since their introduction by Ostrowski [28] M-matrices have been studied in many papers and they have found many applications. The term Z-matrix was introduced by Fiedler–Pták [11].

Results on the existence, uniqueness and nonnegativity of a solution x of the equation $(\lambda I - A)x = b$ for a given nonnegative vector b appear in many places (e.g. Berman-Plemmons [4] in the case of a nonsingular M-matrix or an irreducible singular M-matrix). Using the Frobenius normal form of A and access relation

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ISSN 0308–1087 print/ISSN 1563–5139 online

http://dx.doi.org/10.1080/03081087.2012.656107
http://www.tandfonline.com

defined by the graph of the matrix, Carlson [8] studied the existence and uniqueness of nonnegative solutions x of this equation in the case of a reducible singular M-matrix, and his results were generalized to all Z-matrices in Hershkowitz–Schneider [19].

The purpose of this article is to prove corresponding results in the max-times algebra of nonnegative matrices, unifying and comparing them with the results in the classical nonnegative linear algebra. We also notice that the basic proof techniques are much more general. In particular, we exploit a generalization of the Frobenius trace-down method [12,32]. This generalization is reminiscent of the universal algorithms developed by Litvinov et al. [24,25,27], based on the earlier works on regular algebra applied to path-finding problems by Backhouse et al. [2,30]. Following this line allows to include other examples of idempotent semirings, such as max–min algebra [15] and distributive lattices [34]. A more general theoretic setup is described in Section 4. It is very close to Cohen et al. [9] and Litvinov et al. [26].

The main object of our study is Z-matrix equation

$$Ax + b = \lambda x \tag{1}$$

over semirings. In the classical nonnegative algebra and max-plus algebra, any $\lambda \neq 0$ is invertible, which allows to reduce (1) to

$$Ax + b = x. \tag{2}$$

In max-plus algebra, this equation is sometimes referred to as discrete Bellman equation, being related to the Bellman optimality principle and dynamic programming [1,17,20]. In particular, it is very well-known that this equation has the least solution. However (to the authors' knowledge) a universal and complete description of solutions of $\lambda x = Ax + b$ or even $x = Ax + b$, which would cover both classical nonnegative and max-plus algebra cases, is not found (surprisingly) in the key monographs on max-plus algebra and related semirings. Such a description is what we try to achieve in this article, see Theorems 3.2 and 3.5. In brief, the results in the case of max times linear algebra are similar to those in classical Z-matrix theory [19], but they are not identical with them. Details are given in the main sections. The situation is analogous to that for the Perron–Frobenius equation $Ax = \lambda x$, as may be seen by comparing the results in Butkovič [7, Sect. 4.5], for max-algebra with those in Hershkowitz and Schneider [18, Sect. 3], for classical nonnegative algebra.

The rest of this article consists of Prerequisites (Section 2), theory of Z-matrix equations (Section 3) and Algebraic generalization (Section 4). Prerequisites are devoted to the general material, mostly about max-plus algebra: Kleene star, Frobenius normal forms and spectral theory in the general (reducible) case. Theory of Z-matrix equations over max-plus algebra stands on two main results. Theorem 3.2 describes the solutions of (2) as combinations of the least solution A^*b and the eigenvector space. We emphasize the algebraic generality of the argument. Theorem 3.5 exploits the Frobenius trace-down method. This method serves both for theoretic purposes (to provide a necessary and sufficient condition for the existence of solutions, and to characterize the support of the least solution) and as an algorithm for calculating it. As an outcome, we get both combinatorial and geometric description of the solutions. The results in max-times algebra are compared with the case of nonnegative matrix algebra [19]. This article ends with

Section 4, devoted to an abstract algebraic setting for which Theorem 3.2 holds, in the framework of semirings, distributive lattices and lattice-ordered groups [5,16].

We use the conventional arithmetic notation $a+b$ and ab for the operations in semirings, to emphasize the common features of the problem over classical nonnegative algebra and in the idempotent semirings, viewing max-times algebra (isomorphic to max-plus algebra) as our leading example.

We note that a complete description of solutions of $x = Ax + b$ was also achieved by Krivulin [21–23], for the case of max-algebra and related semirings with idempotent addition ($a \oplus a = a$). His proof of Theorem 3.2, recalled here in a remark following that theorem, is different from the one found by the authors. We show that Krivulin's proof also works both for max-algebra and nonnegative linear algebra, and admits further algebraic generalizations. The case of reducible matrix A and general support of b has also been investigated, see [21, Theorem 2] or [23, Theorem 3.2], which can be seen as a corollary of Theorem 3.5 of this article with application of the max-plus spectral theory, see Theorem 2.2.

2. Prerequisites

2.1. *Kleene star and the optimal path problem*

The main motivation of this article is to unite and compare the Z-equation theory in the classical nonnegative linear algebra, and the max-times linear algebra. Algebraically, these structures are semirings [16] (roughly speaking, 'rings without subtraction', see Section 4 for a rigorous definition). Thus we are focused on:

Example 1 (Max-times algebra) Nonnegative numbers, endowed with the usual multiplication $\times$ and the unusual addition $a+b := \max(a,b)$.

Example 2 (Usual nonnegative algebra) Nonnegative numbers, endowed with usual arithmetics $+$, $\times$.

Some results in this article will have a higher level of generality, which we indicate by formulating them in terms of a 'semiring $\mathcal{S}$'. Namely, this symbol '$\mathcal{S}$' applies to a more general algebraic setting provided in Section 4, covering the max-times algebra and the nonnegative algebra. Before reading the last section, it may be assumed by the reader that $\mathcal{S}$ means just 'max-times or usual nonnegative'.

The matrix algebra over a semiring $\mathcal{S}$ is defined in the usual way, by extending the arithmetical operations to matrices and vectors, so that $(A+B)_{ij} = a_{ij} + b_{ij}$ and $(AB)_{ik} = \sum_j a_{ij} b_{jk}$ for matrices A, B of appropriate sizes. The unit matrix (with 1's on the diagonal and 0's off the diagonal) plays the usual role.

Denote $N = \{1, \ldots, n\}$. For $x \in \mathcal{S}^n$, $x > 0$ means $x_i > 0$ for every i. Similarly $A > 0$ for $A \in \mathcal{S}^{n \times n}$. We also denote:

$$A^* = I + A + A^2 + \cdots = \sup_{k \geq 0}(I + A + \cdots + A^k). \tag{3}$$

In (3), we have exploited the nondecreasing property of addition. A^* is also called the *Kleene star*, and is related to the optimal path problem in the following way.

The *digraph associated with* $A = (a_{ij}) \in \mathcal{S}^{n \times n}$ is $D_A = (N, E)$, where $E = \{(i,j); a_{ij} > 0\}$. The weight of a path on D_A is defined as the product of the weights of the arcs, i.e., the corresponding matrix entries. It is easy to check (using the distributivity law)

that $(A^k)_{ij}$ is the sum of the weights of all paths of length k connecting i to j. Further, an entry $(A^*)_{ij}$ collects in a common summation (possibly divergent and formal) all weights of the paths connecting i to j, when $i \neq j$.

Note that $A^* = (I - A)^{-1}$ in the case of the classical arithmetics, and A^* solves the optimal path problem in the case of the max-times algebra (because the summation is maximum).

Thus the Kleene star can be described in terms of paths or access relations in D_A. For $i \neq j$, we say that i *accesses* j, denoted $i \to j$, if there is a path of nonzero weight connecting i to j, equivalently, $(A^*)_{ij} \neq 0$. We also postulate that $i \to i$. The notion of access is extended to subsets of N, namely $I \to J$ if $i \to j$ for some $i \in I$ and $j \in J$.

Both in max-plus algebra and in nonnegative algebra the Kleene star series may diverge to infinity (in other words, be unbounded). In both cases the convergence is strongly related to the largest eigenvalue of A (w.r.t. the eigenproblem $Ax = \lambda x$), which we denote by $\rho(A)$. This is also called the *Perron root* of A. A necessary and sufficient condition for the convergence is $\rho(A) < 1$ in the case of the ordinary algebra, and $\rho(A) \leq 1$ in the case of the max-times algebra. In the max-times algebra (but not in the usual algebra) A^* can be always truncated, meaning $A^* = I + A + \cdots + A^{n-1}$ where n is the dimension of A, in the case of convergence. This is due to the finiteness of D_A and the optimal path interpretation, see [1,7] for more details.

In the case of max-times algebra, $\rho(A)$ is equal to the *maximum geometric cycle mean* of A, namely

$$\rho(A) = \max\left\{\sqrt[k]{a_{i_1 i_2} a_{i_2 i_3} \ldots a_{i_k i_1}};\ i_1, \ldots, i_k \in N, k = 1, 2, \ldots, \right\}.$$

This quantity can be computed in $O(n^3)$ time by Karp's algorithm, see e.g. [1,7].

2.2. *Frobenius normal form*

$A = (a_{ij}) \in \mathbb{R}_+^{n \times n}$ is called *irreducible* if $n = 1$ or for any $i, j \in N$ there are $i_1 = i$, $i_2, \ldots, i_k = j$, such that $a_{i_1 i_2} a_{i_2 i_3} \ldots a_{i_{k-1} i_k} > 0$; A is called *reducible* otherwise. In other words, a matrix is called irreducible if the associated graph is strongly connected. Note that if $n > 1$ and A is irreducible then $A \neq 0$. Hence the assumption 'A irreducible, $A \neq 0$' merely means that A is irreducible but not the 1×1 zero matrix. (It is possible to extend these notions to general semirings with no zero divisors, but we will not require these in this article.)

In order to treat the reducible case for max-times algebra and the (classical) nonnegative linear algebra, we recall some standard notation and the Frobenius normal form (considering it for general semirings will be of no use here). If

$$1 \leq i_1 < i_2 < \cdots < i_k \leq n, \quad K = \{i_1, \ldots, i_k\} \subseteq N$$

then A_{KK} denotes the *principal submatrix*

$$\begin{pmatrix} a_{i_1 i_1} & \ldots & a_{i_1 i_k} \\ \ldots & \ldots & \ldots \\ a_{i_k i_1} & \ldots & a_{i_k i_k} \end{pmatrix}$$

of the matrix $A = (a_{ij})$ and x_K denotes the subvector $(x_{i_1}, \ldots, x_{i_k})^T$ of the vector $x = (x_1, \ldots, x_n)^T$.

If $D=(N,E)$ is a digraph and $K\subseteq N$ then $D(K)$ denotes the *induced subgraph* of D, that is

$$D(K) = (K, E \cap (K \times K)).$$

Observe that $\rho(A)=0$ if and only if D_A is acyclic.

Every matrix $A = (a_{ij}) \in \mathbb{R}_+^{n\times n}$ can be transformed by simultaneous permutations of the rows and columns in linear time to a *Frobenius normal form* [29]

$$\begin{pmatrix} A_{11} & 0 & \dots & 0 \\ A_{21} & A_{22} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ A_{r1} & A_{r2} & \dots & A_{rr} \end{pmatrix}, \tag{4}$$

where $A_{11},\dots,A_{rr}$ are irreducible square submatrices of A, corresponding to the partition $N_1\cup\dots\cup N_r=N$ (i.e. A_{ij} is a shortcut for $A_{N_iN_j}$). The sets $N_1,\dots,N_r$ will be called *classes* (*of* A). It follows that each of the induced subgraphs $D_A(N_i)$ $(i=1,\dots,r)$ is strongly connected and an arc from N_i to N_j in D_A exists only if $i\geq j$.

If A is in the Frobenius normal form (4) then the *reduced graph*, denoted $R(A)$, is the digraph whose nodes are the classes $N_1,\dots,N_r$ and the set of arcs is

$$\{(N_i, N_j); (\exists k \in N_i)(\exists \ell \in N_j) a_{k\ell} > 0\}).$$

In addition we postulate that each class has a self-loop (useful if FNF contains trivial classes consisting of one diagonal zero entry). In the max-times algebra and the nonnegative matrix algebra, the nodes of $R(A)$ are marked by the corresponding greatest eigenvalues (Perron roots) $\rho_i := \rho(A_{ii})$.

Simultaneous permutations of the rows and columns of A are equivalent to calculating $P^{-1}AP$, where P is a generalized permutation matrix. Such transformations do not change the eigenvalues, and the eigenvectors before and after such a transformation only differ by the order of their components. So when solving the eigenproblem, we may assume without loss of generality that A is in a Frobenius normal form, say (4).

2.3. *Eigenvalues and eigenvectors in max-times algebra*

It is intuitively clear that all eigenvalues of A are among the unique eigenvalues of diagonal blocks. However, not all of these eigenvalues are also eigenvalues of A. The following key result describing the set $\Lambda(A)$ of all eigenvalues of A in **max-times algebra** (i.e. set of all λ such that $Ax=\lambda x$ has a nonzero solution x in max-times algebra) appeared for the first time independently in Gaubert's thesis [13] and Bapat et al. [3], see also Butkovič [7].

THEOREM 2.1 (cf. [3,7,13]) *Let* (4) *be a Frobenius normal form of a matrix* $A \in \mathbb{R}_+^{n\times n}$. *Then*

$$\Lambda(A) = \{\rho_j;\ \rho_j \geq \rho_i \textit{ for all } N_i \longrightarrow N_j\}. \tag{5}$$

The same result holds in the nonnegative linear algebra, with the nonstrict inequality replaced by the strict one.

If a diagonal block A_{jj} has $\rho_j \in \Lambda$, it still may not satisfy the condition in Theorem 2.1 and may therefore not provide any eigenvectors. So it is necessary to identify classes j that satisfy this condition and call them *spectral*. Thus $\rho_j \in \Lambda(A)$ if N_j is spectral, but not necessarily the other way round. We can immediately deduce that all initial blocks are spectral, like in the nonnegative linear algebra. Also, it follows that the number of eigenvalues does not exceed n and obviously, $\rho(A) = \max_i \rho_i$, in accordance with $\rho(A)$ being the greatest eigenvalue.

We are now going to describe, for $\lambda \in \Lambda$, the eigencone $V(A, \lambda)$ of all vectors x such that $Ax = \lambda x$. Denote by J_λ the union of all classes N_i which have access to the spectral classes corresponding to this eigenvalue. By (5), $\rho_i \leq \lambda$ for all such classes. Now we define the *critical graph* $C_A(\lambda) = (N_c, E_c)$ comprising all nodes and edges on *critical cycles* of the submatrix $A_{J_\lambda J_\lambda}$, i.e., such cycles where λ is attained. This graph consists of several strongly connected components, and let $T(A, \lambda)$ denote a set of indices containing precisely one index from each component of $C_A(\lambda)$. In the following, $A'(J_\lambda)$ will denote the $n \times n$ matrix, which has $A_{J_\lambda J_\lambda}/\lambda$ as submatrix, and zero entries everywhere else.

Theorem 2.2 (cf. [3,7,13]) *Let $A \in \mathbb{R}_+^{n \times n}$ and $\lambda \in \Lambda(A)$. Then*

(a) *For any eigenvector $v \in V(A, \lambda)$ there exists $\alpha_j \in \mathbb{R}_+$ such that v is the max-times linear combination*

$$v = \sum_{j \in T(A,\lambda)} \alpha_j (A'(J_\lambda))^*_{\cdot j}. \tag{6}$$

(b) *For any two indices j and k in the same component of $C_A(\lambda)$, columns $(A'(J_\lambda))^*_{\cdot j}$ and $(A'(J_\lambda))^*_{\cdot k}$ are proportional.*

(c) *Vectors $(A'(J_\lambda))^*_{\cdot j}$ for $j \in T(A, \lambda)$ form a basis of $V(A, \lambda)$, that is, they generate $V(A, \lambda)$ in the sense of a) and none of them can be expressed as a max-times linear combination of the others.*

Remark 1 An analogous description of $V(A, \lambda)$ in nonnegative matrix algebra is called Frobenius–Victory theorem [12,35], see [32, Theorem 3.7]. Namely, to each spectral node of $R(A)$ with eigenvalue λ, there corresponds a unique eigenvector, with support equal to the union of classes having access to the spectral node. These eigenvectors are the extreme rays of the cone, i.e. they form a 'basis' in analogy with Theorem 2.2.

Moreover, these extreme rays are linearly independent as it may be deduced from their supports. In parallel, it can be shown that the generators of Theorem 2.2 are strongly regular, see [6] for definition (i.e. independent in a stronger sense).

However, extremals in the nonnegative case do not come from $A^* = (I - A)^{-1}$ and, to the authors' knowledge, no explicit algebraic expression for these vectors is known.

3. Theory of Z-matrix equations

3.1. *General results*

In the following, we describe the solution set of $x = Ax + b$ as combinations of the least solution A^*b and the eigenvectors of A. The results of this section hold for the max-times algebra, nonnegative linear algebra and an abstract algebraic setup

(reassuring that A^*b satisfies $x = Ax + b$, and $v = \inf_k A^k x$, for x such that $Ax \le x$, satisfies $Av = v$), which will be provided in Section 4.

We start with a well-known fact, that

$$A^*b := \sup_k (b + Ab + A^2b + \cdots + A^{k-1}b) \tag{7}$$

is the least solution to $x = Ax + b$. We will formulate it in the form of an equivalence. Note that the supremum (7) may exist even if A^* does not exist. (In this sense, A^*b is rather a symbol than a result of matrix-vector multiplication. On the other hand, one can complete a semiring with the greatest element '$+\infty$' and regard A^*b as a matrix-vector product.)

THEOREM 3.1 (Well-known, cf. [1,17]) *Let $A \in \mathcal{S}^{n\times n}$, $b \in \mathcal{S}^n$. The following are equivalent*:

(i) *$x = Ax + b$ has a solution*,
(ii) *$x = Ax + b$ has a least solution*.
(iii) *A^*b converges*.

If any of the equivalent statement holds, A^*b is the least solution of $x = Ax + b$.

Proof (i) $\Rightarrow$ (iii) Let x be a solution to $x = Ax + b$. Then

$$\begin{aligned} x &= Ax + b \\ &= A(Ax + b) + b \\ &= A(A(Ax + b) + b) + b = \cdots. \end{aligned}$$

Therefore for any $k \ge 1$ we have

$$x = A^k x + \left(A^{k-1} + A^{k-2} + \cdots + I\right)b. \tag{8}$$

This shows that the expressions in (7) are bounded from above by x, hence the supremum exists.

(iii) $\Rightarrow$ (ii) We verify that

$$\begin{aligned} AA^*b + b &= A \sup_k (b + Ab + \cdots + A^{k-1}b) + b \\ &= \sup_k (b + Ab + \cdots + A^k b) = A^*b, \end{aligned} \tag{9}$$

treating sup as a limit and using the continuity of the matrix-vector multiplication.

(From the algebraic point of view, we used the distributivity of (max-algebraic, nonnegative) matrix multiplication with respect to sup's of ascending chains, and the distributivity of + with respect to such sup's. Further details on this will be given in Section 4.)

(ii) $\Rightarrow$ (i) Trivial. ■

We proceed with characterizing the whole set of solutions. (See also Remark 2 for an alternative short proof of the first part.)

THEOREM 3.2 *Let $A \in \mathcal{S}^{n\times n}$, $b \in \mathcal{S}^n$ be such that $x = Ax + b$ has a solution. Then*

(a) *The set of all solutions to $x = Ax + b$ is $\{v + A^*b;\ Av = v\}$.*
(b) *If for any x such that $Ax \le x$ we have $\inf_k A^k x = 0$, then A^*b is the unique solution to $x = Ax + b$.*

Proof (a) First we need to verify that any vector of the form $v + A^*b$, where v satisfies $Av = v$, solves (2). Indeed,

$$A(v + A^*b) + b = Av + (AA^*b + b) = v + A^*b,$$

where we used that $Av = v$ and that A^*b is a solution of (2), see Theorem 3.1. It remains to deduce that each solution of (2) is as defined above.

Let x be a solution to (2), and denote $y^{(k)} := A^k x$ and

$$z^{(k)} := \left(A^{k-1} + A^{k-2} + \cdots + I\right)b.$$

We have seen in (8) that

$$x = y^{(k)} + z^{(k)}, \quad \text{for all } k \geq 1. \tag{10}$$

Since $Ax \leq x$ it follows that the sequence $y^{(k)}$ is nondecreasing. The sequence of $z^{(k)}$ is nonincreasing.

Both in max-times and in the nonnegative case, we conclude that $v = \lim_{k\to\infty} y^{(k)}$ exists and (by the continuity of A as operator) we have $Av = v$. We also obtain that $A^*b = \lim_{k\to\infty} z^{(k)}$, and finally

$$x = \lim_{k\to\infty} y^{(k)} + \lim_{k\to\infty} z^{(k)} = v + A^*b, \tag{11}$$

where v satisfies $Av = v$. *The theorem is proved*, both for max-times algebra and nonnegative linear algebra.

In a more general semiring context (see Section 4), it remains to show that $x := \inf_k y^{(k)} + \sup_k z^{(k)}$ is the same as $y^{(k)} + z^{(k)}$ for all k. After showing this we are done, since $\sup_k z^{(k)} = A^*b$, and also

$$A \inf_{k\geq 0} y^{(k)} = A\left(\inf_{k\geq 0} A^k x\right) = \inf_{k\geq 1} A^k x = \inf_{k\geq 0} y^{(k)}, \tag{12}$$

so that we can set $v := \inf_k y^{(k)}$, it satisfies $Av = v$. (From the algebraic point of view, we have used the distributivity of matrix multiplication with respect to inf's of descending chains. Further details will be given in Section 4.)

Using the distributivity of + with respect to inf we obtain

$$x = \inf_k \left(y^{(k)} + \sup_l z^{(l)}\right) \geq x, \tag{13}$$

since this is true of any term in the brackets. Using the distributivity with respect to sup we obtain

$$x = \sup_k (\inf_k y^{(k)} + z^{(l)}) \leq x, \tag{14}$$

for analogous reason. Combining (13) and (14) we obtain

$$x = x = \inf_k y^{(k)} + \sup_k z^{(k)} = v + A^*b,$$

which yields a general proof of part (a).

For part (b), recall that $y^{(k)} := A^k x$, and that x satisfies $Ax \leq x$. ■

These results also apply to equations $\lambda x = Ax + b$ when λ is invertible: it suffices to divide this equation by λ.

Remark 1 The solution set of $x = Ax + b$ is convex over $\mathcal{S}$, since it contains with any two points x, y all convex combinations $\lambda x + \mu y$, $\lambda + \mu = 1$. Further, *both in max-times semiring and in the nonnegative algebra*, A^*b is the only extreme point: it cannot be a convex combination of two solutions different from it. The eigenvectors of A are *recessive rays*, i.e., any multiple of such vectors can be added to any solution, and the result will be a solution again. Moreover, only eigenvectors have this property. Indeed, assume that z is a solution, $z + \mu v$ where $\mu \neq 0$ satisfies

$$z + \mu v = A(z + \mu v) + b,$$

but $Av \neq v$. In the usual algebra this is impossible. In max-times, assume that $(Av)_i \neq v_i$ for some i, then one of these is nonzero. As $z_i = (Az)_i + b_i$ is finite, taking large enough μ will make μv_i or $\mu(Av)_i$ the only maximum on both l.h.s. and r.h.s., in which case $z + \mu v$ will not be a solution. Thus, in both theories the eigencone of A with eigenvalue 1 is the recessive cone of the solution set of $x = Ax + b$. In the max-times case it is generated by the fundamental eigenvectors as in Theorem 2.2. Thus we have an example of the tropical theorem of Minkowski, representing closed max-times convex sets in terms of extremal points and recessive rays, as proved by Gaubert and Katz [14].

Remark 2 In this remark we recall the proof of Theorem 3.2 (a) given by Krivulin, see [21, Lemma 7] or [23, Lemma 3.5]. We slightly change the original proof to make it work also for nonnegative linear algebra. Let x be a solution of $x = Ax + b$ and define $\underline{w}$ as the least vector w satisfying $x = u + w$ where $u := A^*b$. It can be defined explicitly by

$$\underline{w}_i = \begin{cases} x_i, & \text{if } x_i > u_i, \\ 0, & \text{if } x_i = u_i, \end{cases} \quad \text{or} \quad \underline{w}_i = \begin{cases} x_i - u_i, & \text{if } x_i > u_i, \\ 0, & \text{if } x_i = u_i, \end{cases} \tag{15}$$

in the case of max-algebra and nonnegative linear algebra, respectively. Now notice that if $x = u + w$ then $x = u + Aw$. Indeed

$$x = A(u + w) + b = (Au + b) + Aw = u + Aw.$$

Hence $\underline{w} \leq A\underline{w}$. Indeed, both $w := \underline{w}$ and $w := A\underline{w}$ satisfy $x = u + w$ but $\underline{w}$ is the least such vector. Defining $\underline{v} := \sup_{n \geq 1} A^n \underline{w}$ we obtain $x = u + \underline{v}$ and $A\underline{v} = \underline{v}$. The algebraic generality of this argument is also quite high, it will be discussed in the last section of this article.

3.2. *Spectral condition, Frobenius trace-down method*

We consider equation $\lambda x = Ax + b$ in max-times algebra and nonnegative linear algebra, starting with the case when A is irreducible. Theorem 3.5 can also be viewed in nonnegative linear algebra, but only after some modification which will be described. Denote $A_\lambda := A/\lambda$ and $b_\lambda := b/\lambda$.

The following is a max-times version of the Collatz–Wielandt identity in the Perron–Frobenius theory.

Lemma 3.3 (Well-known, cf. [7,13]) *Let* $A \in \mathbb{R}_+^{n\times n}, A \neq 0$. *Then* $Ax \leq \lambda x$ *has a solution* $x > 0$ *if and only if* $\lambda \geq \rho(A)$, $\lambda > 0$.

Proof Let $x > 0$ be a solution, then $Ax \neq 0$ and so $\lambda > 0$. If $\rho(A) = 0$ there is nothing to prove, so we may suppose $\rho(A) > 0$. Let $\sigma = (i_1, \ldots, i_k, i_{k+1} = i_1)$ be any cycle with nonzero weight. Then

$$\begin{aligned} a_{i_1 i_2} x_{i_2} &\leq \lambda x_{i_1}, \\ a_{i_2 i_3} x_{i_3} &\leq \lambda x_{i_2}, \\ &\cdots \\ a_{i_k i_1} x_{i_1} &\leq \lambda x_{i_k}. \end{aligned}$$

After multiplying out and simplification we get $\lambda \geq \sqrt[k]{a_{i_1 i_2} a_{i_2 i_3} \ldots a_{i_k i_1}}$ and so $\lambda \geq \rho(A)$.

Suppose now $\lambda \geq \rho(A)$, $\lambda > 0$. Then $\rho(A_\lambda) \leq 1$ and so $A_\lambda^* = I + A_\lambda + \cdots + A_\lambda^k$ for every $k \geq n-1$, yielding $A_\lambda A_\lambda^* \leq A_\lambda^*$. Let u be any positive vector in $\mathbb{R}_+^n$. Take $x = A_\lambda^* u$, then $x > 0$ because $A_\lambda^* u \geq u$ and

$$A_\lambda x = A_\lambda A_\lambda^* u \leq A_\lambda^* u = x.$$

■

Lemma 3.4 (Well-known, cf. [7,28]) *If* $A \in \mathbb{R}_+^{n\times n}$ *is irreducible*, $b \in \mathbb{R}_+^n, b \neq 0$ *and* $\lambda > 0$ *then the following are equivalent*:

(i) $\lambda x = Ax + b$ *has a solution*.
(ii) A_λ^* *converges*.
(iii) $\lambda \geq \rho(A)$ (*max-times algebra*), $\lambda > \rho(A)$ (*nonnegative linear algebra*).

All solutions of $\lambda x = Ax + b$ (*if any*) *are positive*.

Proof In the case of nonnegative matrix algebra, this lemma follows from the results of Ostrowski's famous paper [28] (where matrices appear as determinants), see also [31, Lemma 5]. For the equivalence between (ii) and (iii) in max-times algebra, refer e.g. [1,7,17]. (Both in max-times algebra and in the nonnegative linear algebra, such equivalence holds also for reducible matrices.) For the reader's convenience we show the equivalence between (i) and (iii) in max-times algebra.

(iii) $\Rightarrow$ (i): If $\lambda \geq \rho(A)$ then $1 \geq \rho(A_\lambda)$, hence A_λ^* and $A_\lambda^* b_\lambda$ converge. In this case, $A_\lambda^* b_\lambda$ is the least solution by Theorem 3.1.
(i) $\Rightarrow$ (iii): If $Ax + b = \lambda x$ then $\lambda > 0$ and $x \neq 0$ since $b \neq 0$. We need to show that $x > 0$, to apply Lemma 3.3.

If $n = 1$ then the result holds. Suppose now $n > 1$, thus $\rho(A) > 0$. Let $B = (\rho(A))^{-1} A$ and $\mu = (\rho(A))^{-1}\lambda$. Then B has $\rho(B) = 1$, it is irreducible and $Bx \leq \mu x$. Therefore $B^* > 0$, thus $B^* x > 0$. But $B^* x \leq \mu x$ and hence $x > 0$. By Lemma 3.3 we obtain that $\lambda \geq \rho(A)$. ■

Remark 3 Note that also for general (reducible) A, if $b > 0$ then for any solution x of $\lambda x = Ax + b$ we have $x > 0$, and hence $\lambda \geq \rho(A)$ by Lemma 3.4. However, this condition is not necessary for the existence of a solution to $Ax + b = \lambda x$, when b has at least one zero component, see Theorem 3.5. If $\lambda < \rho(A)$ then some entries of A_λ^* are $+\infty$ and it is not obvious from Theorem 3.2 whether a finite solution exists since the

product $A_\lambda^* b_\lambda$ may in general (if λ is too low) contain $+\infty$. However if $0.(+\infty)$ is defined as 0 and the condition of Theorem 3.5 (iv) holds, then the $+\infty$ entries of A_λ^* in $A_\lambda^* b_\lambda$ will always be matched with zero components of b_λ, and consequently $A_\lambda^* b_\lambda$ will be a finite nonnegative vector.

Now we consider the general (reducible) case. The next result appears as the main result of this article, describing the solution sets to Z-equations in max-times algebra and nonnegative linear algebra.

THEOREM 3.5 *Let $A \in \mathbb{R}_+^{n\times n}$ be in FNF with classes $N_j, j=1,\dots,s$. Let $b \in \mathbb{R}_+^n, \lambda \geq 0$. Denote $J=\{j; N_j \to \mathrm{supp}(b)\}$ and $\overline{\rho} = \max_{j\in J} \rho_j$ (for the case when $b=0$ and $J=\emptyset$ assume that* $\max \emptyset = 0$). *The following are equivalent*:

(i) *System $\lambda x = Ax + b$ has a solution.*
(ii) *System $\lambda x = Ax + b$ has the least solution.*
(iii) *$x^{(0)} = A_\lambda^* b_\lambda$ converges.*
(iv) *If $j \in J$ then $(A_{jj})_\lambda^*$ converges.*
(v) *$\overline{\rho} \leq \lambda$ (max-times), or $\overline{\rho} < \lambda$ (nonnegative linear algebra).*

If any of the equivalent statements hold, then

(a) *x^0 is the least solution of $\lambda x = Ax + b$. For this solution, $x_{N_i}^0 \neq 0$ when $i \in J$ and $x_{N_i}^0 = 0$ when $i \notin J$. The solution x^0 is unique if and only if λ is not an eigenvalue of A.*
(b) *Any solution x of* (1) *can be expressed as $x = x^0 + v$ where v satisfies $Av = \lambda v$.*

Proof We first treat the trivial case, when $b=0$. In this case $x^0=0$ is a solution, $J=\emptyset, \overline{\rho}=0 \leq \lambda$ and thus all the equivalent statements (i)–(iv) are true; (a) and (b) hold trivially with $x^0=0$.

We now suppose $b \neq 0$. Consequently, $\lambda > 0$, and assume w.l.o.g. that $\lambda = 1$. The equivalence of (i)–(iii) was manifested in Theorem 3.1, and part b) was proved in Theorem 3.2. The equivalence of (iv) and (v) follows from Lemma 3.4. It remains to show the equivalence of (i) and (iv), that the minimal solution has a prescribed support, and the spectral criterion for uniqueness.

We show that (i) implies (iv). For simplicity we use the same symbol 'J' for the set of indices in the classes of J. Denote $I := \{1,\dots,n\}\backslash J$. We have

$$\begin{pmatrix} x_I \\ x_J \end{pmatrix} = \begin{pmatrix} A_{II} & 0 \\ A_{JI} & A_{JJ} \end{pmatrix} \begin{pmatrix} x_I \\ x_J \end{pmatrix} + \begin{pmatrix} 0 \\ b_J \end{pmatrix}, \tag{16}$$

and hence x_I is a solution of $A_{II}x_I = x_I$, and x_J is a solution of $x_J = A_{JJ}x_J + A_{JI}x_I + b_J$. Further, denote $b_J := A_{JI}x_I + b_J$ and let J consist of the classes $N_1,\dots,N_t (t \leq s)$. Then

$$A(J) = \begin{pmatrix} A_{11} & 0 & 0 & 0 \\ A_{21} & A_{22} & 0 & 0 \\ \cdots & \cdots & \ddots & 0 \\ A_{t1} & A_{t2} & \cdots & A_{tt} \end{pmatrix}.$$

We now proceed by an inductive argument, showing that $(A_{jj})^*$ converges for all $j=1,\dots,t$, and that all components in $x_{N_1},\dots,x_{N_t}$ are positive. This argument is a max-algebraic version of the Frobenius trace-down method.

As the base of induction, we have $x_{N_1} = A_{11}x_{N_1} + b_{N_1}$. In this case, the class N_1 is final, so b_{N_1} and hence b_{N_1} should have some positive components. Using Lemma 3.4, we conclude that $(A_{11})^*$ converges and x_{N_1} is positive.

Induction step Suppose that for some l, all components of $(x_{N_1}, \ldots, x_{N_l})$ solving

$$\begin{aligned} x_{N_1} &= A_{11}x_{N_1} + b_{N_1}, \\ x_{N_2} &= A_{21}x_{N_1} + A_{22}x_{N_2} + b_{N_2}, \\ \cdots &= \cdots \\ x_{N_l} &= A_{l1}x_{N_1} + \cdots + A_{ll}x_{N_l} + b_{N_l} \end{aligned} \tag{17}$$

are positive. We show that the same holds if we add the next equation

$$x_{N_{l+1}} = A_{l+1,1}x_{N_1} + \cdots + A_{l+1,l+1}x_{N_{l+1}} + b_{N_{l+1}}, \tag{18}$$

and that $(A_{l+1,l+1})^*$ converges. We have two cases: either $b_{N_{l+1}}$ has nonzero components so that N_{l+1} intersects with supp(b), or if not, N_{l+1} should access a class which intersects with supp(b). In this case, one of the submatrices $A_{l+1,1}, \ldots, A_{l+1,l}$ is not identically 0. As all components of $x_{N_1}, \ldots, x_{N_l}$ are positive, this shows that the sum on the r.h.s. of (18) excluding $A_{l+1,l+1}x_{N_{l+1}}$ has some positive components in any case. Using Lemma 3.4, we conclude that $(A_{l+1,l+1})^*$ converges and $x_{N_{l+1}}$ is positive.

Now we show that (iv) implies (i), and moreover, that there is a solution with prescribed support structure. To do so, we let $x_I = 0$ in (16). Then it is reduced to $x_J = A_{JJ}x_J + b_J$, and we have to show the existence of a positive solution x_J. The proof of this follows the lines of the Frobenius trace-down method described above, making the inductive assumption that (17) has a positive solution $(x_{N_1}, \ldots, x_{N_l})$ and using Lemma 3.4 to show that (18) can be solved with a positive $x_{N_{l+1}}$. Strictly speaking, in this case we have b instead of b in (17) and (18), but this does not make any change in the argument.

Let the conditions (i)–(v) be satisfied. Since letting $x_I = 0$ in (16) produces a solution (see above), the support of the least solution is contained in J. However, the support of *any* solution should contain J by the argument in the proof of (i) $\Rightarrow$ (iv).

Evidently, solution x^0 is unique if λ is not an eigenvalue of A. To show the converse (in max-times algebra), note that for any nonzero v there is a large enough α such that some component αv_i is greater than x_i^0, hence $x^0 + \alpha v \neq x^0$. (Note that the converse would be evident in the usual nonnegative algebra.)

The proof is complete. ■

Remark 4 (cf. [2,25]) The Frobenius trace-down method of Theorem 3.5 can also be viewed as a generalized block-triangular elimination algorithm for obtaining the least solution A^*b (assumed w.l.o.g. that $\lambda = 1$). Namely, if $(x_{N_1} \ldots x_{N_l})$ is the least solution of (17), then computing

$$x_{N_{l+1}} := (A_{l+1,l+1})^*(A_{l+1,1}x_{N_1} + \cdots + A_{l+1,l}x_{N_l} + b_{N_{l+1}}) \tag{19}$$

yields the least solution $(x_{N_1}, \ldots, x_{N_l}, x_{N_{l+1}})$ of the enlarged system (17) and (18) with b instead of $\tilde{b}$. Indeed, if we suppose that there is another solution $(x'_{N_1}, \ldots, x'_{N_{l+1}})$, then $x'_{N_i} \geq x_{N_i}$, and it follows from (19) that $x'_{N_{l+1}} \geq x_{N_{l+1}}$. As an algorithm for finding the least solution of $x = Ax + b$, it is valid even for more general semirings than the setting of Section 4, provided that a solution to $x = Ax + b$ exists.

Remark 5 The description of the support of x^0 as in Theorem 3.5 (a) can be obtained directly from the path interpretation of A^*, using that $x^0 = A^*b$ (when this is finite). Indeed, write $b = \sum_{k \in \mathrm{supp}(b)} \beta_k e_k$ where e_k is the kth unit vector and β_k are all positive. Hence $x^0 = A^*b = \sum_{k \in \mathrm{supp}(b)} \beta_k A^*_{\cdot k}$. It can be now deduced from the path interpretation of A^* that $x^0_l > 0$ whenever l accesses k from supp(b). This argument also shows that the description of the support of $x^0 = A^*b$ is valid over any semiring with no zero divisors. With zero divisors, the access condition for $x^0_l > 0$ may be no longer sufficient.

We have represented any solution x of $x = Ax + b$ in the form $x^0 + v$, where $x^0 = A^*b$ and $v = \inf_k A^k x$. Below we give an explicit formula for v in the case of max-times algebra, see, e.g., Dhingra and Gaubert [10].

Let C be the set of critical nodes (i.e., nodes of the critical graph) corresponding to the eigenvalue 1 (see Section 2.3). For any critical cycle $(i_1, \ldots, i_k)$ we either obtain $x_{i_l} = 0$ for all $l = 1, \ldots, k$, or both $x_{i_l} \neq 0$ and $a_{i_l i_{l+1}} x_{i_{l+1}} = x_{i_l}$ for all l (after multiplying all inequalities $a_{i_l i_{l+1}} x_{i_{l+1}} \leq x_{i_l}$ and cancelling $x_{i_1} \ldots x_{i_k}$ it turns out that any strict inequality causes $a_{i_1 i_2} \ldots a_{i_k i_1} < 1$). This implies $(Ax)_C = x_C$, for the critical subvectors of Ax and x. Applying A to $x^{(k)} = A^{(k)}x$ which also satisfies $Ax^{(k)} \leq x^{(k)}$ we obtain that $(A^k x)_C = x_C$ for any k, and hence also $v_C = x_C$.

It remains to determine the noncritical part of v. For this we expand the non-critical part of $Av = v$ as $v_N = A_{NC} v_C + A_{NN} v_N$. Forming A_{NN} corresponds to eliminating all spectral nodes of eigenvalue 1 from the reduced graph $R(A)$. The nonspectral nodes with eigenvalue 1 will remain nonspectral, hence A_{NN} does not have eigenvalue 1, and $(A_{NN})^* A_{NC} v_C$ is the only solution. Combining with the previous argument we conclude that

$$v_C = x_C, \quad v_N = (A_{NN})^* A_{NC} x_C. \tag{20}$$

3.3. *Max-times algebra and nonnegative linear algebra*

We can make further comparison with the survey of Schneider [32, Sect. 4], and with the work of Hershkowitz and Schneider [19] describing solutions of Z-matrix equations in nonnegative linear algebra. It can be seen that:

(1) In the nonnegative linear algebra, Theorem 3.5 extends the statements of [32, Theorem 4.3] and [32, Theorem 4.12] (due to Victory [35]). In particular, it gives an explicit formula for the least solution.
(2) Frobenius trace-down method is also used in the proof of [19, Proposition 3.6], and condition (v) of Theorem 3.5 (with the strict inequality) is equivalent to [19 condition (3.12)].
(3) As observed in [19, Theorem 3.16], in the case of nonnegative algebra, x^0 is the only vector with support J, because supp(b) cannot be accessed by the spectral classes of $R(A)$ in the case of solvability. However, this is not the case in the max-times algebra, making it possible that all spectral classes of $R(A)$ access supp(b). This condition is necessary and sufficient for all solutions of $\lambda x = Ax + b$ to have the same support as A^*b.

(4) It can be observed that geometric and combinatorial descriptions of the solution set in the usual nonnegative algebra, as provided by [19, Theorem 3.20 and Corollary 3.21], can be deduced from Theorem 3.5, with an application of Frobenius–Victory theorem, see remark after Theorem 2.2. Max-times analogues of these results of [19] can also be easily formulated.

We next give an example illustrating similarity and difference between the two theories. Let A be the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 2 \end{pmatrix}.$$

We note that this matrix is essentially the same as in [19, Example 3.22], that is, we have replaced $I-A$ by A and its (reduced) graph $R(A)$, given below on Figure 1, differs from the one in that Example 3.22 only by the markings of the nodes.

Let $b \in \mathbb{R}_+^7$. It follows from Theorem 3.5 (iv) that there exists a solution x to the max-times equation $Ax+b=x$ if and only if $\mathrm{supp}(b) \subseteq \{1,3,4,6\}$. In the usual nonnegative algebra, the condition is more restrictive: $\mathrm{supp}(b) \subseteq \{4,6\}$.

We choose

$$b = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}^T$$

as in [19, Example 3.22]. Then $\mathrm{supp}(b)=\{4\}$ and the minimal solution x^0 of $Ax+b=x$ has support $\{4, 6\}$ and equals

$$x^0 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}^T$$

in both theories.

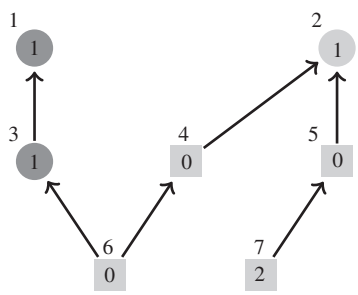

Figure 1. The marked (reduced) graph of matrix A. Circles correspond to the nodes with the greatest Perron root 1, the darker ones being spectral. Each node is marked by its Perron root (inside) and by its number (outside).

In max-times algebra, {1} and {3} are spectral nodes, and the eigenvector cone for the eigenvalue 1 is generated by

$$v^1 = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}^T$$

and

$$v^2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}^T,$$

see Theorems 2.1 and 2.2. In the usual nonnegative algebra, {3} is the only spectral node and any eigenvector is a multiple of v^2.

In max-times algebra, the maximal support of a solution is $\{1, 3, 4, 6\}$. For example take

$$y^1 = \begin{pmatrix} 2 & 0 & 3 & 1 & 0 & 3 & 0 \end{pmatrix}^T,$$

the max-times 'sum' of x^0, $2v^1$ and $3v^2$. In the usual nonnegative algebra, the maximal support is {3, 4, 6}, take

$$y^2 = x^0 + v^2 = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 2 & 0 \end{pmatrix}^T,$$

as in [19]. Note that neither y^1 is a solution in the usual sense, nor y^2 is a solution in the max-times sense.

Observe that for given A, b, if the usual Perron roots of all blocks in FNF are the same as the max-times roots (as in the example above), the existence of a solution in nonnegative algebra implies the existence in max-algebra (but not conversely). Examples of vectors for which a solution exists with A as above in max-algebra but not in nonnegative algebra are given by v^1 and v^2.

In the case of existence, minimal solutions have the same support in both theories.

4. Algebraic generalizations

Here we describe general setting in which Theorems 3.1 and 3.2 of Section 3.1 hold.

Recall that a set $\mathcal{S}$ is called a *semiring* if it is equipped with operations of addition $+$ and multiplication $\cdot$ satisfying the following laws:

(a) Addition is commutative $a + b = b + a$ $\forall a, b \in \mathcal{S}$.
(b) Multiplication distributes over addition $a(b + c) = ab + ac$ $\forall a, b, c \in \mathcal{S}$.
(c) Both addition and multiplication are associative: $a + (b + c) = (a + b) + c$, $a(bc) = (ab)c$ $\forall a, b, c \in \mathcal{S}$.
(d) There are elements $\mathbf{0}$ and $\mathbf{1}$ such that $a + \mathbf{0} = a$, $a\mathbf{1} = \mathbf{1}\ a = a$, and $a\mathbf{0} = \mathbf{0}\ a = \mathbf{0}$ for all $a \in \mathcal{S}$.

The max-times algebra and the usual nonnegative algebra are semirings (also note that any ring and any field is a semiring), see also other examples below. We consider a semiring $\mathcal{S}$ endowed with a *partial order* $\leq$, i.e. binary relation $\leq$ such that (1) $a \leq b$, $b \leq c$ imply $a \leq c$, (2) $a \leq b$, $b \leq a$ imply $a = b$, (3) $a \leq a$. In the case of idempotent addition ($a + a = a$), one defines a *canonical order* by $a \leq b \Leftrightarrow a + b = b$.

To model both max-times algebra and the usual nonnegative algebra, we may assume that the following axioms are satisfied.

(A1) Any countable ascending chain (i.e., linearly ordered subset) in $\mathcal{S}$ bounded from above has supremum, and any countable descending chain has infimum.
(A2) Addition is nondecreasing: $a+b \geq a$ and $a+b \geq b$.
(A3) Both operations distribute over any supremum or infimum of any such chain in $\mathcal{S}$, i.e.

$$\begin{aligned} a + \sup_{\mu} b_\mu &= \sup_{\mu}(a + b_\mu), \quad & a' \cdot \sup_{\mu} b_\mu &= \sup_{\mu}(a' \cdot b_\mu), \\ c + \inf_{\mu} d_\mu &= \inf_{\mu}(c + d_\mu), \quad & c' \cdot \inf_{\mu} d_\mu &= \inf_{\mu}(c' \cdot d_\mu), \end{aligned} \tag{21}$$

for any countable bounded ascending chain $\{b_\mu\} \subseteq \mathcal{S}$, countable descending chain $\{d_\mu\} \subseteq \mathcal{S}$, elements a, c, a', $c' \in \mathcal{S}$.

Axiom A2 implies that the semiring is *nonnegative*: $a \geq 0$ for all a, and *antinegative*: $a+b=0$ implies $a=b=0$. Axiom A3 implies that both arithmetic operations are monotone.

The operations of $\mathcal{S}$ extend to matrices and vectors in the usual way. Moreover we can compute matrix powers A^k for $k \geq 0$, where we assume $A^0 = I$, the identity matrix, where all diagonal entries equal to $\mathbf{1}$ and all off-diagonal entries equal to $\mathbf{0}$. The extension of notions of associated digraph and access relations is also evident, provided that there are no zero divisors.

Note that partial order in $\mathcal{S}$ is extended to $\mathcal{S}^n$ and $\mathcal{S}^{m \times n}$ (matrices with m rows and n columns over $\mathcal{S}$) componentwise. The monotonicity of addition and multiplication is preserved. Moreover, distributivity (21) also extends to matrices and vectors:

$$\begin{aligned} A + \sup_{\mu} B^\mu &= \sup_{\mu}(A + B^\mu), \quad & A' \cdot \sup_{\mu} B^\mu &= \sup_{\mu}(A' \cdot B^\mu), \\ C + \inf_{\mu} D^\mu &= \inf_{\mu}(C + D^\mu), \quad & C' \cdot \inf_{\mu} D^\mu &= \inf_{\mu}(C' \cdot D^\mu). \end{aligned} \tag{22}$$

Here $\{B^\mu\}$, $\{D^\mu\}$ are chains of matrices (ascending and descending, respectively), where $\{B^\mu\}$ is bounded from above.

Indeed, the distributivity of addition is verified componentwise. Let us verify the sup-distributivity for multiplication. Let n be the number of columns of C'. We have:

$$(C' \cdot \sup_{\mu \in \mathcal{N}} D^\mu)_{ik} = \sum_{j=1}^{n} c'_{ij}(\sup_{\mu \in \mathcal{N}} d^\mu_{jk}) = \sup_{\kappa} \sum_{j=1}^{n} c'_{ij} d^{\kappa(j)}_{jk}, \tag{23}$$

where $\mathcal{N}$ denotes a countable set and the last supremum is taken over all mappings κ from $\{1, \ldots, n\}$ to the natural numbers. The last equality is due to the scalar sup-distributivity. Now denote $\nu := \max_{j=1}^{n} \kappa(j)$ and observe that

$$\sum_{j=1}^{n} c'_{ij} d^{\kappa(j)}_{jk} \leq \sum_{j=1}^{n} c'_{ij} d^{\nu}_{jk}, \tag{24}$$

since d^ν_{jk} are ascending chains. This implies that in the last supremum of (23) we can restrict maps κ to identity, obtaining that

$$\sup_{\kappa} \sum_{j=1}^{n} c'_{ij} d^{\kappa(j)}_{jk} = \sup_{\nu} \sum_{j=1}^{n} c'_{ij} d^{\nu}_{jk} = \sup_{\nu}(C'D)_{ik}. \tag{25}$$

Thus the matrix sup-distributivity also holds. The inf-distributivity can be checked along the same lines replacing infimum by supremum, and ascending chains by descending chains.

It can be checked that axioms A1–A3 and matrix distributivity (22) provide sufficient ground for the proofs of Theorems 3.1 and 3.2.

For the alternative proof of Theorem 3.2 given in Remark 2 the system A1–A3 has to be modified. Note that the main part of the proof after (15) does not need anything but the existence of sups of bounded ascending chains and the distributivity of addition over such sups. It is only the starting representation $x = u + \underline{w}$, where $u = A^*b$ and $\underline{w}$ is the least vector w satisfying $x = u + w$, which may need more than that.

We impose A1, A2 and the part of A3 asserting the distributivity of addition and multiplication with respect to sup's of ascending chains, which is needed for Theorem 3.1, that is, for $u = A^*b$ to be the least solution of $x = Ax + b$. We also require that there is at least one vector w satisfying $x = u + w$. This will hold if we impose the following axiom.

(A4) For all a, $c \in \mathcal{S}$ such that $a \leq c$ there is $b \in \mathcal{S}$ such that $a + b = c$.

In addition, in order to get the least w satisfying $x = u + w$, we impose the distributivity of $+$ with respect to arbitrary inf's. Notice that in the case of an idempotent semiring we define the order canonically ($a \leq b \Leftrightarrow a + b = b$), and the axioms A2 and A4 are satisfied automatically.

Now we consider some examples of semirings where Theorems 3.1 and 3.2 are valid. In particular, axioms A1–A3 are satisfied for these examples.

Examples 1,2 Max-times algebra, classical nonnegative algebra (see Prerequisites).

Example 3 (Max-min algebra) Interval [0, 1] equipped with $a\ b := \min(a, b)$ and $a + b := \max(a, b)$.

Example 4 (Łukasiewicz algebra) Interval $[0, 1]$ equipped $ab := \max(0, a + b - 1)$ and $a + b := \max(a, b)$.

Example 5 (Distributive lattices) Recall that a *lattice* is a partially ordered set [5] where any two elements a, b have the least upper bound $a \vee b := \sup(a, b)$ and the greatest lower bound $a \wedge b := \inf(a, b)$. A lattice is called *distributive* if the following laws hold:

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c), \quad a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c). \tag{26}$$

When a lattice also has the lowest element ϵ and the greatest element $\top$, it can be turned into a semiring by setting $ab := a \wedge b$, $a + b := a \vee b$, $\mathbf{0} = \epsilon$ and $\mathbf{1} = \top$. To ensure that the axioms A1 and A3 hold, we require that the lattice is *complete*, i.e., that $\vee_\alpha a_\alpha$ and $\wedge_\beta b_\beta$ exist for all subsets $\{a_\alpha\}$ and $\{b_\beta\}$ of the lattice, and that the distributivity can be extended:

$$a \vee \wedge_\beta b_\beta = \wedge_\beta (a \vee b_\beta), \quad b \wedge \vee_\alpha a_\alpha = \vee_\alpha (b \wedge a_\alpha). \tag{27}$$

Max-min algebra is a special case of this example.

Example 6 (Idempotent interval analysis) Suppose that $a_1, a_2 \in \mathcal{S}$ where $\mathcal{S}$ satisfies A1–A3, and consider the semiring of ordered pairs (a_1, a_2), where $a_1 \leq a_2$ and the

operations of $\mathcal{S}$ are extended componentwise. This semiring, satisfying A1–A3, is the basis of idempotent interval analysis as introduced in [27].

Example 7 (Extended order complete vector lattices) We can consider a semiring of all sequences $(a_1, a_2, \ldots)$ where $a_i \in \mathcal{S}$ and $\mathcal{S}$ satisfies A1–A3, with the operations extended componentwise. A generalization of Example 6 is then a semiring of all ordered sequences $a_1 \leq a_2 \leq \ldots$ where $a_i \in \mathcal{S}$.

Example 8 (Semirings of functions) Further extension of Example 7 to functions on a continuous domain is also evident (following [9,26]). As an example of a subsemiring of functions satisfying A1–A3, one may consider convex functions on the real line, equipped with the operations of componentwise max (as addition) and componentwise addition (as multiplication). In the spirit of max-plus semiring, we allow a function to take $-\infty$ values (which are absorbing). To verify A1–A3, recall that a function f is convex if the set $\{(x, t) \mid t \geq f(x)\}$ is convex (providing connection to the well-known properties of convex sets). In particular, the addition corresponds to the intersection of convex sets, and the multiplication corresponds to the Minkowski sum of convex sets. Note that the inf of descending chain of convex functions is also computed componentwise. As another example, we can consider a semiring, where an element is a class of functions on a continuous domain different only on a countable subset. Then, all countable sups or infs are well-defined, since any two members of the class corresponding to such sup or inf will differ only on a countable subset, and axioms A1–A4 are verified componentwise, as above.

Note that the Kleene star (3) always converges in Examples 3–5. Moreover, it can be truncated for $k \geq n$, for an $n \times n$ matrix, so that $A^* = I + A + \cdots + A^{n-1}$, which follows from the optimal path interpretation, the finiteness of associated digraph, and because the matrix entries do not exceed $\mathbf{1}$. Hence A^*b is well-defined for any A, b, and $x = Ax + b$ is always solvable, with the solution set described by Theorem 3.2. In Examples 6–8 the convergence of Kleene star should hold for the matrices corresponding to each component of the semiring (for the last 'subexample', excluding a countable subset of components).

Finally we observe that theorems formally like Theorems 3.1 and 3.2 of the present article also hold in the case a linear operator leaving invariant a proper cone in $\mathbb{R}^n$, see [33, Theorem 3.1], where an analogue of Theorem 3.5 is also proved.

Acknowledgements

This research was supported by EPSRC grant RRAH12809 (all authors) and partially by RFBR-CRNF grant 11-01-93106 (S. Sergeev).

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