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Calculating essential terms of a characteristic maxpolynomial

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Abstract

Let us denote $a \oplus b = \max(a, b)$ and $a \otimes b = a + b$ for $a, b \in \mathbf{R} \cup \{-\infty\}$ and extend this pair of operations to matrices and vectors in the same way as in conventional linear algebra. We present a polynomial algorithm for finding all essential terms of the characteristic maxpolynomial $\chi_A(x) = \mathrm{per}(A \oplus x \otimes I)$ of matrices A with entries from $\mathbf{Q} \cup \{-\infty\}$. In the cases when all terms are essential this algorithm also solves the following problem: Given an $n \times n$ matrix A and $k \in \{1, \ldots, n\}$, find the maximal optimal value for the assignment problem over all $k \times k$ principal submatrices of A.

Keywords: max-algebra, characteristic maxpolynomial, assignment problem.

1 Introduction

A wide class of problems in different research areas, like graph theory, automata theory, scheduling theory etc. can be expressed using an attractive formulation language by setting up an algebra of, say, real numbers in which the operations of addition and multiplication are replaced by the selection of the maximum of the two numbers and arithmetical addition, respectively. Monographs [1], [2] and [9] can be used as a comprehensive guide in this field. Among other major works in the field are [3], [5], and [6]. Specifically, significant effort has been devoted to building up a theory similar to that of linear algebra, for instance to study systems of linear equations [8], eigenvalue problems, independence, rank, regularity and dimension. It turns out that there is only a thin barrier separating these concepts and combinatorial properties of matrices.

Let us denote $a \oplus b = \max(a, b)$ and $a \otimes b = a + b$ for $a, b \in \overline{\mathbf{R}} = \mathbf{R} \cup \{-\infty\}$. The iterated product $a \otimes a \otimes \ldots \otimes a$, in which the letter a appears k-times will be denoted by $a^{(k)}$. Let us extend the pair of operations $(\oplus, \otimes)$ to matrices and vectors in the same way as in conventional linear algebra. That is if $A = (a_{ij})$, $B = (b_{ij})$ and $C = (c_{ij})$ are matrices or vectors over $\overline{\mathbf{R}}$ of compatible sizes then we write $C = A \oplus B$ if $c_{ij} = a_{ij} \oplus b_{ij}$ for all i, j and $C = A \otimes B$ if $c_{ij} = \sum_k^{\oplus} a_{ik} \otimes b_{kj}$ for all i, j.

For any set X and positive integer n the symbol $X(n, n)$ will denote the set of all $n \times n$ matrices over X. The letter I stands for a square matrix of an appropriate order whose diagonal entries are 0 and off-diagonal ones are $-\infty$.

Principal submatrix of $A = (a_{ij}) \in \overline{\mathbf{R}}(n, n)$ is as usual any matrix of the form

$$\begin{pmatrix} a_{i_1 i_1} & a_{i_1 i_2} & \cdots & a_{i_1 i_k} \\ a_{i_2 i_1} & a_{i_2 i_2} & \cdots & a_{i_2 i_k} \\ \vdots & \vdots & & \vdots \\ a_{i_k i_1} & a_{i_k i_2} & \cdots & a_{i_k i_k} \end{pmatrix}$$

where $1 \le i_1 < i_2 < \ldots < i_k \le n$. This matrix will be denoted by $A(i_1, i_2, \ldots, i_k)$.

The ***permanent*** of $A = (a_{ij}) \in \overline{\mathbf{R}}(n, n)$ is defined as an analogue of the classical one:

$$\mathrm{per}(A) = \sum\nolimits^{\oplus}_{\pi \in P_n} w(A, \pi)$$

where P_n stands for the set of all permutations of the set $N = \{1, \ldots, n\}$ and

$$w(A, \pi) = \prod\nolimits^{\otimes}_{i \in N} a_{i, \pi(i)} \, .$$

In conventional notation $\mathrm{per}(A) = \max\limits_{\pi \in P_n} \sum\limits_{i \in N} a_{i, \pi(i)}$, which is the optimal value of the ***classical assignment problem***. There are a number of efficient solution methods for finding $\mathrm{per}(A)$, one of the best known is the Hungarian method of computational complexity $O(n^3)$.

Characteristic maxpolynomial of $A = (a_{ij}) \in \overline{\mathbf{R}}(n, n)$ has been defined in [4] as $\chi_A(x) = \mathrm{per}(A \oplus x \otimes I)$, that is the permanent of the matrix

$$\begin{pmatrix} a_{11} \oplus x & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} \oplus x & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \oplus x \end{pmatrix}.$$

It follows from this definition that $\chi_A(x)$ is of the form

$$(1.1) \qquad \delta_0 \oplus (\delta_1 \otimes x) \oplus \ldots \oplus (\delta_{n-1} \otimes x^{(n-1)}) \oplus x^{(n)}$$

or, briefly $\sum^{\oplus}_{i=1,\ldots,n} \delta_i \otimes x^{(i)}$ where $\delta_n = 0$ and, by convention, $x^{(0)} = 0$.

Example 1.1

Let A be the matrix

$$\begin{pmatrix} 1 & 3 & 2 \\ 0 & 4 & 1 \\ 2 & 5 & 0 \end{pmatrix}.$$

Then

$$\chi_A(x) = \mathrm{per}\begin{pmatrix} 1 \oplus x & 3 & 2 \\ 0 & 4 \oplus x & 1 \\ 2 & 5 & 0 \oplus x \end{pmatrix} =$$

$$= (1 \oplus x) \otimes (4 \oplus x) \otimes (0 \oplus x) \oplus (3 \otimes 1 \otimes 2) \oplus (2 \otimes 0 \otimes 5) \oplus$$
$$\oplus\ (2 \otimes (4 \oplus x) \otimes 2) \oplus ((1 \oplus x) \otimes 1 \otimes 5) \oplus (3 \otimes 0 \otimes (0 \oplus x)) =$$
$$= x^{(3)} \oplus 4 \otimes x^{(2)} \oplus 6 \otimes x \oplus 8 \, .$$

It has been proved in [4] that for $k = 0, 1, \ldots, n-1$

$$(1.2) \qquad \delta_k = \sum\nolimits^{\oplus}_{B \in \mathcal{P}_k(A)} \mathrm{per}(B)$$

where $\mathcal{P}_k(A)$ is the set of all principal submatrices of A of order $n-k$. Hence $\delta_0 = \mathrm{per}(A)$ and $\delta_{n-1} = \max(a_{11}, a_{22}, \ldots, a_{nn})$. Obviously, $\delta_k = -\infty$ if all $B \in \mathcal{P}_k(A)$ have $\mathrm{per}(B) = -\infty$ in which case the term $\delta_k \otimes x^{(k)}$ is omitted from $\chi_A(x)$ by convention. Note that $\chi_A(x)$ may reduce to just $x^{(n)}$, for instance if $a_{ij} = -\infty$ for all $i \ge j$. For the method presented in this paper it will be essential to know the smallest value of k for which δ_k is finite ("the lowest order term"). We will answer this question in Section 2.

The characteristic maxpolynomial (1.1) written using conventional notation is

$$\chi_A(x) = \max\{\delta_0, \delta_1 + x, \delta_2 + 2x, \delta_3 + 3x, \ldots, \delta_{n-1} + (n-1)x, nx\}.$$

Hence $\chi_A(x)$ is the upper envelope of $n+1$ linear functions and thus a convex function. If for some $k \in \{0, \ldots, n\}$

$$\delta_k \otimes x^{(k)} \le \sum\nolimits^{\oplus}_{i \ne k} \delta_i \otimes x^{(i)}$$

then the term $\delta_k \otimes x^{(k)}$ is called inessential, otherwise it is called essential (see Figures 1 and 2). Hence

$$\chi_A(x) = \sum\nolimits^{\oplus}_{i \ne k} \delta_i \otimes x^{(i)}$$

holds for all $x \in \mathbf{R}$ if $\delta_k \otimes x^{(k)}$ is inessential, and therefore inessential terms may be ignored if $\chi_A(x)$ is considered as a function. In Figure 1 below all terms of a characteristic maxpolynomial of a 3×3 matrix are drawn, all are essential. A similar situation appears in Figure 2 but the quadratic term is here inessential.

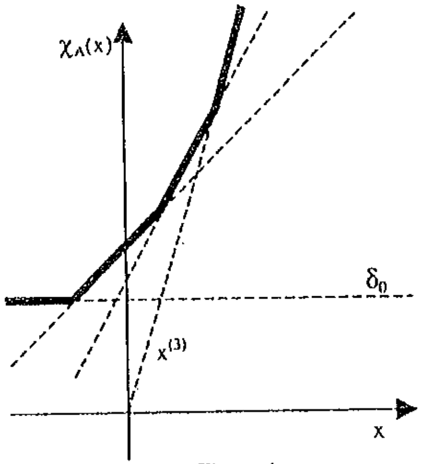

Figure 1

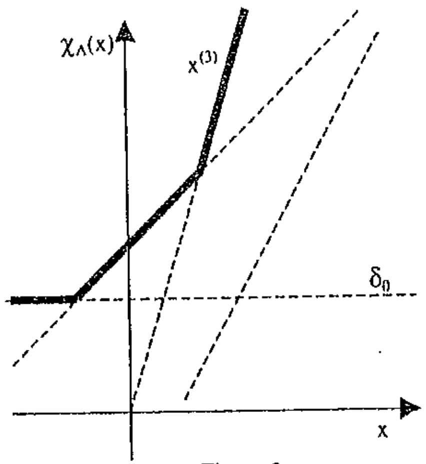

Figure 2

This paper is motivated by the fact that although per(B) can easily be found for any matrix B, δ_k cannot be computed from (1.2) efficiently since the number of matrices in $\mathcal{P}_k(A)$ is $\binom{n}{k}$.

Another motivation is related to the following two combinatorial optimisation problems:

OPTIMAL AP-SUBMATRIX (OAPSM): Given a matrix $A \in \overline{\mathbf{R}}(n, n)$ and $k \in \{1, \dots, n\}$, find the biggest optimal assignment problem value over all $k \times k$ submatrices of A.

OPTIMAL AP-PRINCIPAL SUBMATRIX (PRINCIPAL OAPSM): Given a matrix $A \in \overline{\mathbf{R}}(n, n)$ and $k \in \{1, \dots, n\}$, find the biggest optimal assignment problem value over all $k \times k$ principal submatrices of A.

Although it is not difficult to solve OAPSM [7], to the authors knowledge no polynomial method is known for PRINCIPAL OAPSM. In Section 4 we will present a polynomial method for finding all essential terms of a characteristic maxpolynomial. It follows from (1.2) that in the case when all terms in the characteristic maxpolynomial are essential this method solves also PRINCIPAL OAPSM in polynomial time.

2 Finding the lowest order term in the characteristic maxpolynomial

As has been mentioned before, for the main result of this paper it will be important to know the lowest order term of the characteristic maxpolynomial.

Theorem 2.1

Let $A = (a_{ij}) \in \overline{\mathbf{R}}(n, n)$. Then $\chi_A(x) = x^{(n)}$ if and only if the digraph $D = (N, E)$ is acyclic where $N = \{1, \dots, n\}$, $E = \{(i, j);\ a_{ij}$ is finite$\}$.

Proof

If D contains a cycle, say $(i_1, \dots, i_k)$, then $per(A(i_1, \dots, i_k)) \geq a_{i_1 i_2} + a_{i_2 i_3} + \dots + a_{i_k i_1} > -\infty$ and so δ_{n-k} is finite.

Conversely, if δ_k is finite for some $k \in \{0, 1, \dots, n-1\}$ then there is $B \in \mathcal{P}_k(A)$ such that $per(B) > -\infty$, say $B = A(i_1, \dots, i_{n-k})$. Let π be a permutation of $\{i_1, \dots, i_{n-k}\}$ with $w(\pi, B) = per(B)$ and let σ be any of the cycles to which π decomposes, say $\sigma = (j_1, \dots, j_r)$ where

$$\{j_1, \dots, j_r\} \subseteq \{i_1, \dots, i_{n-k}\}.$$

Then all of $a_{j_1 j_2}, a_{j_2 j_3}, \dots, a_{j_r j_1}$ are finite and thus $(j_1, \dots, j_r)$ is a cycle in D. ∎

Due to Theorem 2.1 we may assume in what follows that $\chi_A(x) \neq x^{(n)}$, that is $\chi_A(x)$ contains at least one term of order less than n.

Let us now denote by A_{min} the smallest finite element of the matrix A and by A_{max} the biggest element of A. Also denote $\underline{\delta} = \min(0, nA_{min})$ and $\overline{\delta} = \max(0, nA_{max})$.

Theorem 2.2

If x_0 is the point in which two different terms from $\chi_A(x)$ have the same value then

$$x_0 \geq \underline{\delta} - \overline{\delta}. \tag{2.1}$$

Proof

Suppose that $\delta_r + rx_0 = \delta_s + sx_0$ for some $r, s \in \{0, 1, \dots, n\}$, $r > s$. Then

$$(r-s)x_0 = \delta_s - \delta_r \tag{2.2}$$

and, clearly, $\delta_s \neq -\infty$. If $A_{min} \leq 0$ then (see (1.2)) $\delta_s \geq (n-s)A_{min} \geq nA_{min} = \underline{\delta}$; if $A_{min} \geq 0$ then $\delta_s \geq (n-s)A_{min} \geq 0 = \underline{\delta}$. Hence $\delta_s \geq \underline{\delta}$ and by a similar argument one can prove that $\delta_r \leq \overline{\delta}$. Therefore (2.2) yields $(r-s)x_0 \geq \underline{\delta} - \overline{\delta}$ and (2.1) now follows since $r - s > 0$ and $\underline{\delta} - \overline{\delta} \leq 0$.

∎

Let us denote $\underline{\delta} - \overline{\delta}$ by δ^*.

Corollary

If $\delta_k \otimes x^{(k)}$ is the lowest order term in $\chi_A(x)$ then $k = \chi_A(\delta^*) - \chi_A(\delta^* - 1)$ and $\delta_k = \chi_A(\delta^*) - k\delta^*$.

Proof

By Theorem 2.2 $\chi_A(x) = \delta_k \otimes x^{(k)}$ for all $x \leq \delta^*$. Hence $\chi_A(\delta^*) \dots \chi_A(\delta^* - 1)$

Note that the task of finding the lowest order term in a characteristic maxpolynomial is equivalent to the task of finding the maximal value of k for which there is a $k \times k$ principal submatrix B with finite per(B). By replacing $-\infty$ by 1 and finite elements by 0 it is easily seen that this is equivalent to the following combinatorial problem: Given a $0-1$ matrix A, find the maximal value of k for which A contains a $k \times k$ principal submatrix with k independent zeros (that is k zeros no two of which are taken either from the same row or the same column).

It should also be noted that if δ_k is finite then some of $\delta_{k+1}, \ldots, \delta_{n-1}$ may still be $-\infty$. For instance for every matrix of the form

$$\begin{pmatrix} \bullet & \bullet & -\infty \\ -\infty & -\infty & \bullet \\ \bullet & -\infty & -\infty \end{pmatrix}$$

where the dots indicate finite elements, δ_0 is finite (namely $a_{12} + a_{23} + a_{31}$) but $\delta_1 = -\infty$ (since the permanent of each of the three principal submatrices of order 2 is $-\infty$).

Example 2.1

Let A be the matrix

$$\begin{pmatrix} 3 & -4 & 1 \\ -\infty & 2 & -\infty \\ -\infty & 0 & -\infty \end{pmatrix}.$$

Then $A_{max} = 3$, $A_{min} = -4$, $\underline{\delta} = -12$, $\overline{\delta} = 9$, $\delta^* = -21$,

$$\chi_A(\delta^*) = \mathrm{per}\begin{pmatrix} 3 & -4 & 1 \\ -\infty & 2 & -\infty \\ -\infty & 0 & -21 \end{pmatrix} = -16,\ \chi_A(\delta^* - 1) = \begin{pmatrix} 3 & -4 & 1 \\ -\infty & 2 & -\infty \\ -\infty & 0 & -22 \end{pmatrix} = -17.$$

Hence for the lowest order term $k = 1$ and $\delta_1 = -16 + 21 = 5$.
It is easily verified that $\chi_A(x) = (5 \otimes x) \oplus (3 \otimes x^{(2)}) \oplus x^{(3)}$. ∎

3 An estimate

In what follows we will restrict our attention to matrices whose finite entries are all integers. Note however, that the method we will develop will be readily applicable to matrices with rational entries since if B is a matrix obtained from A by multiplying each entry by a constant, say c, we have that $\chi_B(x) = c\chi_A(x)$. Therefore the task of finding the characteristic maxpolynomial for a rational matrix can be converted to the same task for integer matrices by multiplying all entries by a common multiple of all denominators appearing in the matrix.
We denote $\mathbf{Z} \cup \{-\infty\}$ by $\overline{\mathbf{Z}}$.

Theorem 3.1

Let $A \in \overline{\mathbf{Z}}(n, n)$ and x and x' be two different points in each of which two of the terms in $\chi_A(x)$ have equal value. Then

$$|x - x'| > \frac{1}{n^2}. \tag{3.1}$$

Proof
Suppose that x is the point of intersection of $\delta_h \otimes x^{(h)}$ and $\delta_m \otimes x^{(m)}$ $(h > m)$ and x', $x' \neq x$, is the point of intersection of $\delta_i \otimes x^{(i)}$ and $\delta_j \otimes x^{(j)}$ $(i > j)$.
Hence

$$x = \frac{\delta_m - \delta_h}{h - m},\ x' = \frac{\delta_j - \delta_i}{i - j}$$

and so

$$|x - x'| = \frac{|(i-j)(\delta_m - \delta_h) - (h-m)(\delta_j - \delta_i)|}{(h-m)(i-j)}. \tag{3.2}$$

All finite entries of A are integers, thus the numerator in (3.2) is a non-zero integer, hence at least 1. At the same time, $h - m \leq n$, $i - j \leq n$ and so $|x - x'| \geq \frac{1}{n^2}$. Equality would imply (see the denominator of (3.2)) that $h = i = n$, $m = j = 0$ in which case the numerator turns to zero, a contradiction, and so (3.1) follows. ∎

4 The method

Here we present a method for finding all essential terms of the characteristic maxpolynomial of a given matrix $A = (a_{ij}) \in \overline{\mathbf{Z}}(n, n)$, that is for finding a subset E of $\{0, 1, \ldots, n\}$ such that

$$\chi_A(x) = \sum\nolimits^{\oplus}_{i \in E} \delta_i \otimes x^{(i)}$$

for all $x \in \mathbf{R}$. Recall that due to the results of Section 2 it may be assumed without loss of generality that $\chi_A(x)$ contains at least one term other than $x^{(n)}$ and that the lowest order term is known.
The method is based on the following two simple observations:
O1: It is easy to evaluate $\chi_A(\bar{x})$ for any particular value $\bar{x}$ since this task is exactly that of finding the optimal value for the assignment problem for the matrix $A \oplus \bar{x} \otimes I$.
O2: If $\bar{x}$ is a point of intersection of two terms of $\chi_A(x)$ then by Theorem 3.1 the interval (a, b) where $a = \bar{x} - \frac{1}{n^2}$ and $b = \bar{x} + \frac{1}{n^2}$ does not contain other

terms passing through $\bar{x}$: if all three points $(a, \chi_A(a))$, $(\bar{x}, \chi_A(\bar{x}))$, $(b, \chi_A(b))$ belong to one line then this line corresponds to the unique essential term passing through $\bar{x}$; if they don't belong to the same line then they determine two essential terms intersecting at $\bar{x}$.

Algorithm ESSENTIAL TERMS

Input: $A = (a_{ij}) \in \overline{\mathbf{Z}}(n, n)$ and the lowest order term of $\chi_A(x)$, say $\tilde{k}x + \tilde{\delta}$.

Output: The set K of all essential terms of $\chi_A(x)$.

<1> K(NOWN) := $\{\tilde{k}x + \tilde{\delta}, nx\}$
C(OUNTER) := 2
Q(UEUE) := $\{(\tilde{k}x + \tilde{\delta}, nx)\}$

<2> If $Q = \emptyset$ then stop
Take any $(kx + \delta, k'x + \delta') \in Q$, $Q := Q - \{(kx + \delta, k'x + \delta')\}$

$$\bar{x} := \frac{\delta - \delta'}{k' - k}$$

<3> Compute $\chi_A(\bar{x})$ and check whether it is equal to $k\bar{x} + \delta_k$ for some $kx + \delta_k \in K$. If yes, then go to <2> else go to <4>.

<4> $a := \bar{x} - \dfrac{1}{n^2}$, $b := \bar{x} + \dfrac{1}{n^2}$

If $\chi_A(\bar{x}) \neq \frac{1}{2}(\chi_A(a) + \chi_A(b))$ then go to <6> else go to <5>.

<5> $k_{new} := \frac{1}{2} n^2 (\chi_A(b) - \chi_A(a))$
$\delta_{new} := \chi_A(\bar{x}) - k_{new}\bar{x}$
$Q := Q \cup \{(k_{new}x + \delta_{new}, kx + \delta);\ kx + \delta \in K\}$
$K := K \cup \{k_{new}x + \delta_{new}\}$
$C := C + 1$
If $C = n + 1$ then stop else go to <2>

<6> $k_{new}(1) := n^2(\chi_A(b) - \chi_A(\bar{x}))$, $k_{new}(2) := n^2(\chi_A(\bar{x}) - \chi_A(a))$
$\delta_{new}(i) := \chi_A(\bar{x}) - k_{new}(i).\bar{x} \quad (i = 1, 2)$
$Q := Q \cup \{(k_{new}(i)x + \delta_{new}(i), kx + \delta);\ i = 1, 2;\ kx + \delta \in K\}$
$K := K \cup \{k_{new}(i)x + \delta_{new}(i);\ i = 1, 2\}$
$C := C + 2$
If $C = n + 1$ then stop else go to <2>

Theorem 4.1

Algorithm ESSENTIAL TERMS is correct and its computational complexity is $O(n^5)$.

Proof

Correctness: Upon termination of the algorithm either $C = n + 1$ or $Q = \emptyset$. In the first case correctness is trivial and we prove it in the second case by

two lines whose value at x is $\chi_A(x)$. Hence we can assume that at the termination of the algorithm

$$\chi_A(x) = F(x) \text{ for all } x \in X. \tag{4.1}$$

We need to prove that $\chi_A(x) = F(x)$ for all $x \in \mathbf{R}$. Clearly, $\chi_A(x) \geq F(x)$ for all $x \in \mathbf{R}$. Suppose that the strict inequality holds for some $x \in \mathbf{R}$, hence also over some interval. This interval is bounded since the first and last lines of $\chi_A(x)$ are in K.
Suppose that (a, b) is a maximal such interval, that is

$$\chi_A(x) > F(x) \text{ for all } x \in (a, b), \tag{4.2}$$

$$\chi_A(a) = F(a),$$
$$\chi_A(b) = F(b).$$

(Note that $\chi_A(x)$ may consist of more than one line over (a, b).)

Set
$$k_1 = \max\{k;\ kx + \delta_k \in K,\ ka + \delta_k = F(a)\},$$
$$k_2 = \min\{k;\ kx + \delta_k \in K,\ kb + \delta_k = F(b)\}$$

and let z be the point of intersection of the lines $k_1x + \delta_{k_1}$ and $k_2x + \delta_{k_2}$, that is

$$z = \frac{\delta_{k_2} - \delta_{k_1}}{k_1 - k_2} \quad \text{(see Figure 3).}$$

Since $F(x) < \chi_A(x) \leq \dfrac{F(b) - F(a)}{b - a}(x - a) + F(a)$ for $x \in (a, b)$, we have $k_2 > \dfrac{F(b) - F(a)}{b - a}$. By substituting for $F(a)$ and $F(b)$ and using simple manipulations we get

$$a < \frac{\delta_{k_2} - \delta_{k_1}}{k_1 - k_2} = z.$$

We can similarly derive that $z < b$. But then $\chi_A(z) > F(z)$ by (4.2), a contradiction to (4.1) since $z \in X$.

Computational complexity: It is easily seen that the individual steps <1> to <6> in the algorithm have the following complexities: $O(1)$, $O(1)$, $O(n^3)$, $O(n^3)$, $O(n)$, $O(n)$ (we assume here that the values of $\chi_A(a)$, $\chi_A(b)$ and $\chi_A(\bar{x})$ are stored and thus they are calculated only once per iteration). One run of the loop <2> to <6> is therefore $O(n^3)$. A new pair of lines appears in Q only if at least one of the lines has been accepted as a new essential term. One term cannot be accepted as an essential term more than once (see <3>) and each of $\frac{1}{2}n(n + 1)$ or less points of intersection of up to $n + 1$ lines will be processed at most once. Hence the number of iterations is $O(n^2)$ and the computational complexity bound follows. ∎

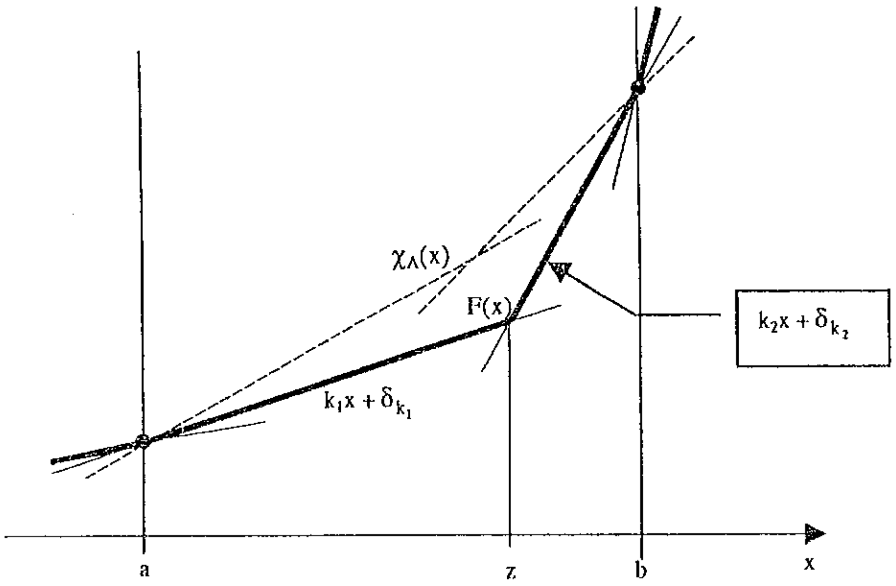

Figure 3

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Polynomial algorithm for linear matrix period in max-plus algebra

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Summary

Linear periodicity of matrices in max-plus algebra is studied. It is proved that the linear factor matrix and the linear period of a matrix A can be computed in $O(n^3)$ time, if A is almost linear periodic. Computation of the coordinate linear period $\mathrm{lper}(a^*_{ij})$ for given indices $i, j \in n$ is shown to be NP-hard. Further, a polynomial algorithm is described, which decides whether a given matrix is almost linear periodic, if the matrix fulfils a condition of incomparability for trivial strongly connected components. In general, this problem is NP-complete.

keywords: linear matrix period, max-plus algebra, NP-completeness

1 Introduction

Discrete dynamic systems and other algebraic structures are often studied using max-plus or max-min matrix operations and digraphs [2, 13, 16]. In the max-min matrix theory, the convergence and periodicity of matrices were studied in [1, 14] and by other authors. Polynomial algorithms and NP-completeness results for matrix and orbit periods in max-min algebra are presented in [5, 6, 7]. Polynomial algorithms for computing the matrix period in max-plus algebra, are described in [3, 4, 11, 12].

A more general notion of a linear matrix period in max-plus algebra was studied in [8]. An algorithm for computing the value of the linear matrix period for a given matrix A with strongly connected digraph was presented. It was also shown, that the problem of linear periodicity of matrices in max-plus algebra is NP-complete, in the general case. In [9, 10], more efficient