An Elimination Method for Finding All Solutions of the System of Linear Equations over an Extremal Algebra

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1. Introduction

In some of recent papers the extremal algebra and related structures have been treated, see e.g. [1]–[3], [5]–[13]. They deal particularly with the systems of linear equations over these structures. For example, in [6], [10] and [12] the systems of "one-sided" linear equations are considered, moreover in [12] the numerical methods for the solution of the corresponding linear programs are developed. Iterative techniques for the solution of the systems of "two-sided" linear equations (or, shortly two-sided systems) in some special cases are derived in [7]. The problem of eigenvectors of a given matrix might be, of course, also considered as a problem of the solution of the two-sided systems in a special case. This problem was solved in [3], [6], [8], [10], [11], [13]. The general two-sided systems as well as the minimization of isotone functions over their solutions sets have been treated in [1], and in [2] there are given necessary conditions for the existence of nontrivial solutions of the homogeneous two-sided systems.

The aim of this paper is to present a method for finding all solutions of the two-sided systems. It is an analogue of finding all solutions of the system of linear inequalities in the classical linear algebra by the successive elimination of inequalities (cf. [4]). The analogy is more evident if the idea of extremal sign of numbers introduced in [9] is used. However, this technique is avoided here because it is not very familiar.

2. Economic motivations

I. The first motivation is based on ideas treated in [5]. In industrial processes it often occurs that the machines do not work independently. There may happen, for example, that some of the machines produce semiproducts which will be used in the next cycle of activity by another machines.

Suppose that n machines act in the way described above and that $a_{ij}(r)$ is the activity duration of the j-th machine for the i-th machine in the r-th cycle, for all $i, j = 1, 2, \ldots, n$ and natural numbers r. Denoting the starting time of the r-th cycle of the machine i by $x_i(r)$ we get the following system of relations:

$$(1) \qquad x_i(r + 1) = \max (x_1(r) + a_{i1}(r), \ldots, x_n(r) + a_{in}(r)) ; \quad i = 1, \ldots, n .$$

Writing $x \oplus y$ instead of $\max (x, y)$ and $x \otimes y$ instead of $x + y$, (1) becomes

$$x_i(r+1) = \sum_{j=1}^{n}{}^{\oplus}\, a_{ij}(r) \otimes x_j(r)$$

or, using the obvious matrix notation:

$$\mathbf{x}(r+1) = \mathbf{A}(r) \otimes \mathbf{x}(r)\,. \tag{2}$$

One can easily check that there holds an "activity equation":

$$\mathbf{x}(r+1) = \mathbf{C}(r) \otimes \mathbf{x}(1)\,, \tag{3}$$

where $\mathbf{C}(r) = \mathbf{A}(r) \otimes \mathbf{A}(r-1) \otimes \ldots \otimes \mathbf{A}(1)$.

Suppose that another system of n machines acts by the same way and that its activity equation is

$$\mathbf{y}(s+1) = \mathbf{D}(s) \otimes \mathbf{y}(1)\,.$$

Problem: When have the machines of each system to start their first cycle to reach that every corresponding pair of machines start at the same time their r-th and s-th cycle, respectively?

In the algebraic terminology we have to find a nontrivial solution (if it exists) of the following system of equations:

$$\mathbf{C} \otimes \mathbf{x} = \mathbf{D} \otimes \mathbf{y}\,, \tag{4}$$

where we have denoted to simplify $\mathbf{C}(r)$ by $\mathbf{C}$, $\mathbf{D}(s)$ by $\mathbf{D}$, $\mathbf{x}(1)$ by $\mathbf{x}$ and $\mathbf{y}(1)$ by $\mathbf{y}$.

II. Now we use the motivation described in [10]. Suppose that the systems S_1, S_2 will work in one of m modes (it is not known in which one). Both systems consist of n and k subsystems, respectively. The breakdown of an arbitrary subsystem influences the work of the whole system in such a way that the breakdown of the j-th subsystem of S_1, resp. S_2 in the i-th mode causes with the probability a_{ij}, resp. b_{ij} the breakdown of the system S_1 and S_2, respectively. We have to find out whether it is possible to determine the breakdown probabilities $x_1, \ldots, x_n$, $y_1, \ldots, y_k$ of the subsystems such that the breakdown probabilities of S_1 and S_2 will be equal in whichever mode they will work. This demand leads to the system of equations

$$\max_{j=1,\ldots,n} a_{ij}x_j = \max_{j=1,\ldots,k} b_{ij}y_j\,; \quad i = 1, \ldots, m\,; \tag{5}$$

which is of the same type as (4) denoting $\max(x, y)$ by $x \oplus y$ and $x \,.\, y$ by $x \otimes y$.

3. Definitions and basic properties

Let $(G, \otimes, \geqq)$ be a nontrivial, commutative, linearly ordered group. Its neutral element will be denoted by 1. Let

$$G^0 = G \cup \{0\}\,,$$

where 0 is the adjoined element and extend $\otimes$ and $\geqq$ on G^0 in such a way that

$$a \otimes 0 = 0 \otimes a = 0 \quad \text{for all} \quad a \in G^0\,,$$

$$a \geqq 0 \qquad \text{for all} \quad a \in G^0\,.$$

Define a binary operation $\oplus$ on G^0 by the formula

$$a \oplus b = \max(a, b) \quad \text{for all} \quad a, b \in G^0 .$$

The triple $(G^0, \oplus, \otimes)$ will be called extremal algebra. The symbol $a > b$ for $a, b \in G^0$ means that $a \geqq b$ and $a \neq b$.

Supposing $m, n \geqq 1$ to be integers we denote the set of all (m, n) matrices over an arbitrary set S by $S(m, n)$. The set $S(m, 1)$ will be denoted shortly by S_m and its elements will be called vectors. Extend $\oplus$, $\otimes$ and $\geqq$ on matrices over G^0 by the obvious way. If $\mathbf{A} \in G^0(m, n)$ then the element of $\mathbf{A}$ in its i-th row and j-th column will be denoted $(\mathbf{A})_{ij}$ and its i-th row by $\mathbf{A}_{i}$. The symbol $\mathbf{A}^T$ means the transposition of the matrix $\mathbf{A}$ and the matrices (incl. vectors) each element of which is 0 will be denoted by $\boldsymbol{0}$. Many properties of matrices over an extremal algebra may be derived from the results presented in [6], [7], [10], [12], [13]. Let us mention some of them useful in our further considerations. Suppose that $k, l, m, n \geqq 1$ are given integers. Then the following formulas hold:

$$\mathbf{A} \otimes (\mathbf{B} \otimes \mathbf{C}) = (\mathbf{A} \otimes \mathbf{B}) \otimes \mathbf{C} \quad \text{for all} \quad \mathbf{A} \in G^0(k, l), \quad \mathbf{B} \in G^0(l, m), \quad \mathbf{C} \in G^0(m, n) ; \tag{6}$$

$$\mathbf{A} \otimes (\mathbf{B} \oplus \mathbf{C}) = (\mathbf{A} \otimes \mathbf{B}) \oplus (\mathbf{A} \otimes \mathbf{C}) \quad \text{for all} \quad \mathbf{A} \in G^0(k, l), \quad \mathbf{B}, \mathbf{C} \in G^0(l, m) ; \tag{7}$$

$$\mathbf{A} \in G(m, n), \quad \mathbf{X} \in G^0_n - \{\boldsymbol{0}\} \Rightarrow \mathbf{A} \otimes \mathbf{X} \in \mathbf{G}_m . \tag{8}$$

4. Homogeneous system of two-sided linear equations

Suppose that an integer $n \geqq 1$ as well as $a_1, \ldots, a_n, b_1, \ldots, b_n \in G^0$ are given. The equation

$$\sum_{j=1}^{n}{}^{\oplus} a_j \otimes x_j = \sum_{j=1}^{n}{}^{\oplus} b_j \otimes x_j \tag{9}$$

is called two-sided linear equation. Let, moreover, an integer $m \geqq 1$ as well as the matrices $\mathbf{A}, \mathbf{B} \in G^0(m, n)$ be given. In what follows we shall denote the set $\{1, 2, \ldots, m\}$ resp. $\{1, 2, \ldots, n\}$ by M and N, respectively. Consider the system of two-sided linear equations

$$\sum_{j \in N}{}^{\oplus} (\mathbf{A})_{ij} \otimes x_j = \sum_{j \in N}{}^{\oplus} (\mathbf{B})_{ij} \otimes x_j ; \; i \in M ,$$

or, in the vector-matrix notation

$$\mathbf{A} \otimes \mathbf{x} = \mathbf{B} \otimes \mathbf{x} . \tag{10}$$

This system is said to be homogeneous. Denote its solution set by S. There is always $\boldsymbol{0} \in S$ and this vector will be called trivial solution of (10).

Proposition 1: If $\mathbf{x}, \mathbf{y} \in S$ and $\boldsymbol{a}, \boldsymbol{b} \in G^0$ then

$$\boldsymbol{a} \otimes \mathbf{x} \oplus \boldsymbol{b} \otimes \mathbf{y} \in S .$$

Proof: The assertion follows immediately from (7).

Proposition 2: For a given equation (9) there exists an integer $w \geqq 1$ and a matrix $\mathbf{T} \in G^0(n, w)$ such that the solution set of (9) is equal to

$$\{\mathbf{T} \otimes \mathbf{z} \mid \mathbf{z} \in G^0_w\} .$$

Proposition 2 will be proved in the next section.

Proposition 3: For a given system (10) there exists an integer $w \geqq 1$ and a matrix $\mathbf{T} \in G^0(n, w)$ such that

$$S = \{\mathbf{T} \otimes \mathbf{z} \mid \mathbf{z} \in G^0_w\} .$$

The matrix $\mathbf{T}$ from Proposition 2, resp. Proposition 3 is said to be the matrix of generators of the equation (9) and the system (10), respectively.

Proof: We may write the system (10) in the form

$$\begin{aligned} \mathbf{A}_1 \otimes \mathbf{x} &= \mathbf{B}_1 \otimes \mathbf{x} , \\ \mathbf{A}_2 \otimes \mathbf{x} &= \mathbf{B}_2 \otimes \mathbf{x} , \\ &\vdots \\ \mathbf{A}_m \otimes \mathbf{x} &= \mathbf{B}_m \otimes \mathbf{x} . \end{aligned}$$

Define the matrices $\mathbf{T}_1, \mathbf{T}_2, \ldots, \mathbf{T}_m$ and $\mathbf{R}_1, \ldots, \mathbf{R}_m$ over G^0 as follows:

1^0 $\mathbf{R}_1 = \mathbf{T}_1 \in G^0(n, w_1)$ is the matrix of generators of the equation

$$\mathbf{A}_1 \otimes \mathbf{x} = \mathbf{B}_1 \otimes \mathbf{x} .$$

2^0 Suppose that the matrices

$$\begin{aligned} &\mathbf{T}_1 \in G^0(n, w_1), && \mathbf{R}_1 \in G^0(n, w_1), \\ &\mathbf{T}_2 \in G^0(w_1, w_2), && \mathbf{R}_2 \in G^0(n, w_2) , \\ &\vdots \\ &\mathbf{T}_{i-1} \in G^0(w_{i-2}, w_{i-1}), && \mathbf{R}_{i-1} \in G^0(n, w_{i-1}) , \end{aligned}$$

$(i \geqq 2)$ are defined. Then $\mathbf{T}_i$ will be the matrix of generators of the equation

(11) $$(\mathbf{A}_i \otimes \mathbf{R}_{i-1}) \otimes \mathbf{y} = (\mathbf{B}_i \otimes \mathbf{R}_{i-1}) \otimes \mathbf{y} .$$

The number of columns of $\mathbf{T}_i$ denote by w_i. Thus, $\mathbf{T}_i \in G^0(w_{i-1}, w_i)$. Set $\mathbf{R}_i = \mathbf{R}_{i-1} \otimes \mathbf{T}_i$.

We shall show that $\mathbf{R}_m$ is the matrix we are looking for, i.e. that

$$S = \{\mathbf{R}_m \otimes \mathbf{z} \mid \mathbf{z} \in G^0_{w_m}\} .$$

Note, at first, that $\mathbf{R}_m = \mathbf{T}_1 \otimes \mathbf{T}_2 \otimes \ldots \otimes \mathbf{T}_m$.

i) Let $x \in S$. Then $\mathbf{A}_1 \otimes \mathbf{x} = \mathbf{B}_1 \otimes \mathbf{x}$ and thus according to Proposition 2 there exists $\mathbf{z}^{(1)} \in G^0_{w_1}$ such that

(12) $$\mathbf{x} = \mathbf{T}_1 \otimes \mathbf{z}^{(1)} .$$

Hence,

$$\mathbf{A}_2 \otimes (\mathbf{T}_1 \otimes \mathbf{z}^{(1)}) = \mathbf{B}_2 \otimes (\mathbf{T}_1 \otimes \mathbf{z}^{(1)}) .$$

But due to (6) the last equality can be written as

$$(\mathbf{A}_2 \otimes \mathbf{R}_1) \otimes \mathbf{z}^{(1)} = (\mathbf{B}_2 \otimes \mathbf{R}_1) \otimes \mathbf{z}^{(1)}$$

and from Proposition 2 applied on (11) with $i = 2$ we have that there exists $\mathbf{z}^{(2)}$ satisfying

$$\mathbf{z}^{(1)} = \mathbf{T}_2 \otimes \mathbf{z}^{(2)} .$$

Thus, using (12) we get

$$\mathbf{x} = (\mathbf{T}_1 \otimes \mathbf{T}_2) \otimes \mathbf{z}^{(2)} = \mathbf{R}_2 \otimes \mathbf{z}^{(2)} .$$

One can easily verify by induction that there exists $\mathbf{z}^{(m)} \in G^0_{w_m}$ such that

$$\mathbf{x} = \mathbf{R}_m \otimes \mathbf{z}^{(m)} .$$

ii) Let, conversely, $z \in G^0_{w_m}$ and take an arbitrary $i \in M$. We have to prove that

$$\mathbf{A}_i \otimes (\mathbf{R}_m \otimes \mathbf{z}) = \mathbf{B}_i \otimes (\mathbf{R}_m \otimes \mathbf{z}) .$$

But from (6) we deduce that it suffices to check the relation

$$(\mathbf{A}_i \otimes \mathbf{R}_{i-1}) \otimes \mathbf{T}_i \otimes (\mathbf{T}_{i+1} \otimes \ldots \otimes \mathbf{T}_m \otimes \mathbf{z}) =$$
$$= (\mathbf{B}_i \otimes \mathbf{R}_{i-1}) \otimes \mathbf{T}_i \otimes (\mathbf{T}_{i+1} \otimes \ldots \otimes \mathbf{T}_m \otimes \mathbf{z})$$

which holds true because $\mathbf{T}_i$ is the matrix of generators of the equation (11), QED.

5. The proof of Proposition 2

Lemma 1. Let $a, b, c, d \in G^0$; $a > b$. Then

$$\{x \in G^0 \mid a \otimes x \oplus c = b \otimes x \oplus d\} = \{x \in G^0 \mid a \otimes x \oplus c = d\} .$$

Proof. Let $a \otimes x \oplus c = b \otimes x \oplus d$. Suppose $x \neq 0$ (otherwise the proof is trivial). It is easy to verify that then $a \otimes x > b \otimes x$ and thus $a \otimes x \oplus c > b \otimes x$. This yields that $a \otimes x \oplus c = d$.

Let $a \otimes x \oplus c = d$. Hence $d \geqq a \otimes x \geqq b \otimes x$ and thus $d = b \otimes x \oplus d$, QED.

We say that an equation is equivalent to another equation if their solution sets are equal. An equation (9) is said to be in the standard form if for all $j \in N$ there holds

(13) $$a_j \neq b_j \Rightarrow \min(a_j, b_j) = 0 .$$

According to Lemma 1 every equation (9) is equivalent to an equation in the standard form. Thus, we may assume without loss of generality that for the coefficients in (9) the implication (13) is true. This situation can be described schematically as follows

$$(0, \ldots, 0, e, \ldots, e, a, \ldots, a, 0, \ldots, 0) \otimes x =$$
$$= (0, \ldots, 0, e, \ldots, e, 0, \ldots, 0, b, \ldots, b) \otimes x .$$

This scheme corresponds to the partition of the index set N into four subsets:

$$I = \{j \in N \mid a_j = b_j = 0\}, \quad J = \{j \in N \mid a_j = b_j \neq 0\},$$
$$K = \{j \in N \mid a_j > b_j\}, \quad L = \{j \in N \mid b_j > a_j\}.$$

Let us define the following system of vectors:

$\mathbf{e}^i = (e_1^i, \ldots, e_n^i)^{\mathrm{T}}$ for all $i \in I$, where

$$e_j^i = 0, \quad \text{if} \quad j \neq i,$$
$$= 1, \quad \text{if} \quad j = i;$$

$\mathbf{r}^i = (r_1^i, \ldots, r_n^i)^{\mathrm{T}}$ for all $i \in J$, where

$$r_j^i = 0, \quad \text{if} \quad j \neq i,$$
$$= a_i^{-1} = b_i^{-1}, \quad \text{if} \quad j = i;$$

$\mathbf{s}^{k,l} = (s_1^{k,l}, \ldots, s_n^{k,l})^{\mathrm{T}}$ for all $k \in K$, $l \in L$, where

$$s_j^{k,l} = 0, \quad \text{if} \quad j \notin \{k, l\},$$
$$= a_k^{-1}, \quad \text{if} \quad j = k,$$
$$= b_l^{-1}, \quad \text{if} \quad j = l;$$

$\mathbf{r}^{i,h} = (r_1^{i,h}, \ldots, r_n^{i,h})^{\mathrm{T}}$ for all $i \in J$, $h \in K \cup L$, where

$$r_j^{i,h} = r_j^i, \quad \text{if} \quad j \neq h,$$
$$= a_h^{-1}, \quad \text{if} \quad j = h \in K,$$
$$= b_h^{-1}, \quad \text{if} \quad j = h \in L;$$

$\mathbf{s}^{k,l,h} = (s_1^{k,l,h}, \ldots, s_n^{k,l,h})^{\mathrm{T}}$ for all $k \in K$, $l \in L$, $h \in L \cup K - \{k, l\}$, where

$$s_j^{k,l,h} = s_j^{k,l}, \quad \text{if} \quad j \neq h,$$
$$= a_h^{-1}, \quad \text{if} \quad j = h \in K - \{k\},$$
$$= b_h^{-1}, \quad \text{if} \quad j = h \in L - \{l\}.$$

Lemma 2. The equation (9) has a nontrivial solution if and only if $I \cup J \cup K \times L \neq \emptyset$.

Proof. If $I \cup J \cup K \times L \neq \emptyset$ then there exists at least one of the vectors $\mathbf{e}^i$, $i \in I$; or $\mathbf{r}^i$, $i \in J$; or $\mathbf{s}^{k,l}$, $k \in K$; $l \in L$; everyone of which is a (nontrivial) solution of (9).

If, on the other hand, $I \cup J \cup K \times L = \emptyset$ then $I = J = \emptyset$ and with respect to the facts that

$$I \cup J \cup K \cup L = N \neq \emptyset \quad \text{and} \quad K \cap L = \emptyset$$

there is either $N = K$ and $L = \emptyset$ or $N = L$ and $K = \emptyset$.

In the first case we get from (9):

$$\max_{i \in N} a_i \otimes x_i = 0$$

and $a_i > 0$ for all $i \in N$, what implies that the unique solution of (9) is $\mathbf{0}$. The second case can be treated by the same way, QED.

Note that the systems (4), resp. (5) can be regarded as special cases of the system (10) in each equation of which there is $J = \emptyset$ taking

$$(G, \otimes, \geqq) = \mathscr{G}_1 = (R, +, \geqq)$$

and

$$\mathscr{G}_2 = (R^+, \cdot, \geqq),$$

respectively, where R is the set of reals, R^+ is the set of positive reals and $\geqq$ is the obvious order of these sets.

Proposition 2 will be proved if we show that $\mathbf{y} \in G_n^0$ is a solution of (9) if and only if it can be written in the form

(14)

$$\mathbf{y} = \sum_{i \in I}^{\oplus} \varepsilon^i \otimes \mathbf{e}^i \oplus \sum_{i \in J}^{\oplus} \varrho^i \otimes \mathbf{r}^i \oplus \sum_{\substack{k \in K \\ l \in L}}^{\oplus} \sigma^{k,l} \otimes \mathbf{s}^{k,l} \oplus \sum_{\substack{i \in J \\ h \in k \cup L}}^{\oplus} \varrho^{i,h} \otimes \mathbf{r}^{i,h} \oplus \sum_{\substack{k \in K \\ l \in L \\ h \in L \cup K - \{k,l\}}}^{\oplus} \sigma^{k,l,h} \otimes \mathbf{s}^{k,l,h},$$

where $\varepsilon^i, \varrho^i, \sigma^{k,l}, \varrho^{i,h}, \sigma^{k,l,h} \in G^0$. Namely, if the sums do not exist then $I \cup J \cup K \times \times L = \emptyset$ and according to Lemma 2 there is $S = \{\mathbf{0}\}$. But then it suffices to take an arbitrary integer $w \geqq 1$ and $\mathbf{T} = \mathbf{0} \in G^0(n, w)$.

It is not difficult to see that all vectors $\mathbf{e}^i$, $i \in I$; $\mathbf{r}^j$, $j \in J$; $\mathbf{s}^{k,l}$, $k \in K$, $l \in L$; $\mathbf{r}^{i,h}$, $i \in J$, $h \in K \cup L$; $\mathbf{s}^{k,l,h}$, $k \in K$, $l \in L$, $h \in L \cup K - \{k, l\}$ are solutions of (9) and thus, according to Proposition 1, every linear combination of these vectors is also a solution of (9). It remains to show that every solution of (9) can be expressed as a linear combination of the mentioned vectors. Let $\mathbf{x} = (x_1, \ldots, x_n)^{\mathrm{T}} \in S$. We shall distinguish three cases (at least one of which has to occur) denoting by $v = \sum_{i \in N}^{\oplus} a_i \otimes x_i = \sum_{i \in N}^{\oplus} b_i \otimes x_i$:

1^0 $v = 0$,

2^0 $v \neq 0$ and $(\exists j \in J)$ $\mathbf{v} = a_j \otimes x_j = b_j \otimes x_j$,

3^0 $v \neq 0$ and $(\exists f \in K)(\exists g \in L)$ $\mathbf{v} = a_f \otimes x_f = b_g \otimes x_g$.

In the first case there must be $x_i = 0$ for all $i \in J \cup K \cup L$. Hence it suffices to take $\varepsilon^i = x_i$ for all $i \in I$ and all other coefficients equal to 0.

Case 2^0. Now there is $a_j = b_j > 0$ and for all $i \in N$ $a_i \otimes x_i \leqq v$ and $b_i \otimes x_i \leqq v$, implying

(15)
$$a_j^{-1} \otimes a_i \otimes x_i \leqq x_j \quad \text{and}$$
$$b_j^{-1} \otimes b_i \otimes x_i \leqq x_j \quad \text{for all} \quad j \in N.$$

Take

$$\varepsilon^i = x_i \quad \text{for all} \quad i \in I,$$
$$\varrho^i = a_i \otimes x_i \quad \text{for all} \quad i \in J,$$
$$\varrho^{j,h} = a_h \otimes x_h \quad \text{for all} \quad h \in K,$$
$$= b_h \otimes x_h \quad \text{for all} \quad h \in L$$

and $\varrho^{i,h}$ for all $i \in J - \{j\}$, $h \in K \cup L$ as well as all other coefficients equal to 0. Let **y** be the vector defined by (14) and take an arbitrary $t \in I$. Then

$$y_t = \sum_{i \in I}^{\oplus} \varepsilon^i \otimes e_t^i = \varepsilon^t \otimes e_t^t = x_t \otimes 1 = x_t \,.$$

Take $t \in J - \{j\}$. Then

$$y_t = \sum_{i \in J}^{\oplus} \varrho^i \otimes r_t^i \oplus \sum_{h \in K \cup L}^{\oplus} \varrho^{j,h} \otimes r_t^{j,h} = \varrho^t \otimes a_t^{-1} \oplus 0 = a_t \otimes x_t \otimes a_t^{-1} = x_t$$

because here $t \notin K \cup L$ and $t \neq j$. Further,

$$y_j = \varrho^j \otimes a_j^{-1} \oplus \sum_{h \in K}^{\oplus} \varrho^{j,h} \otimes r_j^j \oplus \sum_{h \in L}^{\oplus} \varrho^{j,h} \otimes r_j^j =$$
$$= x_j \oplus \sum_{h \in K}^{\oplus} a_h \otimes x_h \otimes a_j^{-1} \oplus \sum_{h \in L}^{\oplus} b_h \otimes x_h \otimes b_j^{-1} = x_t \,,$$

due to (15). Take $t \in K$. Then

$$y_t = \sum_{\substack{i \in J \\ h \in K \cup L}}^{\oplus} \varrho^{i,h} \otimes r_t^{i,h} = \sum_{h \in K}^{\oplus} \varrho^{j,h} \otimes r_t^{j,h} = \varrho^{j,t} \otimes r_t^{j,t} =$$
$$= a_t \otimes x_t \otimes a_t^{-1} = x_t \,.$$

By the same way it can be shown that $y_t = x_t$ for $t \in L$.

Case 3°. Here $a_f, b_g > 0$ and for all $i \in N$ there is

(16) $$a_f^{-1} \otimes a_i \otimes x_i \leqq x_f \quad \text{and}$$
$$b_g^{-1} \otimes b_i \otimes x_i \leqq x_g \,.$$

Take now
$$\varepsilon^i = x_i \quad \text{for all} \quad i \in I \,,$$
$$\varrho^i = a_i \otimes x_i \quad \text{for all} \quad i \in J \,,$$
$$\sigma^{f,g} = a_f \otimes x_f = b_g \otimes x_g \,,$$
$$\sigma^{f,g,h} = a_h \otimes x_h \,, \quad \text{if} \quad h \in K$$
$$= b_h \otimes x_h \,, \quad \text{if} \quad h \in L$$

and all other coefficients equal to 0. Let again **y** be the vector defined by (14) and take an arbitrary $t \in I$. Then

$$y_t = \sum_{i \in I}^{\oplus} \varepsilon^i \otimes e_t^i = \varepsilon^t \otimes e_t^t = x_t \otimes 1 = x_t \,.$$

Let $t \in J$. Then

$$y_t = \sum_{i \in J}^{\oplus} \varrho^i \otimes r_t^i = \varrho^t \otimes a_t^{-1} = a_t \otimes x_t \otimes a_t^{-1} = x_t \,.$$

Let $t \in K - \{f\}$. Then

$$y_t = \sigma^{f,g} \otimes s_t^{f,g} \oplus \sum_{h \in K \cup L - \{f,g\}}^{\oplus} \sigma^{f,g,h} \otimes s_t^{f,g,h} =$$
$$= 0 \oplus \sigma^{f,g,t} \otimes s_t^{f,g,t} = a_t \otimes x_t \otimes a_t^{-1} = x_t \,,$$

because $t \notin \{f, g\}$. Further,

$$y_f = \sigma^{f,g} \otimes s_f^{f,g} \oplus \sum_{h \in K \cup L - \{f,g\}}^{\oplus} \sigma^{f,g,h} \otimes s_f^{f,g,h} =$$

$$= a_f \otimes x_f \otimes a_f^{-1} \oplus \sum_{h \in K \cup L - \{f,g\}}^{\oplus} a_h \otimes x_h \otimes a_f^{-1} = x_f ,$$

due to (16). The subcase $t \in L$ can be proved analogically, QED.

6. Non-homogeneous systems of linear equations

Consider the system

$$(17) \qquad \mathbf{A} \otimes \mathbf{x} \oplus \mathbf{c} = \mathbf{B} \otimes \mathbf{x} \oplus \mathbf{d} ,$$

where $\mathbf{A}, \mathbf{B} \in G^0(m, n)$; $\mathbf{c}, \mathbf{d} \in G_n^0$ and the "corresponding" homogeneous system

$$(18) \qquad (\mathbf{A}, \mathbf{c}) \otimes \mathbf{z} = (\mathbf{B}, \mathbf{d}) \otimes \mathbf{z}$$

(written block-wise). The following assertion describes a trivial fact about solutions of these systems.

Proposition 4: Let $\mathbf{x} \in G_n^0$. Then $\mathbf{x}$ is a solution of (17) if and only if the vector $\begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix}$ (written block-wise) is a solution of (18).

This enables us to use the method derived above in order to solve the non-homogeneous systems of linear equations; for details see numerical Examples 3 and 4. Naturally, there may occur that (18) has a nontrivial solution, while (17) has no solution. The following Proposition gives a sufficient condition to avoid such a situation.

Proposition 5: Let $\mathbf{A} \oplus \mathbf{B} \in G(m, n)$ and $\mathbf{c} \oplus \mathbf{d} \in G_m$. Then (17) is solvable if and only if (18) has a nontrivial solution.

Proof: Let $\mathbf{x} \in G_n^0$ be a solution of (17). Then the vector $\mathbf{z} = \begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix}$ is a (nontrivial) solution of (18).

Let $\mathbf{z} = (z_1, \ldots, z_{n+1})^{\mathrm{T}} \in G_{n+1}^0 - \{\mathbf{0}\}$ be a solution of (18). Firstly we show that without loss of generality we may assume $z_{n+1} \in G$. Suppose $z_{n+1} = 0$ and let $\mathbf{v} = (v_1, \ldots, v_m)^{\mathrm{T}} \in G_n^0$ be the vector defined by the formula

$$(\mathbf{A}, \mathbf{c}) \otimes \mathbf{z} = \mathbf{v} = (\mathbf{B}, \mathbf{d}) \otimes \mathbf{z} .$$

Then there holds

$$\mathbf{v} = \mathbf{v} \oplus \mathbf{v} = (\mathbf{A} \oplus \mathbf{B}, \mathbf{c} \oplus \mathbf{d}) \otimes \mathbf{z}$$

and thus, according to (8) and due to the assumptions we get that $\mathbf{v} \in G_m$. This yields that $q \in G$, where

$$q = \min_{i \in M} (c_i \oplus d_i)^{-1} \otimes v_i .$$

But the last formula implies that $c_i \otimes q \leqq v_i$ and $d_i \otimes q \leqq v_i$ for all $i \in M$ and hence the vector $\mathbf{y} = (z_1, \ldots, z_n, q)^{\mathrm{T}}$ is also a solution of (18).

According to Proposition 1 the vector

$$z_{n+1}^{-1} \otimes \mathbf{y} = (z_{n+1}^{-1} \otimes z_1, \ldots, z_{n+1}^{-1} \otimes z_n, 1)^{\mathrm{T}}$$

is a solution of (18), too, what implies that the vector $(z_{n+1}^{-1} \otimes z_1, \ldots, z_{n+1}^{-1} \otimes z_n)^{\mathrm{T}}$ is a solution of (17), QED.

7. Numerical examples

The proofs of Propositions 2 and 3 are constructive and their ideas can be used for finding the matrix of generators of a given homogeneous system. We shall demonstrate this fact on the first two examples. The second two examples will illustrate how to solve the non-homogeneous systems. In all four examples there is $(G, \otimes, \geqq) = \mathscr{G}_2$. Note that the number of columns of the matrix of generators is very large, in general, but in some cases it may be founded by a practically good limit (e.g. for the problem of eigenvectors).

Example 1. Consider the system (10) with

$$\mathbf{A} = \begin{pmatrix} 3 & 2 & 0 \\ 0 & 0 & 2 \\ 2 & 1 & 3 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 3 & 0 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{pmatrix}.$$

Here $m = n = 3$ and for the first equation there is $I = \emptyset$, $J = \{1\}$, $K = \{2\}$, $L = \{3\}$ and thus

$\boldsymbol{r}^1 \ = (\frac{1}{3}, 0, 0)^{\mathrm{T}}$,

$\boldsymbol{s}^{2,3} = (0, \frac{1}{2}, 1)^{\mathrm{T}}$,

$\boldsymbol{r}^{1,2} = (\frac{1}{3}, \frac{1}{2}, 0)^{\mathrm{T}}$; $\boldsymbol{r}^{1,3} = (\frac{1}{3}, 0, 1)^{\mathrm{T}}$, implying that

$$\mathbf{T}_1 = \mathbf{R}_1 = \begin{pmatrix} \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \quad \text{and} \quad w_1 = 4\,.$$

Further we find out that

$\mathbf{A}_2 \otimes \mathbf{R}_1 = (0, 0, 2) \otimes \mathbf{R}_1 = (0, 2, 0, 2)$,

$\mathbf{B}_2 \otimes \mathbf{R}_1 = (1, 0, 2) \otimes \mathbf{R}_1 = (\frac{1}{3}, 2, \frac{1}{3}, 2)$.

Hence for the equation

$$(\mathbf{A}_2 \otimes \mathbf{R}_1) \otimes \mathbf{y} = (\mathbf{B}_2 \otimes \mathbf{R}_1) \otimes \mathbf{y}$$

we get $I = \emptyset$, $J = \{2, 4\}$, $K = \emptyset$, $L = \{1, 3\}$ and thus

$\boldsymbol{r}^2 \ = (0, \frac{1}{2}, 0, 0)^{\mathrm{T}}$; $\boldsymbol{r}^4 \ = (0, 0, 0, \frac{1}{2})^{\mathrm{T}}$,

$\boldsymbol{r}^{2,1} = (3, \frac{1}{2}, 0, 0)^{\mathrm{T}}$; $\boldsymbol{r}^{2,3} = (0, \frac{1}{2}, 3, 0)^{\mathrm{T}}$,

$\boldsymbol{r}^{4,1} = (3, 0, 0, \frac{1}{2})^{\mathrm{T}}$; $\boldsymbol{r}^{4,3} = (0, 0, 3, \frac{1}{2})^{\mathrm{T}}$, implying that

$$\mathbf{R}_2 = \mathbf{T}_1 \otimes \mathbf{T}_2 = \begin{pmatrix} 0 & \frac{1}{6} & 1 & 1 & 1 & 1 \\ \frac{1}{4} & 0 & \frac{1}{4} & \frac{3}{2} & 0 & \frac{3}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \quad \text{and} \quad w_2 = 6\,.$$

We see that fourth and the sixth column of $\mathbf{R}_2$ are equal and the third one is the sum of the first and the fifth. Therefore it suffices to take instead of $\mathbf{R}_2$ the matrix

$$\mathbf{R}'_2 = \begin{pmatrix} 0 & \frac{1}{6} & 1 & 1 \\ \frac{1}{4} & 0 & \frac{3}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

Then we find out that for the system

$$(\mathbf{A}_3 \otimes \mathbf{R}'_2) \otimes \mathbf{y} = (\mathbf{B}_3 \otimes \mathbf{R}'_2) \otimes \mathbf{y}$$

there is $I = \emptyset$, $J = \{1, 2\}$, $K = \{3, 4\}$, $L = \emptyset$ and thus

$r^1 \ = (\frac{2}{3}, 0, 0, 0)^{\mathrm{T}}$; $\ r^2 \ = (0, \frac{2}{3}, 0, 0)^{\mathrm{T}}$,

$r^{1,3} = (\frac{2}{3}, 0, \frac{1}{2}, 0)^{\mathrm{T}}$; $\ r^{1,4} = (\frac{2}{3}, 0, 0, \frac{1}{2})^{\mathrm{T}}$,

$r^{2,3} = (0, \frac{2}{3}, \frac{1}{2}, 0)^{\mathrm{T}}$; $\ r^{2,4} = (0, \frac{2}{3}, 0, \frac{1}{2})^{\mathrm{T}}$,

implying that

$$\mathbf{R}_3 = \mathbf{R}'_2 \otimes \mathbf{T}_3 = \begin{pmatrix} 0 & \frac{1}{9} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{6} & 0 & \frac{3}{4} & \frac{1}{6} & \frac{3}{4} & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} \quad \text{and} \quad w_3 = 6\,.$$

After deleting the dependent columns (the 4-th and the 5-th) and multiplying the others by appropriate constants we conclude that $\mathbf{x}$ is a solution of (10) if and only if it satisfies the relation

$$\mathbf{x} = \begin{pmatrix} 0 & 1 & 6 & 3 \\ 1 & 0 & 9 & 0 \\ 2 & 3 & 4 & 2 \end{pmatrix} \otimes \mathbf{z}\,,$$

where $\mathbf{z}$ is an arbitrary element of G_4^0.

Example 2. Consider the system (10) with

$$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Here $m = n = 2$ and for the first equation we have $I = \emptyset$, $J = \{2\}$, $K = \{1\}$, $L = \emptyset$ and thus

$r^2 = (0, 1)^{\mathrm{T}}$; $\ r^{2,1} = (\frac{1}{2}, 1)^{\mathrm{T}}$, implying that

$$\mathbf{R}_1 = \mathbf{T}_1 = \begin{pmatrix} 0 & \frac{1}{2} \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad w_1 = 2\,.$$

Further we compute that

$$\mathbf{A}_2 \otimes \mathbf{R}_1 = (2, 2)\,; \quad \mathbf{B}_2 \otimes \mathbf{R}_1 = (0, \tfrac{1}{2})\,.$$

Hence, in the equation

$$(\mathbf{A}_2 \otimes \mathbf{R}_1) \otimes \mathbf{y} = (\mathbf{B}_2 \otimes \mathbf{R}_1) \otimes \mathbf{y}$$

there is $I = J = L = \emptyset$, $K = \{1, 2\}$ and thus, according to Lemma 2 the considered system has only trivial solution.

Remark. In [6] a procedure called $\mathscr{A}$-test has been developed which can be used to delete columns of the matrix being linear combinations of some of the other columns.

Example 3. Take the system (17) with

$$\mathbf{A} = \begin{pmatrix} 3 & 2 \\ 0 & 0 \\ 2 & 1 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 3 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}, \quad \mathbf{d} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

We see that the system treated in Example 1 is just the corresponding homogeneous system. Hence we deduce that all solutions $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ of (17) are of the form

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 6 & 3 \\ 1 & 0 & 9 & 0 \end{pmatrix} \otimes \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix},$$

where z_1, z_2, z_3, z_4 are arbitrary elements of G^0 satisfying

$$2 \otimes z_1 \oplus 3 \otimes z_2 \oplus 4 \otimes z_3 \oplus 2 \otimes z_4 = 1. \tag{19}$$

Since (19) is solvable (the set of all solutions of (19) can be found by methods given in [6] or in [12]), we conclude that (17) is solvable.

Example 4. Let in (17) be

$$\mathbf{A} = \begin{pmatrix} 0 & 0 \\ 2 & 1 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \mathbf{d} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

For the corresponding homogeneous system we find out that $\mathbf{R}_2 = (0, 1, 0)^{\mathrm{T}}$ and thus all its solutions are of the form $(0, z, 0)^{\mathrm{T}}$, $z \in G^0$. Since among these solutions there does not exist anyone with the last component equal to 1, we conclude that (17) is not solvable.

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Souhrn

ELIMINAČNÁ METÓDA PRE NÁJDENIE VŠETKYCH RIEŠENÍ SÚSTAVY LINEÁRNYCH ROVNÍC NAD EXTREMÁLNOU ALGEBROU

Peter Butkovič, Gábor Hegedüs

V článku sa dokazuje, že tak ako v lineárnej algebre pre každú homogénnu sústavu lineárnych rovníc nad extremálnou algebrou existuje konečná množina vektorov (generátorov), ktorej lineárny obal sa rovná množine riešení takejto sústavy.

Popisuje sa spôsob, ktorým možno množinu generátorov nájsť a využiť ju na explicitné vyjadrenie všetkých riešení ľubovoľnej (nehomogénnej) sústavy lineárnych rovníc.