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Solution of Systems of Linear Extremal Equations

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1. Introduction

In many recent papers (some of them see in references) there are considered "linear" models where the sign + represents in fact not only addition but other binary operation like maximum or minimum of two numbers, union or symmetric difference of two sets and other. The role of multiplication can be, of course, also different, e.g. minimum or addition of two numbers, Latin multiplication etc. A survey of the most often searched structures one can find e.g. in [5]. Some applications in industry are given for example in [2] and [3].

In this article we shall investigate only cases when + is maximum (or minimum) and multiplication plays its own role. In accordance with [7] the corresponding linear systems of equations we shall call linear extremal systems. The systematic theory of such a model will not be presented here and only some useful properties of these systems will be given.

We shall distinguish three types of linear extremal systems: (I) all unknowns are at the same side of equations; (II) the unknowns are divided into two groups the members of the first group being on the left-hand side of each equation and the unknowns of the second one together with an absolute term being on the right-hand sides, respectively; (III) unknowns and absolute terms are on both sides but in every equation at most on one side.

In all three types we suppose coefficients and absolute terms to be nonnegative. Evidently, type (I) is a special case of type (II) which is a special case of type (III). It can be shown (see [1]) that the last type is general enough, i.e. every system of linear extremal equations and inequalities can be transformed by means of slack variables to the type (III).

Clearly, in current linear situations it is not necessary to distinguish the three types mentioned above. But in our case solutions of those types are rather different.

2. Formalisation

We shall use the following notation. Let q, m, n be natural numbers. Then $Q = \{1, 2, \ldots, q\}$, $M = \{1, 2, \ldots, m\}$, $N = \{1, 2, \ldots, n\}$. Everywhere $\mathbf{X}$, $\mathbf{Y}$, ... are real column vectors. $\mathbf{X}^{\mathrm{T}}$ denotes the transposition of a vector $\mathbf{X}$. Further we denote:

$$\mathfrak{M}_n = \{\mathbf{X} \mid \mathbf{X}^{\mathrm{T}} = [X_1, \ldots, X_n], X_j \geqq 0 \quad \text{for all} \quad j \in N\},$$

$$\mathfrak{M}_n^+ = \{\mathbf{X} \mid \mathbf{X} \in \mathfrak{M}_n, X_j > 0 \quad \text{for all} \quad j \in N\},$$

$$\mathbf{X} \mathbin{\bar{o}} \mathbf{Y} = \max_{j \in N} X_j Y_j,$$

$$X \underset{\circ}{} Y = \min_{j \in N} X_j Y_j\,, \quad \text{for all} \quad X, Y \in \mathfrak{M}_n\,.$$

Supposing $\mathbf{A}$ is a nonnegative matrix of the type (q, n) we denote by A_i the i-th row vector and A^j the j-th column vector of the matrix $\mathbf{A}$ (for all $i \in Q$, $j \in N$). Assume $X \in \mathfrak{M}_n$. We define $\mathbf{A} \,\bar{\circ}\, X$, $\mathbf{A} \,\underset{\circ}{}\, X$ as follows:

$$(\mathbf{A} \,\bar{\circ}\, X) \in \mathfrak{M}_q\,, \quad (\mathbf{A} \,\bar{\circ}\, X)_i = A_i \,\bar{\circ}\, X\,, \quad \text{for} \quad i \in Q\,,$$
$$(\mathbf{A} \,\underset{\circ}{}\, X) \in \mathfrak{M}_q\,, \quad (\mathbf{A} \,\underset{\circ}{}\, X)_i = A_i \,\underset{\circ}{}\, X\,, \quad \text{for} \quad i \in Q\,.$$

If moreover $\mathbf{C} = (C^1, C^2, \ldots, C^l)$ is a nonnegative matrix of the type (n, l) then $\mathbf{A} \,\bar{\circ}\, \mathbf{C}$ is such a matrix of the type (q, l) that $(\mathbf{A} \,\bar{\circ}\, \mathbf{C})^j = \mathbf{A} \,\bar{\circ}\, C^j$.

Partial ordering on $\mathfrak{M}_n$ will be defined as usual:

$$X \leqq Y \quad \text{if} \quad X_j \leqq Y_j \quad \text{for all} \quad j \in N\,,$$

and

$$X < Y \quad \text{if} \quad X_j < Y_j \quad \text{for all} \quad j \in N\,.$$

From the elementary properties of the operation $\bar{\circ}$ recall that for $X, Y \in \mathfrak{M}_n$ and nonnegative matrix $\mathbf{A}$

(1) $$X \leqq Y \quad \text{implies} \quad \mathbf{A} \,\bar{\circ}\, X \leqq \mathbf{A} \,\bar{\circ}\, Y\,.$$

Now suppose

$$X = [X_1, X_2, \ldots, X_n]^{\mathrm{T}} \in \mathfrak{M}_n\,,$$
$$Y = [Y_1, Y_2, \ldots, Y_n]^{\mathrm{T}} \in \mathfrak{M}_n\,.$$

We define $X \oplus Y$, $X \oplus' Y$ as follows:

$$X \oplus Y \in \mathfrak{M}_n\,, \quad (X \oplus Y)_j = \max\{X_j, Y_j\} \quad \text{for all} \quad j \in N\,,$$
$$X \oplus' Y \in \mathfrak{M}_n\,, \quad (X \oplus' Y)_j = \min\{X_j, Y_j\} \quad \text{for all} \quad j \in N\,.$$

For an arbitrary real number $\alpha \geqq 0$ we denote by αX the vector $[\alpha X_1, \alpha X_2, \ldots \ldots, \alpha X_n]^{\mathrm{T}}$.

Let $\mathbf{L} = (L_{ij})$ and $\mathbf{R} = (R_{ij})$ be nonnegative matrices of types (q, m) and (q, n), respectively, $\mathbf{B} \in \mathfrak{M}_q$. The three types of linear extremal systems may be now written as follows.

(I) $\mathbf{L} \,\bar{\circ}\, X = \mathbf{B}$,

(II) $\mathbf{L} \,\bar{\circ}\, X = \mathbf{R} \,\bar{\circ}\, Y \oplus \mathbf{B}$,

(III) $\mathbf{L} \,\bar{\circ}\, X = \mathbf{R} \,\bar{\circ}\, X \oplus \mathbf{B}$, $m = n$, $L_{ij} R_{ij} = 0$ for all $i \in Q$, $j \in N$.

In all three types the signs $\bar{\circ}$ and $\oplus$ may be replaced by $\underset{\circ}{}$ and $\oplus'$, respectively.

All relevant properties of the system (I) are well known and described for example in [7], [8],[9]. But the types (II) and (III) according to author's knowledge were not treated yet with exception of [1] and partly of [6].

We shall derive some basic properties of types (II) and (III) by means of known properties of the system (I).

3. A conjugate

Now suppose $\mathbf{X} \in \mathfrak{M}_n^+$ and $\mathbf{A}$ be a positive matrix of the type (q, n). We denote by $\mathbf{X}^-$ the vector from the set $\mathfrak{M}_n^+$ defined by the following relation:

$$(\mathbf{X}^-)_j = \frac{1}{X_j}, \quad j \in N$$

and by $\mathbf{A}^-$ positive matrix of the type (q, n) defined as follows

$$(\mathbf{A}^-)_{ij} = \frac{1}{A_{ij}} \quad \text{for} \quad i \in Q, j \in N .$$

Let $\mathbf{L}$, $\mathbf{R}$ be positive matrices of the types (q, n), (q, m) respectively and $\mathbf{B} \in \mathfrak{M}_q^+$. We define the sets $\mathbf{S}$ and $\mathbf{S}^-$ as follows:

$$\mathbf{S} = \left\{ \begin{pmatrix} \mathbf{X} \\ \mathbf{Y} \end{pmatrix} \in \mathfrak{M}_{n+m}^+ \;\middle|\; \mathbf{L} \mathbin{\bar{\circ}} \mathbf{X} = \mathbf{R} \mathbin{\bar{\circ}} \mathbf{Y} \oplus \mathbf{B} \right\},$$

$$\mathbf{S}^- = \left\{ \begin{pmatrix} \mathbf{U} \\ \mathbf{V} \end{pmatrix} \in \mathfrak{M}_{n+m}^+ \;\middle|\; \mathbf{L}^- \mathbin{\underset{\circ}{}} \mathbf{U} = \mathbf{R}^- \mathbin{\underset{\circ}{}} \mathbf{V} \oplus' \mathbf{B}^- \right\}.$$

The statement of the next theorem follows immediately from results in [8].

Theorem 1. Under the assumptions made about $\mathbf{L}$, $\mathbf{R}$, $\mathbf{B}$ it holds: $\mathbf{X} \in \mathbf{S}$ if and only if $\mathbf{X}^- \in \mathbf{S}^-$.

4. Basic properties of the system (I)

In what follows it will be useful to utilize well known properties of the system (I). We shall now briefly show some of them. For proofs and related problems see e.g. [7], [8], [9].

We suppose $\mathbf{L}$ to have only nonzero columns. We denote for all $j \in N$

$$Q_j = \{ i \in Q \mid L_{ij} = \max_{k \in Q} L_{kj} \} ,$$

$$^-X_j = \frac{1}{\max\limits_{k \in Q} L_{kj}} \quad \text{and} \quad {}^-\mathbf{X} = [{}^-X_1, \ldots, {}^-X_n]^{\mathrm{T}} ,$$

$$\mathbf{P} = \{ \mathbf{X} \in \mathfrak{M}_n \mid \mathbf{A} \mathbin{\bar{\circ}} \mathbf{X} = \mathbf{B} \} .$$

According to reasons given in [7] we assume in next theorems $\mathbf{B} \in \mathfrak{M}_q^+$ and therefore in the following three theorems without loss of generality $\mathbf{B} = [1, 1, \ldots, 1]^{\mathrm{T}}$. The last vector will be denoted further by $\tilde{1}$.

Theorem 2. $\mathbf{P} \neq \emptyset$ if and only if

(2) $$\bigcup_{j \in N} Q_j = Q .$$

Theorem 3. If $\mathbf{P} \neq \emptyset$ then

a) $^{-}\mathbf{X} \in \mathbf{P}$,

b) $\mathbf{X} \in \mathfrak{M}_n$ is an element of $\mathbf{P}$ if and only if $\mathbf{X} \leqq {}^{-}\mathbf{X}$ and for the set $N_1 = \{j \in N \mid X_j = {}^{-}X_j\}$ it holds

$$\bigcup_{j \in N_1} Q_j = Q .$$

Theorem 4. $^{-}\mathbf{X}$ is the unique solution of (I) if and only if the system

$$\{Q_j \mid j \in N\} \tag{3}$$

is a minimal covering of Q, i.e. (3) covers Q but its every proper subsystem does not.

Definition. A nonnegative matrix $\mathbf{L} = (L_{ij})$ of the type (q, n) is said to have the i-th row covered if there exists an index $k \in N$ so that

$$L_{ik} = \max_{r \in Q} L_{rk} .$$

By means of this concept we can interpret Theorem 2 by words: The system (I) has a solution if and only if the matrix $\mathbf{L}$ has all rows covered.

This enables us to formulate the solvability condition for the system (I) in the case when $\mathbf{B}$ is an arbitrary positive vector by the next statement.

Corollary of Theorem 2. The system (I) has a solution if and only if the matrix created by means of multiplication of the i-th row of matrix $\mathbf{L}$ by $1/B_i$, for each $i \in Q$, has all rows covered.

The mentioned matrix can be, naturally, written in the form

$$\operatorname{diag}\{D_1, D_2, \ldots, D_q\} \mathbin{\bar{\circ}} \mathbf{L} ; \quad D_i = \frac{1}{B_i}, \quad i \in Q .$$

Note that in the last expression the sign $\bar{\circ}$ might be replaced by the obvious product of two matrices.

5. A basic property of the system (III)

Now we show a property of (III) in a certain sense similar to that of system (I). For an arbitrary $\mathbf{X} \in \mathfrak{M}_n$ and for all $j \in N$ we denote

$$Q^{\mathbf{L}}(X_j) = \{i \in Q \mid L_{ij}X_j = \max_{k \in N} L_{ik}X_k\} ,$$
$$Q^{\mathbf{B}}(\mathbf{X}) = \{i \in Q \mid \mathbf{L}_i \mathbin{\bar{\circ}} \mathbf{X} = B_i\} ,$$
$$Q^{\mathbf{R}}(X_j) = \{i \in Q \setminus Q^{\mathbf{B}}(\mathbf{X}) \mid R_{ij}X_j = \max_{k \in N} R_{ik}X_k\} .$$

Evidently, if $\mathbf{X}$ is a solution of (III) then

$$\bigcup_{j \in N} Q^{\mathbf{L}}(X_j) = Q = Q^{\mathbf{B}}(\mathbf{X}) \cup \bigcup_{j \in N} Q^{\mathbf{R}}(X_j) . \tag{4}$$

The opposite, however, does not hold. A counterexample: For $\mathbf{X} = [1, 1]^T$ there is

$$\begin{pmatrix} 0 & 3 \\ 2 & 0 \end{pmatrix} \bar{o}\, \mathbf{X} \neq \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \bar{o}\, \mathbf{X} \oplus \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

but (4) is true.

Theorem 5. If $^-\mathbf{X}$ is a solution of (III) then every vector $\mathbf{X} \in \mathfrak{M}_n$, $\mathbf{X} \leqq {}^-\mathbf{X}$ such that the set

$$N_1 = \{j \in N \mid X_j = {}^-X_j\}$$

satisfies the next condition is also a solution of (III):

$$\bigcup_{j \in N_1} Q^{\mathsf{L}}({}^-X_j) = Q = Q^{\mathsf{B}}({}^-\mathbf{X}) \cup \bigcup_{j \in N_1} Q^{\mathsf{R}}({}^-X_j)\,.$$

The proof follows immediately from assumptions and has the same background as proofs made in [7] concerning the system (I).

6. Uniqueness of solution

In this part we give a sufficient condition for the solution of the system (III) to be unique.

In the remaining parts of the article we shall use continually this notation:

$$E(\mathbf{X}) = \mathbf{L}\, \bar{o}\, \mathbf{X}\,.$$

$E_i(\mathbf{X})$ means the i-th component of the vector $E(\mathbf{X})$.

First we shall assume $\mathbf{B} \in \mathfrak{M}_q^+$, i.e. without loss of generality $\mathbf{B} = \tilde{1}$.

Theorem 6. Let $\mathbf{B} = \tilde{1}$ and $^-\mathbf{X} \in \mathfrak{M}_n$ be a vector satisfying

$$\mathbf{R}\, \bar{o}\, {}^-\mathbf{X} < \tilde{1}\,. \tag{5}$$

Then $^-\mathbf{X}$ is the unique solution of (III) if and only if $^-\mathbf{X}$ is the unique solution of the system

$$\mathbf{L}\, \bar{o}\, \mathbf{X} = \tilde{1}\,. \tag{6}$$

Proof. First suppose $^-\mathbf{X}$ not to be the unique solution of (6). It follows then from Theorem 3 that there exists a solution of the system (6) in each neighbourhood of $^-\mathbf{X}$. However,

$$\mathbf{T} = \{\mathbf{Z} \in \mathfrak{M}_n \mid \mathbf{R}\, \bar{o}\, \mathbf{Z} < \tilde{1}\}$$

is an open set in $\mathfrak{M}_n$. Thus there exists a neighbourhood of $^-\mathbf{X}$ being a subset of $\mathbf{T}$. An arbitrary solution of (6) lying in this neighbourhood (different from $^-\mathbf{X}$) is evidently another solution of (III). Hence $^-\mathbf{X}$ is not the unique solution of (6).

It remains to prove the opposite statement. Suppose $^-\mathbf{X}$ to be the unique solution of (6) (this implies $^-\mathbf{X}$ to be a solution of (III)). According to Theorem 4 it means

$$^{-}X_j = \frac{1}{\max\limits_{k\in Q} L_{kj}}$$

and that's why the condition $\mathbf{R} \mathbin{\bar{o}} {}^{-}\mathbf{X} < \tilde{1}$ implies

$$\frac{R_{ij}}{\max\limits_{k\in Q} L_{kj}} < 1 \quad \text{for all} \quad i \in Q, j \in N\,,$$

or

$$\frac{R_{ij}}{L_{kj}} < 1 \quad \text{for all} \quad i \in Q, j \in N, k \in Q_j\,. \tag{7}$$

In order to get a contradiction suppose $\mathbf{Y} \neq {}^{-}\mathbf{X}$, $\mathbf{Y} \in \mathfrak{M}_n$ be also a solution of (III). Then evidently $E(\mathbf{Y}) \geqq \tilde{1}$ and $E(\mathbf{Y}) \neq \tilde{1}$ (according to the fact that ${}^{-}\mathbf{X}$ is the unique solution of (6)). That's why there exists $i_1 \in Q$ satisfying $E_{i_1}(\mathbf{Y}) > 1$. Let $j_1, j_2 \in N$ be such indices that

$$L_{i_1j_1} Y_{j_1} = E_{i_1}(\mathbf{Y}) = R_{i_1j_2} Y_{j_2} > 1\,. \tag{8}$$

(Note $i_1 \in Q_{j_1}$.) Since $R_{i_1j_2}{}^{-}X_{j_2} < 1$ we get owing to (8)

$$Y_{j_2} > {}^{-}X_{j_2}\,.$$

Take an arbitrary $i_2 \in Q_{j_2}$.

Hence

$$E_{i_2}(\mathbf{Y}) \geqq L_{i_2j_2} Y_{j_2} > L_{i_2j_2}{}^{-}X_{j_2} = 1\,.$$

Recall $L_{ij}R_{ij} = 0$ for all $i \in Q$, $j \in N$. Thus (8) implies $L_{i_1j_2} = 0$ and consequently $j_1 \neq j_2$. From (7) it follows that

$$\frac{R_{i_1j_2}}{L_{i_2j_2}} < 1$$

and thus $R_{i_1j_2} Y_{j_2} < L_{i_2j_2} Y_{j_2}$.

Let $j_3 \in N$ be such an index that

$$E_{i_2}(\mathbf{Y}) = R_{i_2j_3} Y_{j_3} \quad (>1)\,.$$

Again there is $j_3 \neq j_2$ and for $i_3 \in Q_{j_3}$

$$L_{i_2j_2} Y_{j_2} \leqq R_{i_2j_3} Y_{j_3} < L_{i_3j_3} Y_{j_3}\,.$$

By the same way we find indices $j_4, j_5, j_6, \ldots (j_4 \neq j_5 \neq j_6 \neq \ldots)$ and $i_3, i_4, \ldots$ so that it holds $i_4 \in Q_{j_4}$, $i_5 \in Q_{j_5}, \ldots$ and $L_{i_1j_1} Y_{j_1} = R_{i_1j_2} Y_{j_2} < L_{i_2j_2} Y_{j_2} \leqq R_{i_2j_3} Y_{j_3} < L_{i_3j_3} Y_{j_3} \leqq R_{i_3j_4} Y_{j_4} < L_{i_4j_4} Y_{j_4} \leqq R_{i_4j_5} Y_{j_5} < \ldots$.

Evidently, there exists an index j_v so that

$$j_v = j_k$$

for some $k \in \{1, 2, \ldots, v - 2\}$.

Hence we get

$$L_{i_k j_k} Y_{j_k} < R_{i_{\nu-1}, j_\nu} Y_{j_\nu}$$

what implies

$$L_{i_k j_k} < R_{i_{\nu-1}, j_\nu}$$

and contradicts $i_k \in Q_{j_k} = Q_{j_\nu}$ and (7), QED.

Now it is easy to give similar sufficient condition in a more general case, when $\mathbf{B} \in \mathfrak{M}_q$. If $\mathbf{B} = [0, 0, \ldots, 0]^T$ then the system (III) evidently cannot have unique nontrivial solution and we may assume without loss of generality that all positive components stand before all zero ones.

Denote as usual $\mathbf{A}(1, 2, \ldots, k)$ the submatrix of matrix $\mathbf{A}$ consisting of its first k rows.

Corollary of Theorem 6. Let in the system (III) be

$$B_1 = B_2 = \ldots = B_k = 1\,,$$
$$B_{k+1} = B_{k+2} = \ldots = B_q = 0\,,$$
$$1 \leqq k \leqq q\,,$$

$^-\boldsymbol{X}$ be a solution of (III) and suppose $^-\boldsymbol{X}$ to be the unique solution of the system

$$\mathbf{L}(1, 2, \ldots, k) \,\bar{\circ}\, \boldsymbol{X} = \tilde{1}$$

and

$$\mathbf{R}(1, 2, \ldots, k) \,\bar{\circ}\, {}^-\boldsymbol{X} < \tilde{1}\,.$$

Then $^-\boldsymbol{X}$ is the unique solution of the system (III).

7. An associate problem

Further we shall deal with (II). Everywhere we suppose matrices $\mathbf{L}$, $\mathbf{R}$ to be positive and $\mathbf{B} \in \mathfrak{M}_q^+$. We can associate with (II) a corresponding "homogeneous" system

(IV) $$\mathbf{L} \,\bar{\circ}\, \boldsymbol{X} = \mathbf{T} \,\bar{\circ}\, \boldsymbol{Z}$$

where $\mathbf{T}$ is a matrix of the type $(q, m + 1)$ with columns

$$\boldsymbol{R}^1, \boldsymbol{R}^2, \ldots, \boldsymbol{R}^m, \boldsymbol{B}$$

and

$$\boldsymbol{Z} = [Z_1, Z_2, \ldots, Z_{m+1}]^T\,.$$

Thus solutions of (IV) are vectors from $\mathfrak{M}_{n+m+1}$ and the vector $[0, 0, \ldots, 0]^T$ is always its (trivial) solution.

Theorem 7. System (II) has a solution if and only if the corresponding system (IV) has a nontrivial solution.

Proof. If $[X_1, X_2, \ldots, X_n, Y_1, \ldots, Y_m]^T$ is a solution of (II) then $[X_1, X_2, \ldots, X_n, Y_1, \ldots, Y_m, 1]^T$ is a nontrivial solution of (IV).

Now suppose $[X_1, \ldots, X_n, Z_1, \ldots, Z_{m+1}]^T$ to be a solution of (IV). Then due to the positivity of **L** and **T** there is $E(\mathbf{X}) \in \mathfrak{M}_q^+$. If there would be $Z_{m+1} = 0$ then every vector $[X_1, \ldots, X_n, Z_1, \ldots, Z_m, Z'_{m+1}]^T$, Z'_{m+1} being an arbitrary number from the interval

$$\left(0, \min_{i \in Q} \frac{E_i}{B_i}\right)$$

is obviously also a solution of (IV). Hence without loss of generality we may assume $Z_{m+1} > 0$.

Since α-multiple of a solution of (IV) for every $\alpha > 0$ is also a solution of (IV) we have that the vector

$$\frac{1}{Z_{m+1}} [X_1, \ldots, X_n, Z_1, \ldots, Z_{m+1}]^T$$

is a solution of (IV). But this implies that the vector

$$\frac{1}{Z_{m+1}} [X_1, \ldots, X_n, Z_1, \ldots, Z_m]^T$$

is a solution of (II), QED.

Note that the just presented proof shows also how to find a solution of (II) if a solution of (IV) is known.

8. Solution of the system (IV)

In this section we shall deal with the following problems:

Problem 1. Decide whether the system (IV) has a solution.

Problem 2. If the answer on the first question is positive find at least one solution of (IV).

First of all notice that if a nonzero vector $[X_1, \ldots, X_n, Z_1, \ldots, Z_{m+1}]^T$ is solution of (IV) then the vector

$$E(\mathbf{X}) = \mathbf{L} \,\bar{\circ}\, \mathbf{X} = \mathbf{T} \,\bar{\circ}\, \mathbf{Z}$$

is an element of $\mathfrak{M}_q^+$ (due to positivity of **L** and **T**).

This implies according to Corollary of Theorem 2 that the matrices

$$\operatorname{diag}\left\{\frac{1}{E_1}, \ldots, \frac{1}{E_q}\right\} \bar{\circ}\, \mathbf{L} \quad \text{and} \quad \operatorname{diag}\left\{\frac{1}{E_1}, \ldots, \frac{1}{E_q}\right\} \bar{\circ}\, \mathbf{T}$$

have all rows covered. The opposite holds trivially, too. Hence we have the next assertion.

Theorem 8. The system (IV) has a nontrivial solution if and only if there exist positive numbers $D_1, \ldots, D_q$ such that matrices

$$\operatorname{diag}\{D_1, \ldots, D_q\} \,\bar{o}\, \mathbf{L} \quad \text{and} \quad \operatorname{diag}\{D_1, \ldots, D_q\} \,\bar{o}\, \mathbf{T}$$

have all rows covered.

Definition. Let $\mathbf{L}$ and $\mathbf{T}$ be matrices with q rows. (Ordered) q-tuple $(D_1, D_2, \ldots, D_q)$ of positive numbers is called an **LT** q-tuple if both matrices

$$\operatorname{diag}\{D_1, \ldots, D_q\} \,\bar{o}\, \mathbf{L} \quad \text{and} \quad \operatorname{diag}\{D_1, \ldots, D_q\} \,\bar{o}\, \mathbf{T}$$

have all rows covered.

Thus the task to solve the system (IV) means to find an **LT** q-tuple $(D_1, \ldots, D_q)$. Then it remains to solve two systems of the type (I):

$$(\operatorname{diag}\{D_1, \ldots, D_q\} \,\bar{o}\, \mathbf{L}) \,\bar{o}\, \mathbf{X} = \tilde{1}$$

and

$$(\operatorname{diag}\{D_1, \ldots, D_q\} \,\bar{o}\, \mathbf{T}) \,\bar{o}\, \mathbf{Z} = \tilde{1}\,.$$

Remark 1. If $(D_1, \ldots, D_q)$ is an **LT** q-tuple then obviously $(\alpha D_1, \ldots, \alpha D_q)$, $\alpha > 0$ is also an **LT** q-tuple. That's why we may always put one of D_i equal to a constant (say 1).

In what follows we concentrate our effort on the explanation of the following two facts:

A. To every permutation of the set Q there corresponds a set of "significant" q-tuples of positive numbers. This set has at most $(n + m + 1)^{q-1}$ elements.

B. For every nontrivial solution of the system (IV) there exists a permutation of the set Q and a significant q-tuple corresponding to this permutation which is an **LT** q-tuple.

After showing these facts it will suffice for the solution of both problems mentioned in the beginning of this section to test all significant q-tuples of each permutation of the set Q to be **LT** q-tuples.

Definition. Significant q-tuple corresponding to a permutation $(i_1, \ldots, i_q)$ of the set Q is every q-tuple $(D_1, \ldots, D_q)$ of positive numbers satisfying the following conditions:

$$D_{i_1} = 1$$

and for every $s = 2, 3, \ldots, q$ there exists either an index $j \in N$ such that

$$D_{i_s} L_{i_s j} = \max_{k=1,\ldots,s-1} D_{i_k} L_{i_k j}$$

or an index $j \in M$ such that

$$D_{i_s} T_{i_s j} = \max_{k=1,\ldots,s-1} D_{i_k} T_{i_k j}\,.$$

Remark 2. It follows immediately from this definition that the number of significant q-tuples corresponding to a permutation lies in the interval

$$(9) \qquad \langle 1, (n + m + 1)^{q-1} \rangle .$$

Hence the number of all significant q-tuples of the system (IV) is not greater than

$$q!\,(n + m + 1)^{q-1} .$$

Example 1.

$$X_1 \oplus 2X_2 = 3Z_1 \oplus 4Z_2 \oplus 5Z_3 ,$$
$$5X_1 \oplus 6X_2 = 7Z_1 \oplus 8Z_2 \oplus 9Z_3 .$$

Here we have $q = 2$, $n = 2$, $m = 2$,

$$\mathbf{L} = \begin{pmatrix} 1 & 2 \\ 5 & 6 \end{pmatrix}, \quad \mathbf{T} = \begin{pmatrix} 3 & 4 & 5 \\ 7 & 8 & 9 \end{pmatrix}.$$

For the permutation $(i_1, i_2) = (1, 2)$ we get $(2 + 3)^{2-1} = 5$ various significant couples: $(1, \frac{1}{5})$, $(1, \frac{1}{3})$, $(1, \frac{3}{7})$, $(1, \frac{1}{2})$, $(1, \frac{5}{9})$. The reader can easy verify that if we join to this system the third equation

$$10X_1 \oplus 11X_2 = 12Z_1 \oplus 13Z_2 \oplus 14Z_3$$

then the new system will have exactly $(2 + 3)^{3-1} = 25$ different significant triples corresponding to the permutation $(1, 2, 3)$.

Note that the lower bound in (9) can be reached if, for example, the row vectors of both matrices $\mathbf{L}$ and $\mathbf{T}$ are created by the same vector.

Theorem 9. If the system (IV) has a nontrivial solution then there exists a permutation $(1, i_2, \ldots, i_q)$ of the set Q such that at least one significant q-tuple corresponding to this permutation is an **LT** q-tuple.

Proof. Suppose

$$[X_1, \ldots, X_n, Z_1, \ldots, Z_{m+1}]^{\mathrm{T}}$$

be a solution of (IV) and $D_i = 1/E_i(X)$ for all $i \in Q$. Hence $(D_1, \ldots, D_q)$ is an **LT** q-tuple and we may assume (according to what has been remarked) $D_1 = 1$. Put $i_1 = 1$, ${}^{-}D_1 = D_1$ and take β_2 the least positive number β such that

$$(D_1, \beta D_2, \ldots, \beta D_q)$$

retains an **LT** q-tuple. It means for all $\alpha < \beta_2$ there exists an uncovered ("critical") row either in the matrix

$$\operatorname{diag}\{D_1, \alpha D_2, \ldots, \alpha D_q\} \mathbin{\bar{\circ}} \mathbf{L}$$

or in the matrix

$$\operatorname{diag}\{D_1, \alpha D_2, \ldots, \alpha D_q\} \mathbin{\bar{\circ}} \mathbf{T} .$$

We take an arbitrary among critical rows (say with the least index) and denote its index by i_2. Naturally, there must exist either an index $j \in N$ with property

$$\beta_2 D_{i_2} L_{i_2 j} = L_{1j}$$

or an index $j \in M$ with property

$$\beta_2 D_{i_2} T_{i_2 j} = T_{1j}\,.$$

Denote

$$^{-}D_{i_2} = \beta_2 D_{i_2}\,.$$

Further take β_3 the least positive number β such that

$$(D_1, \beta D_2, \ldots, \beta_2 D_{i_2}, \ldots, \beta D_q)$$

retains an **LT** q-tuple and take i_3 the least index among those of critical rows. Obviously, there must again exist either an index $j \in N$ with property

$$\beta_3 D_{i_3} L_{i_3 j} = \max\left(L_{1j}, \beta_2 D_{i_2} L_{i_2 j}\right)$$

or an index $j \in M$ with property

$$\beta_3 D_{i_3} T_{i_3 j} = \max\left(T_{1j}, \beta_2 D_{i_2} T_{i_2 j}\right).$$

Denote

$$^{-}D_{i_3} = \beta_3 D_{i_3}\,.$$

Repeat this procedure until we obtain i_q. It follows from the construction presented above that the q-tuple $(^{-}D_1, {}^{-}D_2, \ldots, {}^{-}D_q)$ is one of significant q-tuples corresponding to the permutation

$$(1, i_2, \ldots, i_q)$$

at the same time being an **LT** q-tuple, QED.

The theorem we have just proved enables us to test whether (IV) is soluble via testing the significant q-tuples corresponding to all permutations of the type $(1, i_2, \ldots, i_q)$ of the set Q to be **LT** q-tuples. If no significant q-tuple would be an **LT** q-tuple then according to this theorem the system (IV) has only trivial solution. If some significant q-tuple would be an **LT** q-tuple then it determines a solution of (IV) by the technique mentioned in the beginning of this section.

These conclusions can be summarized in the following algorithm. The author is aware of the fact that the importance of this algorithm is only theoretical (with the exception of very small dimensions). The aim, however, is to create a theoretical basis for further research.

9. The algorithm and an example

Algorithm

1. For every permutation $(1, i_2, \ldots, i_q)$ of the set Q test all q-tuples $(\beta_1, \beta_2, \ldots, \beta_q)$ satisfying following relations to be **LT** q-tuples:

$$\beta_1 = 1 ,$$

$$\beta_2 \in Q_2 = \left\{ \frac{\beta_1 L_{i_1 j}}{L_{i_2 j}} \,\middle|\, j \in N \right\} \cup \left\{ \frac{\beta_1 T_{i_1 j}}{T_{i_2 j}} \,\middle|\, j \in M \right\} ,$$

$$\beta_3 \in Q_3 = \left\{ \frac{\beta_1 L_{i_1 j} \oplus \beta_2 L_{i_2 j}}{L_{i_3 j}} \,\middle|\, j \in N \right\} \cup \left\{ \frac{\beta_1 T_{i_1 j} \oplus \beta_2 T_{i_2 j}}{T_{i_2 j}} \,\middle|\, j \in M \right\} ,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\beta_q \in Q_q = \left\{ \frac{\sum\limits_{h=1}^{q-1}{}^{\oplus} \beta_h L_{i_h j}}{L_{i_q j}} \,\middle|\, j \in N \right\} \cup \left\{ \frac{\sum\limits_{h=1}^{q-1}{}^{\oplus} \beta_h T_{i_h j}}{T_{i_q j}} \,\middle|\, j \in M \right\} .$$

2. If there exists an **LT** q-tuple $(1, \beta_2, \ldots, \beta_q)$ among q-tuples tested sub 1 then the system (IV) has nontrivial solutions and one of them can be found by the successive solution (by the technique of Section 4) of two systems:

$$(\operatorname{diag}\{1, \beta_2, \ldots, \beta_q\} \mathbin{\bar{o}} \mathbf{L}) \mathbin{\bar{o}} \mathbf{X} = \tilde{1} ,$$
$$(\operatorname{diag}\{1, \beta_2, \ldots, \beta_q\} \mathbin{\bar{o}} \mathbf{T}) \mathbin{\bar{o}} \mathbf{Z} = \tilde{1} .$$

If such an **LT** q-tuple does not exist then the system (IV) has only trivial solution.

Example 2.

$$X_1 \oplus 5X_2 = 3Z_1 \oplus 5Z_2 \oplus 2Z_3 ,$$
$$2X_1 \oplus 3X_2 = 6Z_1 \oplus Z_2 \oplus Z_3 ,$$
$$X_1 \oplus 2X_2 = 2Z_1 \oplus 3Z_2 \oplus Z_2 .$$

Here we have $n = 2$, $m = 2$, $q = 3$,

$$\mathbf{L} = \begin{pmatrix} 1 & 5 \\ 2 & 3 \\ 1 & 2 \end{pmatrix} , \quad \mathbf{T} = \begin{pmatrix} 3 & 5 & 2 \\ 6 & 1 & 1 \\ 2 & 3 & 1 \end{pmatrix} .$$

Firstly we test significant triples corresponding to the permutation $(1, 2, 3)$: $\beta_1 = 1$

$$\beta_2 = \tfrac{1}{2} , \quad \beta_3 \in \{\tfrac{3}{2}, \tfrac{5}{3}, 2, \tfrac{5}{2}\} ,$$
$$\beta_2 = \tfrac{5}{3} , \quad \beta_3 \in \{\tfrac{5}{3}, \tfrac{5}{2}, \tfrac{10}{3}, 5\} .$$

The values of β_3 corresponding to $\beta_2 \in \{2, 5\}$ would be useless to compute because the first row cannot be covered for these values of β_2. One can easy verify that none of significant triples given above is an **LT** triple.

But take the permutation $(2, 3, 1)$. Computing one by one its significant triples we find out that $(1, 1, 2)$ is an **LT** triple.

That's why the system has nontrivial solutions and one of them can be found by the successive solution of systems:

$$X_1 \oplus 5X_2 = 1\,,$$
$$2X_1 \oplus 3X_2 = 1\,,$$
$$2X_1 \oplus 4X_2 = 1$$

and

$$3Z_1 \oplus 5Z_2 \oplus 2Z_3 = 1\,,$$
$$6Z_1 \oplus Z_2 \oplus Z_3 = 1\,,$$
$$4Z_1 \oplus 6Z_2 \oplus 2Z_3 = 1\,.$$

Thus solution of the given system is e.g. vector

$$[\tfrac{1}{2}, \tfrac{1}{5}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{2}]^{\mathrm{T}}\,.$$

10. System (IV) with $q = 2$

It is not necessary to use the general procedure described in previous sections in order to solve system (IV) with $q = 2$, i.e. the system of two linear extremal equations of the type (IV).

Next assertion gives a fundamental information about solutions of such systems.

Theorem 10. Let in (IV) be $q = 2$. Then

(a) it has a nontrivial solution if and only if

$$\mathscr{D} = \left\langle \min_{j\in N} \frac{L_{2j}}{L_{1j}},\quad \max_{j\in N} \frac{L_{2j}}{L_{1j}} \right\rangle \cap \left\langle \min_{j\in N} \frac{T_{2j}}{T_{1j}},\quad \max_{j\in N} \frac{T_{2j}}{T_{1j}} \right\rangle \neq \emptyset\,;$$

(b) the couple $(1, D)$ is an **LT** couple if and only if

$$D^{-1} \in \mathscr{D}\,. \tag{10}$$

Proof.

(a) Denote

$$\mathscr{L} = \{[1, E]^{\mathrm{T}} \in \mathfrak{M}_2^+ \mid (\exists \mathbf{X} \in \mathfrak{M}_n)\,([1, E]^{\mathrm{T}} = \mathbf{L} \mathbin{\bar{o}} \mathbf{X})\}\,,$$
$$\mathscr{R} = \{[1, E]^{\mathrm{T}} \in \mathfrak{M}_2^+ \mid (\exists \mathbf{Z} \in \mathfrak{M}_{m+1})\,([1, E]^{\mathrm{T}} = \mathbf{T} \mathbin{\bar{o}} \mathbf{Z})\}\,.$$

Obviously, the system has a nontrivial solution if and only if $\mathscr{L} \cap \mathscr{R} \neq \emptyset$.

It remains to show: $[1, E]^{\mathrm{T}} \in \mathscr{L}$ if and only if

$$E \in \left\langle \min_{j\in N} \frac{L_{2j}}{L_{1j}},\quad \max_{j\in N} \frac{L_{2j}}{L_{1j}} \right\rangle$$

and $[1, E]^{\mathrm{T}} \in \mathscr{R}$ if and only if

$$E \in \left\langle \min_{j\in N} \frac{T_{2j}}{T_{1j}},\quad \max_{j\in N} \frac{T_{2j}}{T_{1j}} \right\rangle.$$

We show only the first equivalence because the proof of the second one can be carried out by the same steps and differs from the first only in denotation.

Denote

$$\mathbf{L}' = (L'_{ij}) = \operatorname{diag}\{1, E^{-1}\} \mathbin{\bar{\circ}} \mathbf{L}\,.$$

Assuming

$$E > \max_{j \in N} \frac{L_{2j}}{L_{1j}}$$

we get for all $j \in N$

$$L'_{2j} < \frac{L_{2j}}{\dfrac{L_{2j}}{L_{1j}}} = L_{1j} = L'_{1j}$$

and thus $\mathbf{L}'$ has the second row not covered.

Similarly if $E < \max_{j \in N} (L_{2j}/L_{1j})$ we get for all $j \in N$

$$L'_{2j} > L'_{1j}$$

and thus in this case $\mathbf{L}'$ has the first row not covered. In both cases the wanted conclusion follows immediately from the Corollary of Theorem 2.

Finally, let

$$E \in \left\langle \min_{j \in N} \frac{L_{2j}}{L_{1j}}, \quad \max_{j \in N} \frac{L_{2j}}{L_{1j}} \right\rangle.$$

Then there exist indices $j_1, j_2 \in N$ (not necessary different) for which it holds

$$\frac{L_{2j_1}}{L_{1j_1}} \leqq E \leqq \frac{L_{2j_2}}{L_{1j_2}}\,.$$

Hence

$$L'_{2j_1} = \frac{L_{2j_1}}{E} \leqq L_{1j_1} = L'_{1j_1}$$

and

$$L'_{1j_2} = L_{1j_2} \leqq \frac{L_{2j_2}}{E} = L'_{2j_2}\,.$$

But this implies $\mathbf{L}'$ has both rows covered and it follows again from the Corollary of Theorem 2 that there exists a vector $\boldsymbol{X} \in \mathfrak{M}_n$ with property

$$\mathbf{L} \mathbin{\bar{\circ}} \boldsymbol{X} = [1, E]^{\mathrm{T}}\,.$$

The assertion (b) follows immediately from the assertion (a) if we take in mind that $(1, D)$ is an **LT** couple if and only if $[1, D^{-1}]^{\mathrm{T}} \in \mathscr{L} \cap \mathscr{R}$, QED.

At last two remarks:

1. The structure of the last system studied is not very complicated and it is not difficult to give formulas expressing all solutions of this system.

2. According to the part (b) of the last theorem it would be possible to simplify the algorithm described in the foregoing section by testing only those q-tuples $(\beta_1, \beta_2, \ldots, \beta_q)$ the components β_j of which fulfil the condition (10) related to the system of two extremal linear equations the first of which is the $\oplus$ sum of equations with indices $1, i_2, \ldots, i_{j-1}$ and the second one is the i_j-th equation. This reduction, however, is useful only from the practical point of view and does not create any theoretical saving of computational work.

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Resumé

RIEŠENIE SÚSTAV LINEÁRNYCH EXTREMÁLNYCH ROVNÍC

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Článok nadväzuje na práce [1], [7], [8], [9], v ktorých boli zavedené a skúmané sústavy extremálnych rovníc.

Autor rozlišuje tri základné typy týchto sústav. Vychádzajúc zo známych vlastností 1. typu sú odvodené vlastnosti sústav 2. a 3. typu.

Sústavám 2. typu sú priradené zodpovedajúce homogénne sústavy a pomocou nich je sformulovaný a dokázaný algoritmus, ktorý a) rozhoduje, či má sústava riešenie, b) ak riešenie existuje, tak nájde aspoň jeden vektor, ktorý je riešením sústavy.

Zvlášť sa popisuje množina riešení sústav dvoch extremálnych rovníc.