Discrete-event dynamic systems: The strictly convex case

R.A. Cuninghame-Green and P. Butkovič
School of Mathematics and Statistics, The University of Birmingham, Birmingham B15 2TT, United Kingdom

Given the observed output g_j of a discrete-event system, a classical problem is to find a matrix realisation $g_j = cA^jb$ with A of least possible dimension. When the sequence g_j is convex and ultimately 1-periodic, a linear-time algorithm suffices to construct such a realisation over the algebra $(R, \max, +)$. When the transient is strictly convex, this realisation is minimal-dimensional.

Keywords: Discrete dynamic systems, max-algebra.

1. Introduction

A classical problem is the following. An unknown system emits a sequence of (real-number) signals

$$G = \{g_j\}_{j=0,1,\dots}$$

at discrete time intervals; find an economical mathematical description of the system, given only this observed sequence.

Approaches to the problem split according to hypotheses as to the underlying process. A substantial amount of theory has been devoted to the case where the process is assumed to be describable through the *state vector* $x(j) \in \mathbb{R}^n$ of the system at time $j = 0, 1, 2, \dots$; change of state is described through a linear transformation

$$x(j) \mapsto x(j+1) = A \otimes x(j), \quad x(0) = b, \tag{1.1}$$

where $A \in \mathbb{R}^{n \times n}$, $b \in \mathbb{R}^{n \times 1}$ and the state is observed through an *observation vector* $c \in \mathbb{R}^{1 \times n}$:

$$g_j = c \otimes x(j). \tag{1.2}$$

In this case, the observed g_j are called *Markov parameters* and the task is to find A, b and c. Any A, b, c satisfying (1.1) and (1.2) is called a *realisation of* the system. Clearly, there are many trivial realisations, but for an economical description, we seek A, b, c of least possible dimension n – a *minimum-dimensional realisation* of the *discrete-event system emitting G.*

When the underlying process is of a physical nature, the algebra in (1.1), (1.2) is conventional linear algebra, describing superpositions and interactions of mechanical, electrical or other physical signals, and the analysis follows well-known classical lines [2]. However, in some synchronous processes of production of information technology, it has been known for some time [3,4] that an appropriate mathematical description is obtained using equations (1.1), (1.2) when the underlying scalar algebra is *max-algebra* in which the operation of addition is replaced by $\max(x, y)$ and multiplication is replaced by $x + y$. Several substantial studies of the properties of this algebraic system exist [5, 13], and the advent of flexible manufacturing systems and computer networking has brought about a recent resurgence of international interest.

In [10], Olsder posed the minimum-dimensional realisation problem for systems based on max-algebra. His solution method used a field-embedding to make a transformation from max-algebra to normal linear algebra, solving the realisation problem and transforming back to max-algebra. To illustrate his method, Olsder considered the particular sequences

$$3,\ 5,\ 8\tfrac{1}{2},\ 12\tfrac{1}{2},\ 16\tfrac{1}{2},\ldots \tag{1.3}$$

$$5,\ 8,\ 11\tfrac{1}{2},\ 15\tfrac{1}{2},\ 19\tfrac{1}{2},\ldots \tag{1.4}$$

The method is computationally rather complex, involving the inversion of matrices whose elements are rational functions; several pages of algebra may be necessitated by even quite small problems.

Olsder's realisation problem has attracted a lot of interest recently, but for the moment it remains unclear whether an exact algorithmic procedure of polynomial complexity can be found for the general case. We discuss aspects of this question in section 3.

However, the particular examples (1.3), (1.4), and others used in related expositions, actually belong to a special class. Figure 1 shows the linear interpolate of (1.3). Since only the values at discrete time instants are relevant, the function depicted in figure 2 will also suffice, and as a maximum of two (not three) linear functions needs fewer parameters to specify. We present an algorithm of linear-time complexity to construct the parametrically most economical such function. Being piecewise linear and convex, it has a simple representation in max-algebra. It is then a simple matter to convert this to a realisation of the form (1.1), (1.2).

For a subclass which contains Olsder's examples (1.3), (1.4), we prove that the resulting realisation is minimum-dimensional.

2. Basic definitions

Consider the triple $\mathcal{M}$:

$$\mathcal{M} = (R, \oplus, \otimes),$$

$$\text{where}\quad x \oplus y = \max(x, y),$$

$$x \otimes y = x + y.$$

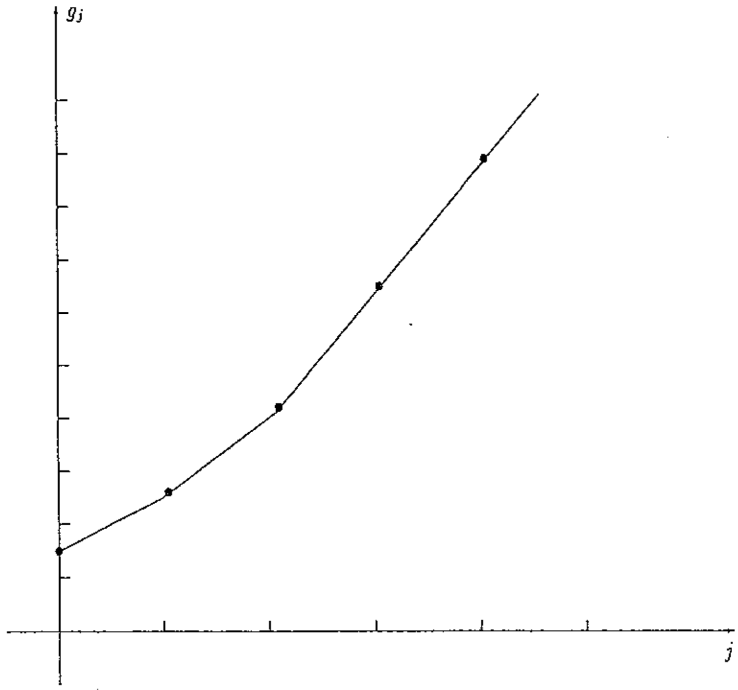

Figure 1.

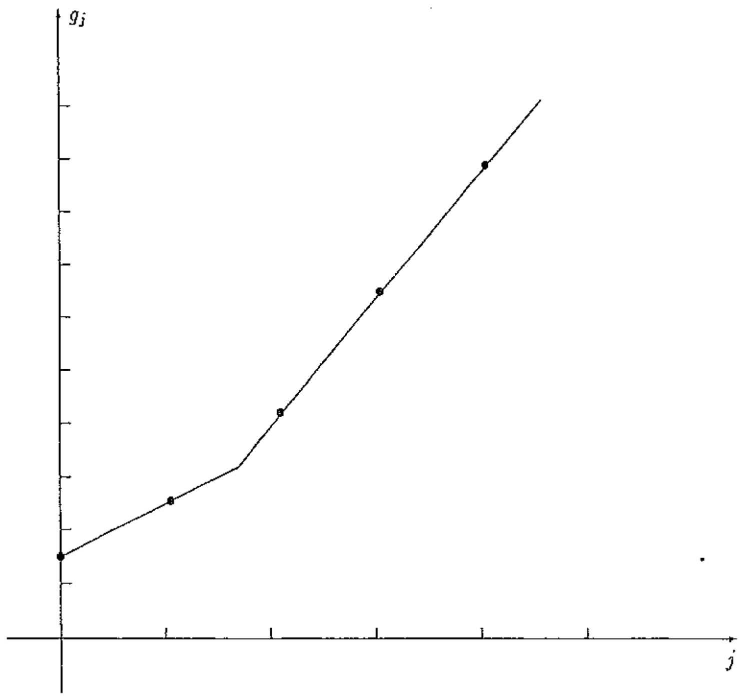

Figure 2.

We call this system *max-algebra* and may readily verify that it has the following properties:

$$x \oplus (y \oplus z) = (x \oplus y) \oplus z,$$
$$x \oplus y = y \oplus x,$$
$$x \otimes (y \otimes z) = (x \otimes y) \otimes z,$$
$$x \otimes y = y \otimes x,$$
$$x \otimes (y \oplus z) = x \otimes y \oplus x \otimes z,$$
$$x \otimes 0 = x,$$
$$x \oplus x = x.$$

Axiom systems of this nature are studied in detail in e.g. [5, 13], where they are used as a starting point for a theory of linear algebra. Frequently, a conventional element $-\infty$ is adjoined, to play the role of a zero. This gives certain algebraic advantages but introduces problems of its own, and in the present work we shall avoid this. All algebra will be finite.

The r-fold "product" of an element x with itself will be written as a "power":

$$x^{(r)} = x \otimes x \otimes \ldots \otimes x \quad (r\text{-times}).$$

Evidently, $x^{(r)}$ is more familiarly denoted rx, so we may consistently introduce zero and negative exponents with the definitions

$$x^{(0)} = 0,$$
$$x^{(-r)} = -rx \quad (r > 0).$$

We can now define a "division" operation inverse to the operation "$\otimes$" by

$$x//y = x \otimes y^{-1}.$$

Being motivated by the fact that $x^{(r)} = rx$ for integer r, we define

$$x^{(a)} = ax$$

for all $x, a \in R$. Clearly,

$$x^{(a)} = a^{(x)}. \tag{2.1}$$

To complete the notation, we need a "sigma" and "pi" to denote iterated "sums" and "products" of indexed expression in max-algebra. Thus, for given terms $\xi_1, \ldots, \xi_n$,

$$\sum_{1}^{n}{}^{\oplus} \xi_j \text{ denotes } \xi_1 \oplus \ldots \oplus \xi_n,$$
$$\prod_{1}^{n}{}^{\otimes} \xi_j \text{ denotes } \xi_1 \otimes \ldots \otimes \xi_n.$$

We extend $\oplus$, $\otimes$ to matrices in the usual way:

If $A = (a_{ij})$, $B = (b_{ij})$ are suitably-dimensioned real matrices, then $A \oplus B = (a_{ij} \oplus b_{ij})$ and $A \otimes B = (c_{ij})$, where

$$c_{ij} = \sum_k^{\oplus} a_{ik} \otimes b_{kj}, \quad \text{for all } i, j.$$

If A is a square matrix, then $A^{(r)}$ ($r \geq 1$ natural) will stand for the iterated product

$$\underbrace{A \otimes A \otimes \ldots \otimes A}_{r\text{-times}}.$$

Finally, $\alpha \otimes A = (\alpha \otimes a_{ij})$ for $\alpha \in R$.

Let A_j denote the jth column of the matrix A ($j = 1, 2, \ldots, n$). The columns of A are called *linearly dependent* if

$$\sum_{j \in U}^{\oplus} \lambda_j \otimes A_j = \sum_{j \in V}^{\oplus} \lambda_j \otimes A_j$$

for some $\lambda_1, \ldots, \lambda_n \in R$; $U \cup V \subseteq \{1, 2, \ldots, n\}$; $U \cap V = \emptyset$; $U, V \neq \emptyset$.

Let $n \geq 1$ be an integer and let P_n (resp. P_n^+, P_n^-) denote the set of all (resp. even, odd) permutations of the set $\{1, 2, \ldots, n\}$. If $\pi \in P_n$, we define

$$w(A, \pi) = \prod_{i=1}^{n} {}^{\otimes} a_{i, \pi(i)},$$

$$\text{per}(A) = \sum_{\pi \in P_n}^{\oplus} w(A, \pi),$$

$$\text{ap}(A) = \{\pi \in P_n ; w(A, \pi) = \text{per}(A)\},$$

$$\text{ap}^+(A) = \text{ap}(A) \cap P_n^+,$$

$$\text{ap}^-(A) = \text{ap}(A) \cap P_n^-.$$

Note that for a given matrix A, the task of finding $\pi \in P_n$, maximising $w(A, \pi)$, is the *classical assignment problem* [12] and thus ap(A) is the set of optimal solutions to this problem.

3. DEDS and their realisations

Let G be a sequence of real numbers

$$\{g_j\}_{j=0}^{\infty}$$

and let $A \in \mathbb{R}^{n \times n}$, $c \in \mathbb{R}^{1 \times n}$, $b \in \mathbb{R}^{n \times 1}$ be such that

$$g_j = c \otimes A^{(j)} \otimes b \quad (j = 0, 1, \ldots),$$

where by convention $A^{(0)} \otimes b = b$.

Then G is said to be *produced by a discrete-event dynamic system* (briefly *system* or *DEDS*) and the triple (A, b, c) is called a *realisation* of the system of *dimension n.*

PROBLEM FORMULATION

Given a sequence produced by a DEDS, find a realisation of the least dimension. Such a realisation will be called a *minimal-dimensional realisation* (abbreviation MDR) of the DEDS.

Since our algebra is finite, it follows from [5, theorem 27-9] that every sequence $G = \{g_j\}_{j=0}^{\infty}$ produced by a DEDS is *ultimately p-λ-periodic*, i.e. there exist $\lambda \in \mathbb{R}$ and non-negative integers p, j_0 such that

$$g_{j+p} = g_j \otimes \lambda^{(p)}, \quad \text{for all } j \geq j_0.$$

Clearly, each such sequence is fully described by its first $j_0 + 1$ terms, p and λ. The value of p is called the *period* of the system. If j_0 is the smallest natural number possessing the above property, then $\{g_j\}_{j=0}^{j_0}$ is called the *transient* of G.

Given the sequence $\{g_j\}_{j=0}^{\infty}$, we denote

$$H_r = \begin{bmatrix} g_0 & g_1 & \cdots & g_r \\ g_1 & g_2 & \cdots & g_{r+1} \\ \vdots & & & \vdots \\ g_r & g_{r+1} & \cdots & g_{2r} \end{bmatrix} \quad \text{for } r = 0, 1, 2, \ldots .$$

We shall require the following known results:

THEOREM 3.1

If a realisation of dimension n exists for the system producing $\{g_j\}_{j=0}^{\infty}$, then for all $r \geq n$ the matrix H_r has linearly dependent columns.

Proof

Follows from the results in [10]. □

THEOREM 3.2

A necessary and sufficient condition that a square matrix B have linearly dependent columns is

$$\text{ap}^+(B) \neq \varnothing \text{ and } \text{ap}^-(B) \neq \varnothing. \quad (3.1)$$

Proof

See [9,1]. □

It was shown in [1] that checking (3.1) is polynomially equivalent to the problem of deciding the existence of an even cycle in a digraph. Some more information on this subject can be found in [8].

However, we shall deal only with a special case for which checking (3.1) can be done in linear time.

THEOREM 3.3

If $\mathrm{ap}^+(H_r) = \emptyset$ or $\mathrm{ap}^-(H_r) = \emptyset$ for some $r > 0$, then there is no realisation of dimension r or less for the system producing $\{g_j\}_{j=0}^{\infty}$.

Proof

Follows immediately from theorems 3.1 and 3.2. (It may also be derived from the results in [7] and [8].). □

The sequence $\{g_j\}_{j=0}^{\infty}$ will be called *convex* if $g_{j+1} - g_j \geq g_j - g_{j-1}$ for all $j = 1, 2, \ldots$.

EXAMPLE 1

Consider the sequence

$$7, 4, 2, 1, 1, 2, 4, 8, 12, \ldots .$$

Here, $j_0 = 6$, $p = 1$, $\lambda = 4$ and the transient is 7, 4, 2, 1, 1, 2, 4. The sequence is convex.

Evidently,

$$H_0 = (7), \quad H_1 = \begin{pmatrix} 7 & 4 \\ 4 & 2 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 7 & 4 & 2 \\ 4 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix}, \quad H_3 = \begin{pmatrix} 7 & 4 & 2 & 1 \\ 4 & 2 & 1 & 1 \\ 2 & 1 & 1 & 2 \\ 1 & 1 & 2 & 4 \end{pmatrix}$$

and it is not difficult to verify that $\mathrm{ap}(H_3) = \{\mathrm{id}\}$ and hence $\mathrm{ap}^-(H_3) = \emptyset$. By theorem 3.3, there is no realisation of order 3 or less.

In order to make the set $\mathrm{ap}(H_4)$ more transparent, it is helpful to transform H_4 by adding suitable constants to rows and columns (which does not change the set of optimal solutions to the assignment problem) so that the resulting matrix $\tilde{H}_4$ is non-positive and $w(\tilde{H}_4, \sigma) = 0$ for some $\sigma \in P_5$. Then, evidently, $\mathrm{ap}(H_4) = \mathrm{ap}(\tilde{H}_4) = \{\pi \in P_5;\ w(\tilde{H}_4, \sigma) = 0\}$. In our example (as indeed for every Hankel matrix), it suffices to subtract from each row and column half of its diagonal element.

In this way, we transform

$$H_4 = \begin{bmatrix} 7 & 4 & 2 & 1 & 1 \\ 4 & 2 & 1 & 1 & 2 \\ 2 & 1 & 1 & 2 & 4 \\ 1 & 1 & 2 & 4 & 8 \\ 1 & 2 & 4 & 8 & 12 \end{bmatrix}$$

to the matrix

$$\tilde{H}_4 = \begin{bmatrix} 0 & -\frac{1}{2} & -2 & -\frac{9}{2} & -\frac{17}{2} \\ -\frac{1}{2} & 0 & -\frac{1}{2} & -2 & -5 \\ -2 & -\frac{1}{2} & 0 & -\frac{1}{2} & -\frac{5}{2} \\ -\frac{9}{2} & -2 & -\frac{1}{2} & 0 & 0 \\ -\frac{17}{2} & -5 & -\frac{5}{2} & 0 & 0 \end{bmatrix}.$$

$\tilde{H}_4$ indicates that

$$\mathrm{ap}^+(\tilde{H}_4) = \{\mathrm{id}\},$$

$$\mathrm{ap}^-(\tilde{H}_4) = \{(1) \circ (2) \circ (3) \circ (4\ \ 5)\}.$$

Hence, we cannot exclude the existence of a realisation of order 4.

The principles of the method we propose for finding an MDR of such a sequence are in fact very straightforward and, before beginning a detailed justification, it may be helpful to consider figure 2 again. The graph shown is the upper envelope of the two linear functions $2j + 3$ and $4j + 1/2$, i.e.

$$g_j = \max(2j + 3, 4j + 1/2).$$

In the notation of max-algebra, this is

$$g_j = 3 \otimes 2^{(j)} \oplus (\tfrac{1}{2}) \otimes 4^{(j)}.$$

If we allowed the use of the "zero" element $-\infty$, so that "diagonal" matrices could be defined in which all off-diagonal entries equal $-\infty$, we could at once rewrite this in the form

$$g_j = c \otimes A^{(j)} \otimes b,$$

where

$$c = (3, 1/2),\ A = \mathrm{diag}(2, 4),\ b = (0, 0)^{\mathrm{T}}.$$

Our work in section 4 is to show how this construction may be implemented without use of $-\infty$, by taking sufficiently small finite off-diagonal elements. In section 5, we discuss the minimal-dimensionality of the resulting realisation.

4. A special case and its realisation

Assume that the sequence $G = \{g_j\}_{j=0}^{\infty}$ produced by the DEDS meets the following two requirements (4.1) and (4.2):

G is ultimately 1-λ-periodic, i.e.

$$g_{j+1} = \lambda \otimes g_j \tag{4.1}$$

for all $j \geq j_0$ and some natural j_0 and real λ; the transient of G is strictly convex, i.e.

$$g_{j+1} - g_j > g_j - g_{j-1} \tag{4.2}$$

for all $j = 1, 2, \ldots, j_0$.

Note that both (1.3), (1.4) and the sequence of example 1 satisfy (4.1), (4.2). It is clear that every sequence satisfying (4.1), (4.2) is convex.

We show now how, given a sequence G satisfying (4.1), (4.2), one can find a realisation of the DEDS producing G.

Suppose $\{g_j\}_{j=0}^{\infty}$ is a sequence satisfying (4.1), (4.2) with transient $\{g_j\}_{j=0}^{j_0}$. Set $N = 1 + \lceil j_0/2 \rceil$ and let $p_s(x)$ $(s = 0, 1, \ldots, N-1)$ be a real function of one variable corresponding to the line determined by the points $[2s, g_{2s}]$ and $[2s+1, g_{2s+1}]$ in the plane, i.e.

$$p_s(x) = (g_{2s+1} - g_{2s})x + (2s+1)g_{2s} - 2sg_{2s+1}. \tag{4.3}$$

It is a matter of routine verification that for arbitrary $r, s \in \{0, 1, \ldots, N-1\}$ we have $p_s(2s) = g_{2s}$, $p_s(2s+1) = g_{2s+1}$; $p_r(x) \leq p_s(x)$, if $2s \leq x \leq 2s+1$; and $p_r(x) \leq p_{N-1}(x)$ if $x \geq j_0$.

THEOREM 4.1

Let $\{g_j\}_{j=0}^{\infty}$ be a sequence satisfying (4.1), (4.2), $p_0(x), \ldots, p_{N-1}(x)$ be defined by (4.3), and $p(x) = \max_{s=0,\ldots,N-1} p_s(x)$. Then

$$\begin{aligned} p(j) &= g_j && \text{for all } j = 0, 1, 2, \ldots \\ &= p_{N-1}(j) && \text{for all } j \geq j_0. \end{aligned}$$

Proof

Follows straightforwardly from the foregoing discussion. □

Given any matrix $A = [a_{st}]$, the element in position (s, t) of $A^{(j)}$ will be denoted $[A^{(j)}]_{st}$. Thus, $[A^{(1)}]_{st} = a_{st}$.

THEOREM 4.2

Given natural numbers N, j_0, and N real numbers $k_1 \leq \ldots \leq k_N$, there exists an $N \times N$ matrix A with the following properties:

$$[A^{(j)}]_{NN} = k_N^{(j)} = \sum_t^{\oplus} [A^{(j)}]_{Nt} \quad (j = 1, 2, \ldots), \tag{4.4}$$

$$[A^{(j)}]_{ss} = k_s^{(j)} = \sum_t^{\oplus} [A^{(j)}]_{st} \quad (j = 1, \ldots, j_0;\ s = 1, \ldots, N). \tag{4.5}$$

Proof

Define $A = [a_{st}]$ by $a_{ss} = k_s$ $(s = 1,\ldots,N)$; $a_{st} = \delta$ $(s, t = 1,\ldots,N; s \neq t)$, where

$$\delta = k_1^{(j_0)} // k_N^{(j_0 - 1)}. \tag{4.6}$$

Since $\delta = k_N \otimes (k_1 // k_N)^{(j_0)} \leq k_N$, the definition of A implies

$$a_{st} \leq k_N \quad (s, t = 1, \ldots, N). \tag{4.7}$$

Now, for any j, s, t; $[A^{(j)}]_{st}$ is of the form

$$[A^{(j)}]_{st} = \sum_{f \ldots h}^{\oplus} a_{sf} \otimes \ldots \otimes a_{ht}, \tag{4.8}$$

the $\oplus$-summands being j-fold $\otimes$-products.

From (4.7), (4.8):

$$[A^{(j)}]_{st} \leq k_N^{(j)} \quad (j = 1, 2, \ldots;\ s, t = 1, \ldots, N). \tag{4.9}$$

For the case $s = t = N$, (4.8) shows that

$$[A^{(j)}]_{NN} \geq a_{NN} \otimes \ldots \otimes a_{NN} = k_N^{(j)} \quad (j = 1, 2, \ldots),$$

which, together with (4.9), proves (4.4).

Suppose that the j-fold $\otimes$-product which determines the value of (4.8) constains u non-diagonal and $j - u$ diagonal elements from A (where u obviously depends on s, t and may equal zero or j). Then from the way A was defined,

$$[A^{(j)}]_{st} \leq \delta^{(u)} \otimes k_N^{(j-u)} = k_1^{(j)} \otimes (k_1 // k_N)^{(j_0 u - j)}, \tag{4.10}$$

where clearly $u \geq 1$ if $s \neq t$. Now, to prove (4.5), assume $j \leq j_0$.

If $s \neq t$, then $u \geq 1$, so $j_0 u \geq j$ and (4.10) implies

$$[A^{(j)}]_{st} \leq k_1^{(j)} \leq a_{ss}^{(j)} \leq \sum_{f \ldots h}^{\oplus} a_{sf} \otimes \ldots \otimes a_{hs} = [A^{(j)}]_{ss}, \tag{4.11}$$

which implies

$$[A^{(j)}]_{ss} = \sum_{t=1,\ldots,N}^{\oplus} [A^{(j)}]_{st} \quad (j = 1, \ldots, j_0;\ s = 1, \ldots, N). \tag{4.12}$$

However, if $s = t$ in (4.8), then u may be zero, in which case the product determining (4.8) must be $a_{ss} \otimes \ldots \otimes a_{ss}$ and we have

$$[A^{(j)}]_{ss} = a_{ss}^{(j)}. \tag{4.13}$$

Otherwise, $u \geq 1$ and the argument leading to (4.11) may be repeated with $s = t$, again implying (4.13). Clearly, (4.12), (4.13) together imply (4.5). □

THEOREM 4.3

Let $G = \{g_j\}_{j=0}^{\infty}$ be a sequence satisfying (4.1), (4.2) and define for $s = 1,\ldots,N$:

$$k_s = g_{2s-1} - g_{2s-2},$$
$$c_s = (2s-1)g_{2s-2} - (2s-2)g_{2s-1}.$$

Then (A, b, c) is a realisation of the DEDS producing G, when $c = (c_1,\ldots,c_N)$, $b = (0,\ldots,0)^{\mathrm{T}}$, and A is as in theorem 4.2.

Proof

Let $p_0(x),\ldots,p_{N-1}(x)$, $p(x)$ be as in theorem 4.1. By choice of b,

$$c \otimes A^{(j)} \otimes b = \sum_{s,t}{}^{\oplus} c_s \otimes [A^{(j)}]_{st} \otimes b_t = \sum_{s}{}^{\oplus} c_s \otimes \sum_{t}{}^{\oplus} [A^{(j)}]_{st}. \tag{4.14}$$

When $j \leq j_0$, (4.14) equals (by (4,5))

$$\begin{aligned}\sum_{s}{}^{\oplus} c_s \otimes k_s^{(j)} &= \max_{s=1,\ldots,N} (c_s + jk_s) \\ &= \max_{s=1,\ldots,N} p_{s-1}(j) \\ &= \max_{s=0,\ldots,N-1} p_s(j) \\ &= p(j) = g_j \qquad \text{by theorem 4.1.}\end{aligned}$$

Now suppose $j > j_0$. For any s $(1 \leq s \leq N)$, let P be the j-fold $\otimes$-product which determines the value of $\sum^{\oplus}[A^{(j)}]_{st}$. Then either P is of the form

$$P = k_s^{(j)} \leq k_s^{(j_0)} \otimes k_N^{(j-j_0)},$$

or P contains $u \geq 1$ non-diagonal and $(j-u)$ diagonal elements, whence, as in (4.10)

$$\begin{aligned}P &\leq \delta^{(u)} \otimes k_N^{(j-u)} \\ &\leq \delta \otimes k_N^{(j-1)} && \text{since } u \geq 1 \text{ and } \delta \leq k_N \\ &= k_1^{(j_0)} \otimes k_N^{(j-j_0)} && \text{by the definition of } \delta \\ &\leq k_s^{(j_0)} \otimes k_N^{(j-j_0)} && \text{since } k_1 \leq k_s.\end{aligned}$$

In any event, then,

$$\begin{aligned}\sum_s^{\oplus} c_s \otimes \sum_t^{\oplus} [A^{(j)}]_{st} &\leq \sum_s^{\oplus} c_s \otimes k_s^{(j_0)} \otimes k_N^{(j-j_0)} \\ &= k_N^{(j-j_0)} \otimes \sum_s^{\oplus} c_s \otimes (j_0)^{(k_s)} \\ &= k_N^{(j-j_0)} \otimes p(j_0) \\ &= k_N^{(j-j_0)} \otimes p_{N-1}(j_0) && \text{by theorem 4.1} \\ &= k_N^{(j-j_0)} \otimes c_N \otimes k_N^{(j_0)} \\ &= c_N \otimes k_N^{(j)} \\ &= c_N \otimes j^{(k_N)} \\ &= p_{N-1}(j). && (4.15)\end{aligned}$$

On the other hand,

$$\begin{aligned}c_N \otimes \sum_t^{\oplus} [A^{(j)}]_{Nt} &= c_N \otimes k_N^{(j)} \quad \text{(using (4.4))} \\ &= c_N \otimes j^{(k_N)} \\ &= p_{N-1}(j). && (4.16)\end{aligned}$$

Clearly, (4.15) and (4.16) show that, for $j > j_0$, (4.14) is just $p_{N-1}(j)$, which equals $p(j)$ when $j \geq j_0$ by theorem 4.1. Hence, for all j,

$$c \otimes A^{(j)} \otimes b = p(j) = g_j. \qquad \square$$

EXAMPLE 1 (continued)

Using theorem 4.2, we easily find a realisation (A, b, c) of the DEDS producing the sequence

$$7, 4, 2, 1, 1, 2, 4, 8, 12, \ldots .$$

Here, $N = 1 + \lceil 6/2 \rceil = 4$ and

$$\begin{aligned}c_1 &= 1 \cdot g_0 - 0 \cdot g_1 = 7, \\ c_2 &= 3g_2 - 2g_3 = 4, \\ c_3 &= 5g_4 - 4g_5 = -3, \\ c_4 &= 7g_6 - 6g_7 = -20;\end{aligned}$$

$$\begin{aligned} k_1 &= 4-7 &&= -3,\\ k_2 &= 1-2 &&= -1,\\ k_3 &= 2-1 &&= 1,\\ k_4 &= 8-4 &&= 4,\\ \delta &= 6(-3)-5.4 &&= -38. \end{aligned}$$

Hence,

$$A = \begin{bmatrix} -3 & -38 & -38 & -38 \\ -38 & -1 & -38 & -38 \\ -38 & -38 & 1 & -38 \\ -38 & -38 & -38 & 4 \end{bmatrix}, \quad c = (7,4,-3,-20), \quad b = (0,0,0,0)^{\mathrm{T}}.$$

Since we have already shown that no realisation of dimension 3 or less exists, we conclude that this is a minimal-dimensional realisation.

5. Minimality of the dimension

In this section, we show that the Hankel matrix corresponding to a convex sequence offers an easy way of checking the criterion given by theorem 3.3. This will enable us to prove that the realisation described in theorem 4.3 is always minimal-dimensional.

PROPOSITION 5.1

If $G = \{g_j\}_{j=0}^{\infty}$ is convex, then

$$g_{j+k} - g_j \le g_{j+k+\ell} - g_{j+\ell} \quad \text{for all } j,k,\ell \ge 0. \tag{5.1}$$

Proof

Straightforward. □

PROPOSITION 5.2

Suppose that $G = \{g_j\}_{j=0}^{\infty}$ is convex and $H_n = (a_{ij})$, where $a_{ij} = g_{i+j-2}$ $(i,j = 1,2,\ldots)$. Then

$$a_{r,n+1} + a_{n+1,s} \le a_{rs} + a_{n+1,n+1} \tag{5.2}$$

for all $r, s = 1, 2, \ldots, n+1$.

Proof

Straightforward, on setting $j = r+s-2$, $k = n-s+1$, $\ell = n-r+1$ (in (5.1)). □

If π is a permutation of the set S and S' is a subset of S such that $\pi(i) \in S'$ for all $i \in S'$, then $\pi|S'$ denotes the permutation of the set S' induced by π.

PROPOSITION 5.3

Suppose that $G = \{g_j\}_{j=0}^{\infty}$ is convex and $\pi \in P_{n+1}$, $\pi(\pi(n+1)) \neq n+1$.

(a) If

$$w(H_n, id) \leq w(H_n, \pi), \tag{5.3}$$

then

$$w(H_{n-1}, id) \leq w(H_{n-1}, \sigma), \quad \text{for some } \sigma \in P_n, \ \sigma \neq id. \tag{5.4}$$

(b) If (5.3) is strict, then also (5.4) is strict.

Proof

Suppose $\pi(n+1) = s$, $\pi(r) = n+1$, $r \neq s$.

(a) Set $\bar{\pi} \in P_{n+1}$ as follows:

$$\bar{\pi}(r) = s,$$
$$\bar{\pi}(n+1) = n+1,$$
$$\bar{\pi}(i) = \pi(i) \ \text{for all } i \neq r, \ i \neq n+1.$$

By proposition 5.2, we then have

$$w(H_n, \bar{\pi}) - w(H_n, \pi) = a_{rs} + a_{n+1,n+1} - a_{r,n+1} - a_{n+1,s} \geq 0$$

and taking $\sigma = \bar{\pi}|\{1, 2, \ldots, n\}$, we obtain

$$\begin{aligned} w(H_{n-1}, \sigma) &= w(H_n, \bar{\pi}) - a_{n+1,n+1} \\ &\geq w(H_n, \pi) - a_{n+1,n+1} \\ &\geq w(H_n, id) - a_{n+1,n+1} \\ &= w(H_{n-1}, id) \end{aligned}$$

and $\sigma \neq id$.

(b) It suffices to follow the lines of the proof of (a) in which the last inequality would be strict. □

THEOREM 5.1

Suppose that $G = \{g_j\}_{j=0}^{\infty}$ is convex. Then for all $n = 0, 1, 2, \ldots$ there holds

(a) $id \in \mathrm{ap}(H_n)$,

and (for $n \geq 1$)

(b) if $\mathrm{ap}(H_0) = \mathrm{ap}(H_1) = \ldots = \mathrm{ap}(H_{n-1}) = \{id\}$ and $\mathrm{ap}(H_n) \neq \{id\}$, then $\mathrm{ap}(H_n) = \{id, (1) \circ (2) \circ \ldots \circ (n-1) \circ (n, n+1)\}$.

Proof

(a) For $n = 0$, the assertion is trivial; for $n \geq 1$, we have $g_0 + g_2 \geq 2g_1$ because $g_2 - g_1 \geq g_1 - g_0$ follows from convexity of G. We now proceed by induction.

Suppose $n > 1$, $id \in \mathrm{ap}(H_j)$, $(j = 0,\ldots,n-1)$ and that

$$w(H_n, id) < w(H_n, \pi)$$

for some $\pi \in P_{n+1}$, $\pi \neq id$.

First we show that $\pi(\pi(n+1)) \neq n+1$. If $\pi(n+1) = r$, $\pi(r) = n+1$ $(1 \leq r \leq n+1)$, then for $\bar{\pi} \in P_{n+1}$ defined by

$$\bar{\pi}(r) = r,$$
$$\bar{\pi}(n+1) = n+1,$$
$$\bar{\pi}(i) = \pi(i) \text{ for all } i \neq r,\ i \neq n+1,$$

there holds (by proposition 5.2, setting $s = r$):

$$w(H_n, \bar{\pi}) - w(H_n, \pi) = a_{rr} + a_{n+1,n+1} - a_{r,n+1} - a_{n+1,r} \geq 0$$

and hence $\sigma = \bar{\pi} \mid \{1, 2,\ldots,n\}$ satisfies

$$\begin{aligned} w(H_{n-1}, \sigma) &= w(H_n, \bar{\pi}) - a_{n+1,n+1} \\ &\geq w(H_n, \pi) - a_{n+1,n+1} \\ &> w(H_n, id) - a_{n+1,n+1} \\ &= w(H_{n-1}, id), \end{aligned}$$

which contradicts the induction hypothesis. Therefore, $\pi(\pi(n+1)) \neq n+1$. Now the statement follows from proposition 5.3(b) and the induction hypothesis.

(b) Suppose $\pi \in \mathrm{ap}(H_n)$, $\pi \neq id$. Then $\pi(n+1) \neq n+1$ because otherwise for $\sigma = \pi \mid \{1, 2,\ldots,n\}$ we would have

$$\sigma \neq id,$$
$$w(H_{n-1}, \sigma) = w(H_n, \pi) - a_{n+1,n+1} = w(H_n, id) - a_{n+1,n+1} = w(H_{n-1}, id)$$

and thus $\sigma \in \mathrm{ap}(H_{n-1})$, a contradiction.

If $\pi(\pi(n+1)) \neq n+1$, then proposition 5.3(a) yields a contradiction with the assumption $\mathrm{ap}(H_{n-1}) = \{id\}$. Hence, we may assume that $\pi(n+1) = r$, $\pi(r) = n+1$, $r \leq n$. We show that $r = n$.

First notice that

$$\pi(i) = i \text{ for all } i \neq r,\ i \neq n+1, \tag{5.5}$$

for, otherwise taking $\bar{\pi} \in P_{n+1}$,

$$\bar{\pi}(r) = r,$$
$$\bar{\pi}(n+1) = n+1,$$
$$\bar{\pi}(i) = \pi(i) \text{ for all } i \neq r,\ i \neq n+1,$$

we would have (by proposition 5.2, setting $r = s$)

$$w(H_n, \bar{\pi}) - w(H_n, \pi) \geq 0,$$

implying, by optimality of π, $w(H_n, \bar{\pi}) = w(H_n, \pi) = w(H_n, id)$ and $w(H_{n-1}, \sigma) = w(H_{n-1}, id)$, $\sigma \neq id$ for $\sigma = \bar{\pi} \mid \{1, 2, \ldots, n\}$, which contradicts $\mathrm{ap}(H_n) = \{id\}$.

Now, from (5.5), and the optimality of π and id,

$$a_{r,n+1} + a_{n+1,r} = a_{rr} + a_{n+1,n+1}, \tag{5.6}$$

or, equivalently,

$$2g_{n+r-1} = g_{2r-2} + g_{2n}, \tag{5.7}$$

which can also be written as

$$g_{2n} - g_{n+r-1} = g_{n+r-1} - g_{2r-2}. \tag{5.8}$$

It follows from the convexity of G that

$$\left.\begin{array}{lcl} g_{2n} - g_{2n-1} & \geq & g_{n+r-1} - g_{n+r-2}, \\ g_{2n-1} - g_{2n-2} & \geq & g_{n+r-2} - g_{n+r-3}, \\ & \vdots & \\ g_{n+r} - g_{n+r-1} & \geq & g_{2r-1} - g_{2r-2}. \end{array}\right\} \tag{5.9}$$

But (5.8) implies that all inequalities in (5.9) must be satisfied as equations.

At the same time

$$g_{2n} - g_{2n-1} \geq g_{2n-1} - g_{2n-2} \geq \ldots \geq g_{n+r} - g_{n+r-1}$$
$$\geq g_{n+r-1} - g_{n+r-2} \geq \ldots \geq g_{2r-1} - g_{2r-2},$$

which yields that all left- and right-hand side values in (5.9) are equal.

Specifically,

$$g_{2r} - g_{2r-1} = g_{2-1} - g_{2r-2},$$

or

$$g_{2r} + g_{2r-2} = 2g_{2r-1}$$

and

$$a_{r+1,r+1} + a_{rr} = a_{r+1,r} + a_{r,r+1}.$$

Hence, if $r < n$, then $\mathrm{ap}(H_r) \neq \{id\}$ because for $\sigma = (1) \circ (2) \circ \ldots \circ (r-1) \circ (r, r+1)$ we have

$$w(H_r, \sigma) = w(H_r, id).$$

Hence, $\pi(n+1) = n$, $\pi(n) = n+1$. □

COROLLARY 1 OF THEOREM 5.1

Let $n \geq 1$ be an integer and $G = \{g_j\}_{j=0}^{\infty}$ be convex. Then $n = \min\{r; \mathrm{ap}(H_r) \neq \{id\}\}$ if and only if n is the smallest natural number satisfying

$$g_{2n} + g_{2n-2} = 2g_{n-1}. \tag{5.10}$$

□

Taken in conjunction with theorem 3.3, relation (5.10) gives an easy criterion for checking the first linear dependence of columns in the Hankel matrices, and hence excluding the existence of low-order realisations. We illustrate this now.

EXAMPLE 1 (continued)

We have

$$G = \{7, 4, 2, 1, 1, 2, 4, 8, 12, \ldots\}.$$

Checking (5.10) for $n = 1, 2, 3, 4$:

$$7 + 2 \neq 8,$$
$$2 + 1 \neq 2,$$
$$1 + 4 \neq 4,$$
$$4 + 12 = 16.$$

This simple calculation confirms our earlier conclusion that there is no realisation of dimension less than 4 of the system producing G.

COROLLARY 2 OF THEOREM 5.1

Let $G = \{g_j\}_{j=0}^{\infty}$ be a sequence satisfying (4.1) and (4.2). Then the realisation of the DEDS producing G described in theorem 4.2 is minimal-dimensional.

Proof

The realisation in theorem 4.2 is of dimension $N = 1 + \lceil j_0/2 \rceil$. It follows from (4.1), (4.2) that the smallest natural number n satisfying (5.10) is $\frac{1}{2}(j_0 + 2)$ (if j_0 is even) or $\frac{1}{2}(j_0 + 3)$ (if j_0 is odd). Using theorem 3.3 and corollary 1 of theorem 5.1, we obtain that there is no realisation of order less than $1 + \lceil j_0/2 \rceil$. □

EXAMPLE 2

For the sequence (1.3), we obtain

$$j_0 = 2, \quad \lambda = 4, \quad N = 2,$$
$$k_1 = 2, \quad k_2 = 4, \quad \delta = 0,$$
$$c_1 = 3, \quad c_2 = 1/2,$$

and hence (A, b, c) is an MDR of the DEDS producing (1.3), where

$$c = (3, 1/2), \quad A = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}, \quad b = (0,0)^{\mathrm{T}}.$$

EXAMPLE 3

For the sequence (1.4), we obtain

$$j_0 = 2, \ \lambda = 4, \ N = 2,$$

$$c = (5, 3\tfrac{1}{2}), \quad A = \begin{bmatrix} 3 & 2 \\ 2 & 4 \end{bmatrix}, \quad b = (0,0)^{\mathrm{T}}.$$

6. Postscript

In a practical situation, the instant j_0 at which the transient is deemed to have finished is essentially a matter of empirical judgement. If the process indeed satisfies conditions (4.1), (4.2), then this can be determined by sequential testing in real time with an amount of calculation which depends only linearly on the length of the transient, and the same is true of the computation of the parameters c_s, k_s and δ. Hence, the proposed method constitutes a linear-time algorithm for constructing an MDR for a DEDS producing a Markov-parameter sequence which is strictly convex in the sense of (4.1), (4.2).

It is clear that the construction works also for any sequence which is convex and ultimately 1-λ-periodic, though the foregoing proof of minimality of dimension is no longer valid. Nevertheless, we conjecture that a suitable modification of the argument will show that the procedure produces an MDR in this case also.

Finally, we note that the piecewise-linear functions involved in the construction are actually *maxpolynomials*. For a full account of such functions, the reader is referred to [6]. Maxpolynomials may be regarded as giving in their own right a mathematical realisation of convex sequences which are ultimately 1-λ-periodic, and the ideas of the present paper may be adapted as a way of finding the most economical such description for such a sequence.

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