Linear Algebra and its Applications 498 (2016) 574–591

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

On tropical supereigenvectors

Peter Butkovič

School of Mathematics, University of Birmingham, Birmingham B15 2TT, United Kingdom

ARTICLE INFO

Article history:
Received 25 November 2015
Accepted 28 February 2016
Available online 3 March 2016
Submitted by R. Brualdi

Dedicated to the memory of Hans Schneider – a great colleague and friend

MSC:
15A18
15A80

Keywords:
Matrix
Eigenvalue
Eigenvector
Subeigenvector
Supereigenvector

ABSTRACT

The task of finding tropical eigenvectors and subeigenvectors, that is non-trivial solutions to $A \otimes x = \lambda \otimes x$ and $A \otimes x \leq \lambda \otimes x$ in the max-plus algebra, has been studied by many authors since the 1960s. In contrast the task of finding supereigenvectors, that is solutions to $A \otimes x \geq \lambda \otimes x$, has attracted attention only recently. We present a number of properties of supereigenvectors focusing on a complete characterization of the values of λ associated with supereigenvectors and in particular finite supereigenvectors. The proof of the main statement is constructive and enables us to find a non-trivial subspace of finite supereigenvectors. We also present an overview of key related results on eigenvectors and subeigenvectors.

1. Introduction

Tropical linear algebra (also called max-algebra or path algebra) is an analogue of linear algebra developed for the pair of operations $(\oplus, \otimes)$ where

$$a \oplus b = \max(a, b)$$

E-mail address: p.butkovic@bham.ac.uk.

http://dx.doi.org/10.1016/j.laa.2016.02.033

and

$$a \otimes b = a + b$$

for $a, b \in \overline{\mathbb{R}} \stackrel{def}{=} \mathbb{R} \cup \{-\infty\}$. This pair is extended to matrices and vectors as in conventional linear algebra. That is if $A = (a_{ij})$, $B = (b_{ij})$ and $C = (c_{ij})$ are matrices of compatible sizes with entries from $\overline{\mathbb{R}}$, we write $C = A \oplus B$ if $c_{ij} = a_{ij} \oplus b_{ij}$ for all i, j and $C = A \otimes B$ if

$$c_{ij} = \bigoplus_k a_{ik} \otimes b_{kj} = \max_k (a_{ik} + b_{kj})$$

for all i, j. If $\alpha \in \overline{\mathbb{R}}$ then $\alpha \otimes A = (\alpha \otimes a_{ij})$. For simplicity we will use the convention of not writing the symbol $\otimes$. Thus in what follows the symbol $\otimes$ will usually not be used and unless explicitly stated otherwise, all multiplications indicated are in max-algebra.

The interest in tropical linear algebra was originally motivated by the possibility of dealing with a class of non-linear problems in pure and applied mathematics, operational research, science and engineering as if they were linear due to the fact that $\left(\overline{\mathbb{R}}, \oplus, \otimes\right)$ is a commutative and idempotent semifield. Besides the main advantage of using linear rather than non-linear techniques, tropical linear algebra enables us to efficiently describe and deal with complex sets [6], reveal combinatorial aspects of problems [5] and view a class of problems in a new, unconventional way. The first pioneering papers appeared in the 1960s [15,16,35], followed by substantial contributions in the 1970s and 1980s such as [17,23, 39,14]. Since 1995 we have seen a remarkable expansion of this research field following a number of findings and applications in areas as diverse as algebraic geometry [30] and [34], geometry [26], control theory and optimization [2], phylogenetic [33], modelling of the cellular protein production [4] and railway scheduling [24]. A number of research monographs have been published [2,7,24,28,29]. A chapter on max-algebra appears in a handbook of linear algebra [25] and a chapter on idempotent semirings is in a monograph on semirings [21].

Tropical linear algebra covers a range of linear-algebraic problems in the max-linear setting, such as systems of linear equations and inequalities, linear independence and rank, bases and dimension of subspaces, polynomials, characteristic polynomials, matrix equations, matrix orbits and periodicity of matrix powers [2,7,17,24]. Among the most intensively studied questions was the *eigenproblem*, that is the question, for a given square matrix A to find all values of $\lambda \in \overline{\mathbb{R}}$ and non-trivial vectors x such that $A \otimes x = \lambda \otimes x$. This and related questions have been answered [17,23,20,3,19] with numerically stable low-order polynomial algorithms. The same is true about the *subeigenproblem* that is solution to $A \otimes x \leq \lambda \otimes x$, which appears to be strongly linked to the eigenproblem (see Section 4). In contrast, until recently [12,38,31] almost no attention has been paid to the *supereigenproblem* that is solution to $A \otimes x \geq \lambda \otimes x$, which is trivial for small values of λ but in general the description of the whole solution set seems to be much more difficult than for the eigenproblem [38,31]. This fact triggers in particular the question of finding

finite supereigenvectors. It is the main aim of this paper to identify (in Theorem 5.8) all values of λ for which a given matrix A has finite supereigenvectors and to find such vectors.

In order to provide a complete picture the results on general and finite supereigenvectors are compared with those for eigenvectors and subeigenvectors. Note that a theory of finite eigenvectors and subeigenvectors is well developed [17,7]. Note also that in the max-times setting, that is for the semifield $(\mathbb{R}_+, \max, .)$ finiteness corresponds to positivity.

We first give in Sections 2–4 a summary of the concepts and known results in tropical linear algebra that will be used in Section 5 to present the results on supereigenvectors.

2. Definitions and notation

Throughout the paper we denote $-\infty$ by ε (the neutral element with respect to $\oplus$) and for convenience we also denote by the same symbol any vector, whose all components are $-\infty$, or a matrix whose all entries are $-\infty$. If $a \in \mathbb{R}$ then the symbol a^{-1} stands for $-a$. We assume everywhere that $n \geq 1$ is an integer and denote $N = \{1, \ldots, n\}$.

A vector or matrix are called *finite* if all their entries are real numbers. A square matrix is called *diagonal* if all its diagonal entries are real numbers and off-diagonal entries are ε. A diagonal matrix with all diagonal entries equal to 0 is called the *unit matrix* and denoted I. Obviously, $AI = IA = A$ whenever A and I are of compatible sizes.

If A is a square matrix then the iterated product $AA \ldots A$ in which the symbol A appears k-times will be denoted by A^k. By definition $A^0 = I$.

It is easily proved that if A, B and C are of compatible sizes then:

$$A \geq B \Longrightarrow AC \geq BC \text{ and } CA \geq CB. \tag{1}$$

Tropical linear algebra often benefits from close links between matrices and digraphs. A *digraph* is an ordered pair $D = (V, E)$ where V is a nonempty finite set (of *nodes*) and $E \subseteq V \times V$ (the set of *arcs*).

Let $D = (V, E)$ be a digraph. A sequence $\pi = (v_1, \ldots, v_p)$ of nodes in D is called a *path* (in D) if $p = 1$ or $p > 1$ and $(v_i, v_{i+1}) \in E$ for all $i = 1, \ldots, p-1$. The number $p-1$ is called the *length* of π and will be denoted by $l(\pi)$. If u is the starting node and v is the endnode of π then we say that π is a $u - v$ path. If there is a $u - v$ path in D then v is said to be *reachable* from u, notation $u \longrightarrow v$. Thus $u \longrightarrow u$ for any $u \in V$.

A path $(v_1, \ldots, v_p)$ is called a *cycle* if $v_1 = v_p$ and $p > 1$ and it is called an *elementary cycle* if, moreover, $v_i \neq v_j$ for $i, j = 1, \ldots, p-1, i \neq j$. If there is no cycle in D then D is called *acyclic*.

A digraph D is called *strongly connected* if $u \longrightarrow v$ for all nodes u, v in D. A subdigraph D' of D is called a *strongly connected component* of D if it is a maximal strongly connected subdigraph of D, that is, D' is a strongly connected subdigraph of D and if D'

is a subdigraph of a strongly connected subdigraph D'' of D then $D' = D''$. Note that a digraph consisting of one node and no arc is strongly connected and acyclic; however, if a strongly connected digraph has at least two nodes then it obviously cannot be acyclic.

If $D = (N, E)$ is a digraph and $K \subseteq N$ then $D[K]$ denotes the *induced subgraph* of D, that is

$$D[K] = (K, E \cap (K \times K)).$$

A *weighted digraph* is $D = (V, E, w)$, where (V, E) is a digraph and w is a real function on E. All definitions for digraphs are naturally extended to weighted digraphs. If $\pi = (v_1, \ldots, v_p)$ is a path in (V, E, w) then the *weight* of π is $w(\pi) = w(v_1, v_2) + w(v_2, v_3) + \ldots + w(v_{p-1}, v_p)$ if $p > 1$ and ε if $p = 1$.

Given $A = (a_{ij}) \in \overline{\mathbb{R}}^{n\times n}$ the symbol D_A will denote the weighted digraph (N, E, w) where $E = \{(i, j); a_{ij} > \varepsilon\}$ and $w(i, j) = a_{ij}$ for all $(i, j) \in E$. The digraph D_A is said to be *associated* with the matrix A. If $\pi = (i_1, \ldots, i_p)$ is a path in D_A then we denote $w(\pi)$ by $w(\pi, A)$ and it now follows from the definitions that $w(\pi, A) = a_{i_1 i_2} + a_{i_2 i_3} + \ldots + a_{i_{p-1} i_p}$ if $p > 1$ and ε if $p = 1$.

If D_A is strongly connected then A is called *irreducible* and *reducible* otherwise.

Given $A \in \overline{\mathbb{R}}^{n\times n}$, the symbol $\lambda(A)$ will stand for the *maximum cycle mean* of A, that is:

$$\lambda(A) = \max_{\sigma} \mu(\sigma, A), \tag{2}$$

where the maximization is taken over all elementary cycles in D_A, and

$$\mu(\sigma, A) = \frac{w(\sigma, A)}{l(\sigma)} \tag{3}$$

denotes the *mean* of a cycle σ. With the convention $\max \emptyset = \varepsilon$ the value $\lambda(A)$ always exists since the number of elementary cycles is finite. However, it is easy to show [7] that $\lambda(A)$ remains the same if the word "elementary" is removed from the definition. It can be computed in $O(n^3)$ time [27], see also [7]. Observe that $\lambda(A) = \varepsilon$ if and only if D_A is acyclic.

Let $A \in \overline{\mathbb{R}}^{n\times n}$ and $\lambda \in \overline{\mathbb{R}}$. A cycle σ in D_A is called λ-*critical* if $\mu(\sigma, A) = \lambda$. We denote by $N_c(A, \lambda)$ the set of λ-*critical nodes*, that is nodes on λ-critical cycles. The λ-*critical digraph* of A is the digraph C_A with the set of nodes N; the set of arcs, notation $E_c(A)$, is the set of arcs of all λ-critical cycles. If $i, j \in N_c(A, \lambda)$ belong to the same λ-critical cycle then i and j are called λ-*equivalent* and we write $i \sim_\lambda j$. Clearly, $\sim_\lambda$ constitutes a relation of equivalence on $N_c(A, \lambda)$. The letter λ or prefix λ- will be omitted when $\lambda = \lambda(A)$.

A matrix $A \in \overline{\mathbb{R}}^{n\times n}$ is called *definite* if $\lambda(A) = 0$ [13,17]. Thus a matrix is definite if and only if all cycles in D_A are nonpositive and at least one has weight zero. It is easy to

check that $\lambda(\alpha A) = \alpha\lambda(A)$ for any $\alpha \in \mathbb{R}$. Hence $(\lambda(A))^{-1} A$ is definite whenever $\lambda(A)$ is finite. The matrix $\lambda^{-1}A$ for $\lambda \in \mathbb{R}$ will be denoted by A_λ.

The (tropical) column span of a matrix A will be denoted by $span\,(A)$ that is for A with columns $A_1, \ldots, A_n$

$$span\,(A) = \left\{ \sum\nolimits_i^{\oplus} \alpha_i A_i ; \alpha_1, \ldots, \alpha_n \in \overline{\mathbb{R}} \right\}.$$

We also define

$$span^+\,(A) = \left\{ \sum\nolimits_i^{\oplus} \alpha_i A_i ; \alpha_1, \ldots, \alpha_n \in \mathbb{R} \right\}.$$

Given $A \in \overline{\mathbb{R}}^{n\times n}$ it is standard [17,2,24,7] in max-algebra to define the infinite series

$$A^+ = A \oplus A^2 \oplus A^3 \oplus \ldots \tag{4}$$

and

$$A^* = I \oplus A^+ = I \oplus A \oplus A^2 \oplus A^3 \oplus \ldots . \tag{5}$$

The matrix A^+ is called the *weak transitive closure* of A and A^* is the *strong transitive closure* of A, also called the *Kleene star*.

It follows from the definitions that every entry of the matrix sequence

$$\left\{ A \oplus A^2 \oplus \ldots \oplus A^k \right\}_{k=0}^{\infty}$$

is a nondecreasing sequence in $\overline{\mathbb{R}}$ and therefore either it is convergent to a real number (when bounded) or its limit is $+\infty$ (when unbounded). If $\lambda(A) \leq 0$ then

$$A^+ = A \oplus A^2 \oplus \ldots \oplus A^k$$

and

$$A^* = I \oplus A \oplus A^2 \oplus \ldots \oplus A^{k-1}$$

for every $k \geq n$ and can be found using the Floyd–Warshall algorithm in $O\left(n^3\right)$ time [7]. If A is also irreducible and $n > 1$ then both A^+ and A^* are finite.

If $\lambda \in \mathbb{R}$ then $(A_\lambda)^+$ will be shortly written as A_λ^+, similarly $(A_\lambda)^*$. If $\lambda = \lambda\,(A)$ then the symbol $\widetilde{A_\lambda^+}$ stands for the matrix consisting of the columns of A_λ^+ with indices $j \in N_c\,(A)$. The following will be useful and is easily proved.

Lemma 2.1. *(See [17,24,7].) Let $A \in \overline{\mathbb{R}}^{n\times n}$, $\lambda = \lambda\,(A) > \varepsilon$ and $A_\lambda^+ = (\gamma_{ij})$. Then $j \in N_c\,(A)$ if and only if $\gamma_{jj} = 0$.*

Let $S \subseteq \overline{\mathbb{R}}^n$. The set S is called a *tropical subspace* if

$$\alpha u \oplus \beta v \in S$$

for every $u, v \in S$ and $\alpha, \beta \in \overline{\mathbb{R}}$. The adjective "tropical" will usually be omitted.

If $S \subseteq \overline{\mathbb{R}}^m$ is a finite set then as a slight abuse of notation we will denote by $span\,(S)$ the set $span\,(A)$, where A is the matrix whose columns are exactly the elements of S. If $span\,(S) = T$ then S is called a *set of generators* for T and T is called *finitely generated*.

Let $v \in \overline{\mathbb{R}}^m$. The *max-norm* or just *norm* of v is the value of the greatest component of v, notation $\|v\|$; v is called *scaled* if $\|v\| = 0$. The set S is called *scaled* if all its elements are scaled.

The set $S = \{v_1, \ldots, v_n\} \subseteq \overline{\mathbb{R}}^m$ is called *dependent* if $v_k \in span\{v_1, \ldots, v_{k-1}, v_{k+1}, \ldots, v_n\}$ for some $k \in N$. Otherwise S is *independent*.

Let $S, T \subseteq \overline{\mathbb{R}}^m$. The set S is called a *basis* of T if it is an independent set of generators for T. The following is of fundamental importance as it shows that every subspace has an essentially unique basis.

Theorem 2.2. *(See [36,37,11,7].) Every non-trivial finitely generated subspace has a unique scaled basis.*

Finally, for $A \in \overline{\mathbb{R}}^{n\times n}$ and $\lambda \in \overline{\mathbb{R}}$ we denote

$$\begin{aligned}
V(A,\lambda) &= \left\{x \in \overline{\mathbb{R}}^n; Ax = \lambda x\right\},\\
V_*(A,\lambda) &= \left\{x \in \overline{\mathbb{R}}^n; Ax \le \lambda x\right\},\\
V^*(A,\lambda) &= \left\{x \in \overline{\mathbb{R}}^n; Ax \ge \lambda x\right\},\\
FV(A,\lambda) &= \{x \in \mathbb{R}^n; Ax = \lambda x\},\\
FV_*(A,\lambda) &= \{x \in \mathbb{R}^n; Ax \le \lambda x\},\\
FV^*(A,\lambda) &= \{x \in \mathbb{R}^n; Ax \ge \lambda x\},\\
\Lambda\,(A) &= \left\{\lambda \in \overline{\mathbb{R}}; V\,(A,\lambda) \ne \{\varepsilon\}\right\}.
\end{aligned}$$

The set $\Lambda\,(A)$ or just Λ will be called the *spectrum* of A.

3. Known results on eigenvectors and subeigenvectors

The tropical *eigenvalue–eigenvector problem* (briefly *eigenproblem*) is the following:

Given $A \in \overline{\mathbb{R}}^{n\times n}$, find all $\lambda \in \overline{\mathbb{R}}$ (eigenvalues) and $x \in \overline{\mathbb{R}}^n$, $x \ne \varepsilon$ (eigenvectors) such that

$$Ax = \lambda x.$$

This problem has been studied since the work of R.A. Cuninghame-Green [16]. A full solution of the eigenproblem in the case of irreducible matrices has been presented by R.A. Cuninghame-Green [17,18] and M. Gondran and M. Minoux [22], see also N.N. Vorobyov [35]. The general (reducible) case was first presented by S. Gaubert [20] and R.B. Bapat, D. Stanford and P. van den Driessche [3]. See also [9] and [7].

Theorem 3.1. *(See [16,17,22].) If $A \in \overline{\mathbb{R}}^{n\times n}$ is irreducible then $\Lambda(A) = \{\lambda(A)\}$ and all eigenvectors of A are finite.*

The value $\lambda(A)$ is usually called the *principal eigenvalue*. For a matrix A to have finite eigenvectors it is not necessary that A is irreducible:

Theorem 3.2. *(See [17].) Let $A = (a_{ij}) \in \overline{\mathbb{R}}^{n\times n}$ and $\lambda(A) > \varepsilon$. Then A has finite eigenvectors if and only if for every $i \in N$ there is a $j \in N_c(A)$ such that $i \to j$. All finite eigenvectors (if any) are associated with $\lambda = \lambda(A)$.*

The following theorem identifies an essentially unique basis of the eigenspace of A corresponding to the principal eigenvalue. Note that this statement for irreducible matrices was already proved in [17]. The case when $\lambda(A) = \varepsilon$ is trivial [7] and will not be discussed here.

Theorem 3.3. *(See [1].) Suppose that $A \in \overline{\mathbb{R}}^{n\times n}, \lambda = \lambda(A) > \varepsilon$ and $g_1, \ldots, g_n$ are the columns of A^+_λ. Then*

$$V(A, \lambda(A)) = span\left(\widetilde{A^+_\lambda}\right)$$

and we obtain a basis of $V(A, \lambda(A))$ by taking exactly one g_j for each equivalence class in $(N_c(A), \sim)$.

Reducible $n \times n$ matrices have up to n eigenvalues. In order to identify all of them A is transformed by simultaneous permutations of the rows and columns (which do not change the spectrum) to a *Frobenius normal form (FNF)*

$$A' = \begin{pmatrix} A_{11} & \varepsilon & \ldots & \varepsilon \\ A_{21} & A_{22} & \ldots & \varepsilon \\ \ldots & \ldots & \ldots & \ldots \\ A_{r1} & A_{r2} & \ldots & A_{rr} \end{pmatrix}, \tag{6}$$

where $A_{11}, \ldots, A_{rr}$ are irreducible square submatrices of A'. This form is unique up to the order of the blocks and simultaneous permutations of the rows and columns within each block. If A is in the Frobenius normal form (6) then the corresponding partition subsets of the node set N of D_A will be denoted as $N_1, \ldots, N_r$ and called *classes (of A)*. It follows that each of the induced subgraphs $D_A[N_i]$ $(i = 1, \ldots, r)$ is strongly connected

and an arc from N_i to N_j in D_A exists only if $i \geq j$. As a slight abuse of language we will also say for simplicity that $\lambda(A_{jj})$ is the eigenvalue of N_j.

If A is in the Frobenius normal form (6) then the *reduced digraph*, notation R_A, is the digraph with nodes $N_1, \ldots, N_r$ and the set of arcs $\{(N_i, N_j); (\exists k \in N_i)(\exists \ell \in N_j) a_{k\ell} > \varepsilon\}$. Observe that R_A is acyclic and represents a partially ordered set. Any class that has no incoming (outcoming) arcs in R_A is called *initial (final)*, similarly for diagonal blocks. Recall that the symbol $N_i \longrightarrow N_j$ means that there is a directed path from N_i to N_j in R_A (and therefore from each node in N_i to each node in N_j in D_A).

It is intuitively clear that all eigenvalues of A in an FNF are among the unique eigenvalues of diagonal blocks. However, in general some of these values are not eigenvalues of A. The following key result appeared for the first time independently in the thesis [20] and report [3], see also [7] and [9].

Theorem 3.4 *(Spectral theorem). Let (6) be a Frobenius normal form of a matrix $A \in \overline{\mathbb{R}}^{n\times n}$. Then*

$$\Lambda(A) = \{\lambda \in \overline{\mathbb{R}}; (\exists j)\, \lambda = \lambda(A_{jj}) = \max_{N_i \to N_j} \lambda(A_{ii})\}.$$

Corollary 3.5. *Every $n \times n$ matrix A has up to n eigenvalues and the greatest eigenvalue is $\lambda(A)$ (which will therefore also be denoted by $\lambda_{\max}$).*

Note that if a diagonal block, say A_{jj} has $\lambda(A_{jj}) \in \Lambda(A)$, it still may not satisfy the condition $\lambda(A_{jj}) = \max_{N_i \to N_j} \lambda(A_{ii})$ and may not provide any eigenvectors. It is therefore necessary to identify blocks that satisfy this condition: If

$$\lambda(A_{jj}) = \max_{N_i \to N_j} \lambda(A_{ii})$$

then A_{jj} (and also N_j or just j) is called *spectral*. Thus $\lambda(A_{jj}) \in \Lambda(A)$ if j is spectral but not necessarily the other way round. We immediately deduce that all initial blocks as well as the blocks with maximum cycle mean $\lambda(A)$ are spectral. The smallest eigenvalue of A that is

$$\min \Lambda(A) = \min \{\lambda(N_i); N_i \text{ spectral}\}$$

will be denoted $\lambda_{\min}$.

We now explain how to find a basis of the eigenspace associated with a general eigenvalue $\lambda \in \Lambda(A)$. Let $A \in \overline{\mathbb{R}}^{n\times n}$ be in the Frobenius normal form (6), $N_1, \ldots, N_r$ be the classes of A and $R = \{1, \ldots, r\}$. The case $\lambda = \varepsilon$ is trivial [7] and will not be discussed here. Suppose that $\lambda \in \Lambda(A), \lambda > \varepsilon$ and denote

$$I(\lambda) = \{i \in R; \lambda(N_i) = \lambda, N_i \text{ spectral}\}.$$

Note that $\lambda(\lambda^{-1}A) = \lambda^{-1}\lambda(A)$ may be positive since $\lambda \leq \lambda(A)$ and thus $A_\lambda^+ = (\gamma_{ij})$ may now include entries equal to $+\infty$. Let us denote

$$N_c'(A,\lambda) = \bigcup_{i\in I(\lambda)} N_c(A_{ii},\lambda) = \left\{ j \in N; \gamma_{jj} = 0, j \in \bigcup_{i\in I(\lambda)} N_i \right\}.$$

Theorem 3.6. *(See [9,7].) Suppose that $A \in \overline{\mathbb{R}}^{n\times n}$ and $\lambda \in \Lambda(A), \lambda > \varepsilon$. Let $g_1, \ldots, g_n$ be the columns of A_λ^+ and $\widetilde{A_\lambda^+}$ consist of $g_j, j \in N_c'(A,\lambda)$. Then*

$$V(A,\lambda) = span\left(\widetilde{A_\lambda^+}\right)$$

and a basis of $V(A,\lambda)$ can be obtained by taking exactly one $g_j, j \in N_c'(A,\lambda)$ for each $\sim_\lambda$ equivalence class.

Corollary 3.7. *The spectrum $\Lambda(A)$ and bases of $V(A,\lambda)$ for all $\lambda \in \Lambda(A)$ can be found in $O(n^3)$ time.*

4. Subeigenvectors

If $A \in \overline{\mathbb{R}}^{n\times n}$ and $\lambda \in \overline{\mathbb{R}}$ then a vector $x \in \overline{\mathbb{R}}^n, x \neq \varepsilon$ satisfying

$$Ax \leq \lambda x \tag{7}$$

is called a *subeigenvector of A with associated subeigenvalue λ.*

The question of existence of subeigenvectors and finite subeigenvectors has been studied for some time and the theorem below summarizes the main results. The case when $\lambda(A) = \varepsilon$ is trivial [7] and will not be discussed here.

Theorem 4.1. *(See [10,7].) Let $A \in \overline{\mathbb{R}}^{n\times n}, \lambda(A) > \varepsilon$. Then*

(a) *$FV_*(A,\lambda) \neq \emptyset$ if and only if $\lambda \geq \lambda(A)$ and $FV_*(A,\lambda) = span^+(A_\lambda^*)$ for $\lambda \geq \lambda(A)$.*
(b) *$V_*(A,\lambda) \neq \{\varepsilon\}$ if and only if $\lambda \geq \lambda_{\min}$ and $V_*(A,\lambda) = span(G)$, for $\lambda \geq \lambda_{\min}$, where G is the matrix consisting of the columns g_j of the matrix A_λ^* with indices $j \in \bigcup_{i\in I_*(\lambda)} N_i$, where*

$$I_*(\lambda) = \{i \in R; \lambda(N_i) \leq \lambda, N_i \text{ spectral}\}.$$

It follows that bases of $FV_*(A,\lambda) \neq \emptyset$ and $V_*(A,\lambda) \neq \{\varepsilon\}$ can be found in $O\left(n^3\right)$ time [32].

5. Supereigenvectors

If $A \in \overline{\mathbb{R}}^{n\times n}$ and $\lambda \in \overline{\mathbb{R}}$ then a vector $x \in \overline{\mathbb{R}}^n, x \neq \varepsilon$ satisfying

$$Ax \geq \lambda x \tag{8}$$

is called a *supereigenvector of A with associated supereigenvalue λ*.

In contrast to eigenvectors and subeigenvectors the questions of existence and full description are more difficult for supereigenvectors, although there is a trivial answer for λ small enough as stated in the next proposition. In what follows we denote $\min_{i=1,\ldots,n} a_{ii}$ by $\overline{\lambda}(A)$ or just $\overline{\lambda}$. Clearly, $\overline{\lambda}(A) \leq \lambda_{\min}$.

Proposition 5.1. *If $\lambda \leq \overline{\lambda}$ then $V^*(A,\lambda) = \overline{\mathbb{R}}^n$.*

Proof. If $\lambda \leq a_{ii}$ for every i then for every x and every i we have

$$\lambda \otimes x_i \leq a_{ii} \otimes x_i \leq (A \otimes x)_i$$

and so $\lambda x \leq Ax$. $\square$

In order to describe the values of λ associated with supereigenvectors we start with a necessary condition for finite supereigenvectors (which later turns out to be insufficient in general).

Lemma 5.2. *If $Ax \geq \lambda x, x$ finite then $\lambda \leq \lambda(A)$.*

Proof. Take any $i = i_1 \in N$. Then

$$\lambda + x_{i_1} \leq a_{i_1 i_2} + x_{i_2}$$

for some $i_2 \in N$. Similarly

$$\lambda + x_{i_2} \leq a_{i_2 i_3} + x_{i_3}$$

for some $i_3 \in N$ and so on. By finiteness and by omitting, if necessary, a few first indices we get for some k:

$$\lambda + x_{i_k} \leq a_{i_k i_1} + x_{i_1}.$$

After adding up and simplifying we have

$$\lambda \leq \frac{a_{i_1 i_2} + \ldots + a_{i_k i_1}}{k} \leq \lambda(A). \quad \square$$

Given $A \in \overline{\mathbb{R}}^{n\times n}$ and non-empty sets $J, K \subseteq N$ the symbol $A(J,K)$ will denote the submatrix of A consisting of entries with row indices from J and column indices from K. The principal submatrix $A(J,J)$ will be written briefly as $A(J)$.

We are ready to characterize all λ associated with supereigenvectors. This result first appeared in [8] and then independently in [38].

Theorem 5.3. *$V^*(A,\lambda) \neq \{\varepsilon\}$ if and only if $\lambda \leq \lambda(A)$.*

Proof. Suppose first $Ax \geq \lambda x, x \neq \varepsilon$. Let $J = \operatorname{supp}(x)$, then

$$A(J)\,x(J) \geq \lambda x(J).$$

By Lemma 5.2 we have $\lambda \leq \lambda(A(J)) \leq \lambda(A)$.

Suppose $\lambda \leq \lambda(A)$. Let $x \in V(A, \lambda(A)), x \neq \varepsilon$, then

$$A \otimes x = \lambda(A) \otimes x \geq \lambda \otimes x. \quad \square$$

Corollary 5.4. *If $\lambda(A) = \varepsilon$ and $V^*(A,\lambda) \neq \{\varepsilon\}$ then $\lambda = \varepsilon$ and $V^*(A,\varepsilon) = \overline{\mathbb{R}}^n$.*

The size of a minimal set of generators of $V^*(A,\lambda)$ can be exponentially large in terms of n [38]. An algorithm for finding a set of generators of $V^*(A,\lambda)$ has been presented in [38]. This method is incrementally polynomial. It can be decided about each generator whether it is an extremal (an element of an essentially unique basis, see Theorem 2.2) of $V^*(A,\lambda)$ in $O\left(n^3\right)$ time [31]. However, the question of finding a basis in an efficient way remains open.

The next two statements show that compared to eigenvectors there is a greater level of freedom in choosing infinite components for a supereigenvector and unlike for eigenvectors we can associate a supereigenvector with any cycle in D_A.

Proposition 5.5. *For every $J \subseteq N, J \neq \emptyset$ there exists an $x \in V^*(A,\lambda), x \neq \varepsilon$, where $\lambda = \lambda(A(J))$ and $x(N-J) = \varepsilon$.*

Proof. Let $J \subseteq N, J \neq \emptyset$. Then by Theorem 3.4 there exists a $z \neq \varepsilon$ such that $A(J)\,z = \lambda(A(J))\,z$. Set $x(J) = z$ and $x(N-J) = \varepsilon$. Hence

$$\begin{aligned}
A \otimes x &= \begin{pmatrix} A(J,J) & A(J,N-J) \\ A(N-J,J) & A(N-J,N-J) \end{pmatrix} \begin{pmatrix} x(J) \\ \varepsilon \end{pmatrix} \\
&= \begin{pmatrix} \lambda(A(J))\,x(J) \\ A(N-J,J)\,x(J) \end{pmatrix} \\
&\geq \lambda(A(J)) \begin{pmatrix} x(J) \\ \varepsilon \end{pmatrix} \\
&= \lambda(A(J))\,x. \quad \square
\end{aligned}$$

Proposition 5.6. *If $A \otimes x \geq \lambda(A(J))\, x$, $x \neq \varepsilon$, where $J = \operatorname{supp}(x)$ then there exists a critical cycle $(i_1, i_2, \ldots, i_k)$ in $D_{A(J)}$ such that*

$$A(C)\, x(C) = \lambda(A(J))\, x(C), \tag{9}$$

where $C = \{i_1, i_2, \ldots, i_k\}$.

Proof. If $\lambda(A(J)) = \varepsilon$ then every cycle is critical and at least one component, say i, of $A(J) \otimes x$ is ε because $A(J)$ has an ε row. Then we can take $C = \{i\}$.

Let us now suppose that $\lambda(A(J)) > \varepsilon$ and denote $\lambda = \lambda(A(J))$. Let $i_1 \in J$. Then

$$\lambda + x_{i_1} \leq \max_j (a_{i_1 j} + x_j) = a_{i_1 i_2} + x_{i_2}$$

for some $i_2 \in J$. Similarly we have

$$\lambda + x_{i_2} \leq \max_j (a_{i_2 j} + x_j) = a_{i_2 i_3} + x_{i_3}$$

for some $i_3 \in J$, and so on. By finiteness and by omitting, if necessary, a few first indices we get for some k:

$$\lambda + x_{i_k} \leq \max_j (a_{i_k j} + x_j) = a_{i_k i_1} + x_{i_1}.$$

After adding up and simplifying we have

$$\lambda \leq \frac{a_{i_1 i_2} + \ldots + a_{i_k i_1}}{k} \leq \lambda(A(J)).$$

Hence none of the inequalities can be strict and (9) follows. □

Our main result is a full characterization of all values of λ associated with *finite* supereigenvectors. Suppose that A is in an FNF (6). Recall that for $i, j \in N$ the symbol $i \to j$ means that there is a path from i to j in D_A and similarly $N_i \to N_j$ means that there is a path from N_i to N_j in R_A. On the other hand $N_i \rightrightarrows N_j$ will mean that there is a path from N_i to N_j in R_A containing only nodes N_t such that $\lambda_t \leq \lambda_j$.

We denote

$$\lambda^*(A) = \min\{\lambda(A_{jj})\,;\, N_j \text{ is a final class}\}.$$

We also use λ^* for $\lambda^*(A)$ if appropriate. Note that in the above definition it does not matter whether N_j is spectral or not.

The following immediate corollary of Theorem 3.2 will be useful for proving our main result, Theorem 5.8 below. In both statements we assume that $A \in \overline{\mathbb{R}}^{n \times n}$ is in the FNF (6) with classes $N_1, \ldots, N_r$, $R = \{1, \ldots, r\}$ and the symbol $\lambda_j, j \in R$ stands for $\lambda(A_{jj})$.

Proposition 5.7. *Let $j \in R$,*

$$\mathcal{M}_j = \{i \in R; N_i \rightrightarrows N_j\},$$
$$M_j = \bigcup_{i \in \mathcal{M}_j} N_i$$

and

$$B^{(j)} = A(M_j).$$

Then $FV\left(B^{(j)}, \lambda_j\right) \neq \emptyset$.

Theorem 5.8. $FV^*(A, \lambda) \neq \emptyset \iff \lambda \leq \lambda^*(A)$.

Proof. Let N_j be final and $\lambda(A_{jj}) = \lambda^*(A)$. Let $A \otimes x \geq \lambda \otimes x, x$ finite. Then $A_{jj} \otimes x(N_j) \geq \lambda \otimes x(N_j), x(N_j)$ finite and so by Lemma 5.2 $\lambda \leq \lambda(A_{jj}) = \lambda^*(A)$.

For the converse it is sufficient to prove that $FV^*(A, \lambda^*) \neq \emptyset$. Let $j \in R$ be such that N_j is final and $\lambda^* = \lambda_j$. By Proposition 5.7 (using the same notation) there exists a finite vector $y^{(j)}$ such that $B^{(j)} \otimes y^{(j)} = \lambda_j \otimes y^{(j)}$. Let $A^{(1)} = A$, $A^{(2)} = A(N - M_j)$ and $l \in R$ be such that N_l is final in $A^{(2)}$ and $\lambda^*\left(A^{(2)}\right) = \lambda_l$. If N_l is also final in $A^{(1)}$ then $\lambda_l \geq \lambda_j$ from the definition of λ_j. If N_l is not final in $A^{(1)}$ then $N_l \rightrightarrows N_j$ and so $\lambda_l > \lambda_j$ (since otherwise l would have been included in $\mathcal{M}_j$). In any case $\lambda_l \geq \lambda_j$ and again by Proposition 5.7 there exists a finite vector $y^{(l)}$ such that

$$B^{(l)} \otimes y^{(l)} = \lambda_l \otimes y^{(l)}.$$

Continue in this way with $A^{(3)} = A(N - M_j - M_l)$ and so on until some $A^{(s)}$ has a finite eigenvector – this is guaranteed to happen when $B^{(s)}$ consists of all the remaining parts of A.

This process creates a sequence of finite vectors $y^{(j)}, y^{(l)}, \ldots$. Set $x = (x_1, \ldots, x_n)^T$ so that $x(M_k) = y^{(k)}$ for all sets M_k created in the process. Then x is finite and

$$A \otimes x = A \otimes \begin{pmatrix} \vdots \\ x(M_k) \\ \vdots \end{pmatrix}$$
$$\geq \begin{pmatrix} \vdots \\ A(M_k) \otimes x(M_k) \\ \vdots \end{pmatrix}$$

$$= \begin{pmatrix} \vdots \\ \lambda_k \otimes x\left(M_k\right) \\ \vdots \end{pmatrix}$$

$$\geq \lambda^* \otimes \begin{pmatrix} \vdots \\ x\left(M_k\right) \\ \vdots \end{pmatrix} = \lambda^* \otimes x.$$

The last inequality follows because $\lambda_k \geq \lambda^*$ for every k. $\square$

It is easily seen that the identification of $\mathcal{M}_j$ in Proposition 5.7 can be done in polynomial time. Therefore the constructive proof above provides a method to find a non-trivial subset of a set of generators of finite supereigenvectors in polynomial time. If (in the notation of this proof) B is the $n \times t$ matrix (for some positive integer t) of the form

$$\begin{pmatrix} \ddots & \varepsilon & \varepsilon \\ \varepsilon & \widetilde{B}^{(j)}_{\lambda_j} & \varepsilon \\ \varepsilon & \varepsilon & \ddots \end{pmatrix}$$

then $span^+\left(B\right) \subseteq FV^*\left(A, \lambda^*\right)$.

Remark 5.9. $\lambda^*\left(A\right)$ ranges over the whole discrete set $\left\{\lambda\left(A_{jj}\right); j \in R\right\}$ and may be smaller or greater than $\lambda_{\min}$, or equal to $\lambda_{\min}$, for instance when

$$A = \begin{pmatrix} \alpha & \varepsilon & \varepsilon \\ 0 & 2 & \varepsilon \\ \varepsilon & 0 & 1 \end{pmatrix}$$

then for any $\alpha \in \mathbb{R}$ we have $\lambda\left(A\right) = \max\left(2, \alpha\right), \lambda_{\min} = 1$ and $\lambda^*\left(A\right) = \alpha$.

Fig. 1 shows a comparison of values of λ associated with general/finite eigenvectors, subeigenvectors and supereigenvectors.

We conclude with an example. Let A be the matrix

$$\begin{pmatrix} 3 & & & & & & \\ & 5 & & & & & \\ & & 1 & & & & \\ 0 & 0 & 0 & 2 & & & \\ & 0 & & 0 & -1 & 1 & \\ & & 0 & & -1 & -2 & \\ 0 & & 0 & & & & 6 \end{pmatrix},$$

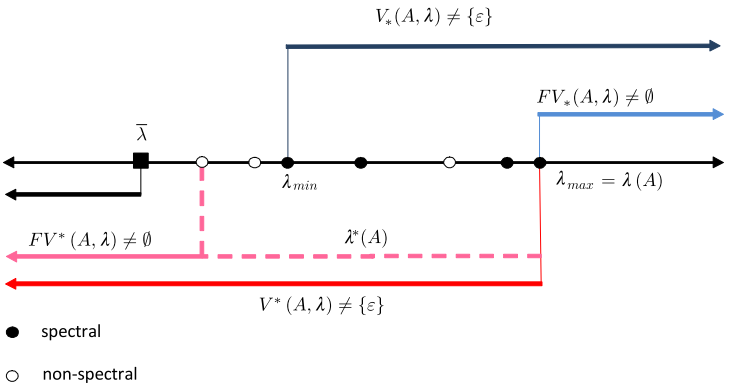

Fig. 1. Values of λ with associated eigenvectors, subeigenvectors and supereigenvectors.

where (and in the matrices below) all missing entries are ε. This matrix is in an FNF with $n = 7$, $r = 6$, $N_i = \{i\}$ for $i = 1, 2, 3, 4$, $N_5 = \{5, 6\}$ and $N_6 = \{7\}$. Clearly, $\lambda_1 = 3$, $\lambda_2 = 5$, $\lambda_3 = 1$, $\lambda_4 = 2$, $\lambda_5 = 0$, $\lambda_6 = 6$, $\Lambda(A) = \{5, 2, 0, 6\}$, $\lambda_{\max} = \lambda(A) = 6$, $\lambda_{\min} = 0, \lambda^*(A) = 1 = \lambda(N_3)$, $\overline{\lambda}(A) = -2$. Initial classes are N_5, N_6, final classes are N_1, N_2 and N_3. The algorithm in the proof chooses first $j = 3$. Hence $\mathcal{M}_3 = \{3, 5\}$, $M_3 = \{3, 5, 6\}$, $\lambda_3 = 1$ and

$$B^{(3)} = A(\{3, 5, 6\}) = \begin{pmatrix} 1 & & \\ & -1 & 1 \\ 0 & -1 & -2 \end{pmatrix}.$$

We then get

$$B^{(3)}_{\lambda_3} = \begin{pmatrix} 0 & & \\ & -2 & 0 \\ -1 & -2 & -3 \end{pmatrix}, \left(B^{(3)}_{\lambda_3}\right)^+ = \begin{pmatrix} 0 & & \\ -1 & -2 & 0 \\ -1 & -2 & -2 \end{pmatrix}$$

and

$$\widetilde{B}^{(3)}_{\lambda_3} = \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix}$$

since only the first column is critical. Hence the eigenspace of $B^{(3)}$ (in which all vectors are finite) is the set of multiples of $y^{(3)} = (0, -1, -1)^T$.

In the next iteration

$$A^{(2)} = \begin{pmatrix} 3 & & & \\ & 5 & & \\ 0 & 0 & 2 & \\ 0 & & & 6 \end{pmatrix},$$

$l = 1$, $\mathcal{M}_1 = \{1, 4\} = M_1$, $\lambda_1 = 3$ and

$$B^{(2)} = A(\{1, 4\}) = \begin{pmatrix} 3 & \\ 0 & 2 \end{pmatrix}.$$

Similarly as before the eigenspace of $B^{(2)}$ (in which all vectors are finite) is the set of multiples of $y^{(2)} = (0, -3)^T$.

In the next iteration

$$A^{(3)} = \begin{pmatrix} 5 & \\ & 6 \end{pmatrix},$$

$l = 2$, $\mathcal{M}_2 = \{2\} = M_2$, $\lambda_2 = 5$ and $y^{(3)} = (0)^T$.

Finally,

$$A^{(4)} = \begin{pmatrix} 6 \end{pmatrix},$$

$l = 6$, $\mathcal{M}_7 = \{7\} = M_6$, $\lambda_6 = 6$ and $y^{(4)} = (0)^T$.

We conclude that finite supereigenvectors of A associated with λ exist if and only if $\lambda \leq 1$ and for any such λ we have $span^+(B) \subseteq FV^*(A, \lambda)$, where

$$B = \begin{pmatrix} & 0 & & \\ & & 0 & \\ 0 & & & \\ & -3 & & \\ -1 & & & \\ -1 & & & \\ & & & 0 \end{pmatrix}.$$

We also observe that $FV^*(A, \lambda) = \mathbb{R}^7$ and $V^*(A, \lambda) = \overline{\mathbb{R}}^7$ for all $\lambda \leq -2$.

6. Conclusions

We have presented an overview of previously proved criteria for the existence of general and finite eigenvectors and subeigenvectors. We have then proved such criteria for general and finite supereigenvectors. A method for finding a non-trivial subset of a set of generators of finite supereigenvectors follows from the proof. However, efficient finding of a set of generators and a basis of the subspace of all general or finite supereigenvectors associated with a given $\lambda \in \mathbb{R}$ remains open.

Acknowledgements

This research was supported by EPSRC grant EP/J00829X/1.

References

[1] M. Akian, S. Gaubert, C. Walsh, Discrete max-plus spectral theory, in: G.L. Litvinov, V.P. Maslov (Eds.), Idempotent Mathematics and Mathematical Physics, in: Contemp. Math., vol. 377, AMS, 2005, pp. 53–77.

[2] F.L. Baccelli, G. Cohen, G.-J. Olsder, J.-P. Quadrat, Synchronization and Linearity, John Wiley, Chichester, New York, 1992.

[3] R.B. Bapat, D. Stanford, P. van den Driessche, The eigenproblem in max algebra, DMS-631-IR, University of Victoria, British Columbia, 1993.

[4] C.A. Brackley, D. Broomhead, M.C. Romano, M. Thiel, A max-plus model of ribosome dynamics during mRNA translation, J. Theoret. Biol. 303 (2012) 128–140.

[5] P. Butkovic, Max-algebra: the linear algebra of combinatorics?, Linear Algebra Appl. 367 (2003) 313–335.

[6] P. Butkovic, Finding a bounded mixed-integer solution to a system of dual inequalities, Oper. Res. Lett. 36 (2008) 623–627.

[7] P. Butkovic, Max-Linear Systems: Theory and Algorithms, Springer Monogr. Math., Springer-Verlag, London, 2010.

[8] P. Butkovic, Supereigenvectors, preprint, School of Mathematics, University of Birmingham, 2012/3.

[9] P. Butkovic, R.A. Cuninghame-Green, S. Gaubert, Reducible spectral theory with applications to the robustness of matrices in max-algebra, SIAM J. Matrix Anal. Appl. 31 (3) (2009) 1412–1431.

[10] P. Butkovic, H. Schneider, Applications of max-algebra to diagonal scaling of matrices, Electron. J. Linear Algebra 13 (2005) 262–273.

[11] P. Butkovic, H. Schneider, S. Sergeev, Generators, extremals and bases of max cones, Linear Algebra Appl. 421 (2007) 394–406.

[12] P. Butkovic, H. Schneider, S. Sergeev, Recognizing weakly stable matrices, SIAM J. Control Optim. 50 (5) (2012) 3029–3051.

[13] B.A. Carré, An algebra for network routing problems, J. Inst. Math. Appl. 7 (1971) 273–294.

[14] G. Cohen, D. Dubois, J.-P. Quadrat, M. Viot, A linear-system-theoretic view of discrete-event processes and its use for performance evaluation in manufacturing, IEEE Trans. Automat. Control AC-30 (3) (1985).

[15] R.A. Cuninghame-Green, Process synchronisation in a steelworks – a problem of feasibility, in: Banbury, Maitland (Eds.), Proc. 2nd Int. Conf. on Operational Research, English University Press, 1960, pp. 323–328.

[16] R.A. Cuninghame-Green, Describing industrial processes with interference and approximating their steady-state behaviour, Oper. Res. Quart. 13 (1962) 95–100.

[17] R.A. Cuninghame-Green, Minimax Algebra, Lecture Notes in Econom. and Math. Systems, vol. 166, Springer, Berlin, 1979.

[18] R.A. Cuninghame-Green, Minimax Algebra and Applications, Adv. Imaging Electron Phys., vol. 90, Academic Press, New York, 1995, pp. 1–121.

[19] L. Elsner, P. van den Driessche, On the power method in max algebra, in: Special Issue Dedicated to Hans Schneider, Madison, WI, 1998, Linear Algebra Appl. 302/303 (1999) 17–32.

[20] S. Gaubert, Théorie des systèmes linéaires dans les dioïdes, Thèse, Ecole des Mines de Paris, 1992.

[21] J.S. Golan, Semirings and Their Applications, Kluwer Acad. Publ., Dordrecht, 1999.

[22] M. Gondran, M. Minoux, Valeurs propres et vecteur propres dans les dioïdes et leur interprétation en théorie des graphes, Bull. Direction Etudes Rech., Ser. C, Math. Inform. 2 (1977) 25–41.

[23] M. Gondran, M. Minoux, Linear algebra of dioïds: a survey of recent results, Ann. Discrete Math. 19 (1984) 147–164.

[24] B. Heidergott, G.-J. Olsder, J. van der Woude, Max Plus at Work: Modeling and Analysis of Synchronized Systems: A Course on Max-Plus Algebra, PUP, 2005.

[25] L. Hogben, et al., Handbook of Linear Algebra, Discrete Math. Appl., vol. 39, Chapman and Hall, 2006.

[26] M. Joswig, Tropical convex hull computations, in: G.L. Litvinov, S.N. Sergeev (Eds.), Proceedings of the International Conference on Tropical and Idempotent Mathematics, in: Contemp. Math., vol. 495, AMS, 2009, pp. 193–212.

[27] R.M. Karp, A characterization of the minimum cycle mean in a digraph, Discrete Math. 23 (1978) 309–311.

[28] V.N. Kolokoltsov, V.P. Maslov, Idempotent Analysis and Its Applications, Kluwer Acad. Publ., Dordrecht, 1997.

[29] W.M. McEneaney, Max-Plus Methods for Nonlinear Control and Estimation, Systems Control Found. Appl., Birkhäuser, 2006.
[30] G. Mikhalkin, Tropical geometry and its application, in: Proceedings of the ICM, Madrid, 2006, pp. 827–852.
[31] S. Sergeev, Extremals of the supereigenvector cone in max algebra: a combinatorial description, Linear Algebra Appl. 479 (2015) 106–117.
[32] S. Sergeev, H. Schneider, P. Butkovic, On visualization scaling, subeigenvectors and Kleene stars in max algebra, Linear Algebra Appl. 431 (2009) 2395–2406.
[33] D. Speyer, B. Sturmfels, Tropical mathematics, Math. Mag. 82 (2009) 163–173.
[34] B. Sturmfels, et al., On the tropical rank of a matrix, in: J.E. Goodman, J. Pach, E. Welzl (Eds.), Discrete and Computational Geometry, in: Math. Sci. Res. Inst. Publ., vol. 52, Cambridge University Press, 2005, pp. 213–242.
[35] N.N. Vorobyov, Extremal algebra of positive matrices, Elektron. Datenverarb. Kybernet. 3 (1967) 39–71 (in Russian).
[36] E. Wagneur, Finitely generated moduloïds: the existence and unicity problem for bases, in: J.L. Lions, A. Bensoussan (Eds.), Analysis and Optimization of Systems, Antibes, 1988, in: Lecture Notes in Control and Inform. Sci., vol. 111, Springer, Berlin, 1988, pp. 966–976.
[37] E. Wagneur, Moduloïds and pseudomodules: 1. Dimension theory, Discrete Math. 98 (1991) 57–73.
[38] X. Wang, H. Wang, The generators of the solution space for a system of inequalities, Linear Algebra Appl. 459 (2014) 248–263.
[39] K. Zimmermann, Extremální Algebra, vol. 46, Výzkumná publikace Ekonomicko-matematické laboratoře při Ekonomickém ústavě ČSAV, Praha, 1976 (in Czech).