Annals of Discrete Mathematics 19 (1984) 41–54

ON PROPERTIES OF SOLUTION SETS OF EXTREMAL LINEAR PROGRAMS

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Two sided systems of linear extremal equations are introduced. The aim is to show an idea of reduction which may be useful in decision-making whether the system is solvable or not and in finding at least one its solution (if it exists). Finally, it is shown how this reduction could be used in order to solve extremally linear programs over solution sets of the introduced systems.

INTRODUCTION

In some recent papers formally linear optimization problems and systems are considered. The operations of addition and multiplication are replaced by a pair of abstract binary operations possessing often two typical properties:

a) the extremality of at least one of the operations (i.e. the result of the operation equals one of the two operands);
b) the invertibility of at most one of the operations.

The research has been partially concentrated on systems of extremally linear equations with variables on the same side of constraint relations as well as on linear programs over their solution sets. Methods for solving these problems have been developed at a rather high level, see e.g. [4], [8], [9]. Another subarea is the theory of eigenproblems treated, for example, in [3], [4], [7], [8], [11]. Under some assumptions concerning the binary operations effective algorithms have been derived, too. Some related questions have been also treated, like linear dependence ([6], [4], [1]), or geometrical aspects ([10], [1]). Exhaustive survey of the research results was made in monographies [4] and [11]. Some economical motivations can be found in [4] and [8].

If the addition is not an invertible operation then, of course, two sided systems of linear equations cannot be transformed on systems with all variables on the same side. The task of solving such systems seems to be sufficiently more difficult than that of solving one sided systems. Some steps in order to solve this problem have been made in [illegible]

The aim of this paper is to present one idea of a reduction process which might be helpful in solving general two sided extremally linear systems as well as linear programs over their solution sets. The main result lies in the fact that a finite subset of the solution set can be explicitly described and that's why it can be used in order to find out whether the system is solvable. Though, the significance of this result is mainly theoretical because of the low computational efficiency of the obtained procedure, it can be used in order to solve systems of small dimensions. Many open problems remain to be solved and some directions for future research are to be found in Conclusions.

EXTREMAL ALGEBRA

Let S be an arbitrary set. An operation

$$\square : S \times S \to S$$

is said to be extremal if

$$a \square b \in \{a,b\}$$

for all $a,b \in S$.

Let $\square, \Delta$ be extremal operations: $S \times S \to S$. We say that $\square$ is complementary to Δ if

$$a \neq b \text{ implies } a \square b \neq a \Delta b$$

for all $a,b \in S$.

Complementarity is, clearly, a symmetric relation.

Let E be an arbitrary set. The triple $E = [E, \oplus, \circ]$ is called extremal algebra if:

$$\oplus : E \times E \to E$$
$$\circ : E \times E \to E$$

and the following assumptions are fulfilled:

1^0 $\oplus$, $\circ$ are associative and commutative;

2^0 $(a \oplus b) \circ c = (a \circ c) \oplus (b \circ c)$, for all $a,b,c \in E$;

3^0 there exists a neutral element $0 \in E$ with respect to $\oplus$;

4^0 $\oplus$ is extremal;

5^0 $\circ$ satisfies exactly one of the following conditions:

(5α): $(E \setminus \{0\}, \circ)$ is a group with the neutral element $1 \neq 0$ (the inverse of an element a will be denoted as usual by a^{-1}),

(5β): $\circ$ is extremal and complementary to $\oplus$.

The symbol $a < b$ means $a \leq b$ and $a \neq b$.

Remark: It follows from the definition of the extremal algebra that $a \circ 0 = 0$ for all $a \in E$. This assertion is a trivial corollary of 3^0 and 5^0 in the case (5β), and is readily proved by contradiction in the case (5α).

Recall now some elementary properties of an extremal algebra E:

$$a \geq 0 \text{ for all } a \in E \tag{1}$$

$$(5\alpha) \text{ satisfied, } c \neq 0 \text{ and } a < b \text{ imply } a \circ c < b \circ c \tag{2}$$

$$a \oplus b \leq c \text{ if and only if } a \leq c \text{ and } b \leq c \tag{3}$$

$$a \leq b \text{ implies } a \circ c \leq b \circ c \tag{4}$$

$$(5\beta) \text{ satisfied and } a \circ b = c \text{ imply } a \geq c \text{ and } b \geq c \tag{5}$$

$$(5\beta) \text{ satisfied implies } a \circ (a \oplus b) = a \tag{6}$$

Lemma 1: Let $k,t \in E$, $k > t$. Then

a) $a \oplus k = b \oplus t$ if and only if $a \oplus k = b$,

b) $a \oplus t = b \oplus t$ implies $a \oplus k = b \oplus k$.

Lemma 2: Let 5(α) be satisfied and $k,t \in E$, $k > t$. Then $k \circ x \oplus a = t \circ x \oplus b$ if and only if $k \circ x \oplus a = b$ for all $a,b,x \in E$.

We verify only Lemma 2.

Assume $x \neq 0$ otherwise the assertion is trivial. Together with (2) it yields

$$t \circ x \oplus b = k \circ x \oplus a \geq k \circ x > t \circ x, \text{ hence } t \circ x \oplus b = b.$$

Supposing $b = k \circ x \oplus a \geq k \circ x > t \circ x$ we get

$$b \oplus t \circ x = b, \text{ QED.}$$

SYSTEMS OF EXTREMAL EQUATIONS

We denote $E_n = \underbrace{E \times \ldots \times E}_{n}$. Elements of E_n will be called vectors.

We can extend the operations $\oplus$, $\circ$ and the relation $\leq$ in a natural way to matrices and vectors over E (denoting the product by $\bar{\circ}$). The symbol X^T means the transposition of the vector X.

The following properties of these operations will be used later (A,B,D are matrices and X,Y,Z column vectors of the appropriate type):

$$A \leq B \text{ implies } A \oplus D \leq B \oplus D \tag{7}$$

$$\text{and } A \,\bar{\circ}\, D \leq B \,\bar{\circ}\, D. \tag{8}$$

$$Y \leq Z \text{ implies } X^T \otimes Y \leq X^T \otimes Z \tag{9}$$

$$\text{and } A \otimes Y \leq A \otimes Z. \tag{10}$$

One can easily verify also the inequality

$$X^T \otimes Y \leq \sum_{j=1}^{n\oplus} x_j \circ \sum_{j=1}^{n\oplus} y_j \tag{11}$$

having denoted $X = (x_1,\ldots,x_n)^T$ and $Y = (y_1,\ldots,y_n)^T$. If $A = (a_{ij})$, $B = (b_{ij})$ are matrices of the same type then $A < B$ denotes the fact that

$$a_{ij} < b_{ij}$$

for all i and j.

Let us write a general system of extremal equations in the form:

$$A^{(1)} \otimes X \oplus B^{(1)} = A^{(2)} \otimes X \oplus B^{(2)} \tag{12}$$

where

$A^{(s)} = (a_{ij}^{(s)})$, $s = 1,2$ are matrices of the type (q,n) over E;

$B^{(s)} = (b_1^{(s)},\ldots,b_q^{(s)})^T \in E_q$, $s = 1,2$

$X \in E_n$.

Let us denote by M the set of all solutions of the system (12) and further

$$J = \{1,2,\ldots,n\},$$

$$Q = \{1,2,\ldots,q\}.$$

In what follows we suppose without loss of generality that

$$B^{(1)} \leq B^{(2)}.$$

Due to Lemma 1 we may assume

$$b_i^{(1)} \neq b_i^{(2)} \text{ implies } b_i^{(1)} = 0.$$

Systems (12) possessing this property are said to be in standard form.

Thus there are only two possibilities for constant terms in each equation of the system in the standard form:

either $$b_i^{(1)} = b_i^{(2)}$$

or $$0 = b_i^{(1)} \neq b_i^{(2)}.$$

The equations with the second property play a slightly more important role in the following parts of the paper and we denote

This set will be called <u>characteristic set</u> of the system (12).

Evidently, the following three propositions are equivalent:

1^o $0 \in M$

2^o $Q_0 = \emptyset$

3^o $B^{(1)} = B^{(2)}$.

For simplicity we denote the vector $B^{(2)} = B^{(1)} \oplus B^{(2)}$ by $B = (b_1, b_2, \dots, b_q)^T$.

REDUCTION OF THE SET M TO A FINITE SUBSET

The following two ideas will be used in order to solve the system (12) and some optimization problems under these constraints.

(I) For every variable x_j there exists a finite set ("set of relevant levels") at least one element of which is the value of the j-th component of some $X \in M$ whenever $M \neq \emptyset$.

(II) Putting $x_j = \bar{x}_j \in E$ for any $j \in J$ we transform the system (12) to a system of the same type with n-1 variables. Naturally, some of the equations may turn to identities.

We denote for all $i \in Q$ and $j \in J$:

$$S_{ij} = \{r \in Q_0 | a_{rj}^{(1)} > 0 \ \& \ a_{rj}^{(1)} \circ b_r^{-1} \geq a_{ij}^{(1)} \circ b_i^{-1}\} \text{ in the case } (5\alpha)$$

$$S_j = \{r \in Q_0 | a_{rj}^{(1)} \geq b_r\} \text{ in the case } (5\beta).$$

Definition: The following sets are called <u>sets of relevant levels</u>:

$$R_j = \{b_i \circ (a_{ij}^{(1)})^{-1} \mid a_{ij}^{(1)} > 0 \ \& \ i \in Q_0 \ \& \ \bigcup_{t=1}^{n} S_{it} = Q_0\}$$

$$\text{if } (5\alpha) \text{ is true and } Q_0 \neq \emptyset,$$

$$R_j = \left\{ \sum_{i \in Q_0}^{\oplus} b_i \right\} \quad \text{if } (5\beta) \text{ is true and } Q_0 \neq \emptyset,$$

$$R_j = \{0\} \quad \text{if } Q_0 = \emptyset.$$

Due to the fact (II) we are able to denote by

$$M(x_{j_1} = \bar{x}_{j_1}, x_{j_2} = \bar{x}_{j_2}, \dots) \text{ resp.}$$

$$Q_0(x_{j_1} = \bar{x}_{j_1}, x_{j_2} = \bar{x}_{j_2}, \dots) \text{ resp.}$$

$$R_j(x_{j_1} = \bar{x}_{j_1},\ x_{j_2} = \bar{x}_{j_2}, \ldots) \quad (j \in J \setminus \{j_1, j_2, \ldots\})$$

the set of solutions, the characteristic set and sets of relevant levels of the system arising from (12) putting successively $x_{j_1} = \bar{x}_{j_1}$, $x_{j_2} = \bar{x}_{j_2}, \ldots$, respectively.

Definition: $\mathcal{B}(M)$ is the set of all $X = (\bar{x}_1, \ldots, \bar{x}_n)^T \in M$ for which there exists a permutation $(j_1, \ldots, j_n)$ of J satisfying

$$\begin{aligned} &\bar{x}_{j_1} \in R_{j_1}, \\ &\bar{x}_{j_2} \in R_{j_2}(x_{j_1} = \bar{x}_{j_1}), \\ &\vdots \\ &\bar{x}_{j_n} \in R_{j_n}(x_{j_1} = \bar{x}_{j_1},\ x_{j_2} = \bar{x}_{j_2},\ \ldots,\ x_{j_{n-1}} = \bar{x}_{j_{n-1}}). \end{aligned} \tag{13}$$

<u>Theorem 1</u>: $M \neq \emptyset$ if and only if $\mathcal{B}(M) \neq \emptyset$.

In order to prove this theorem we show by means of some Lemmas that every $X \in M$ can be reduced to a vector with properties of a certain type. These reductions will enable us to find an element of $\mathcal{B}(M)$ we are looking for.

Denote by A_i the i-th row vector of the matrix A.

Note that for $i \in Q_o$ there is always (supposing $X \in M$)

$$A_i^{(1)} \otimes X \geqslant b_i > 0. \tag{14}$$

<u>Definition</u>: Let $X \in M$. The vector $\mathrm{red}(X) = \rho(X) \otimes X$ is called <u>reduction of the vector X</u> if

$$\rho(X) = \sum_{i \in Q_o}^{\oplus} b_i \otimes (A_i^{(1)} \otimes X)^{-1} \qquad \text{in the case } (5\alpha) \text{ and } Q_o \neq \emptyset,$$

$$\rho(X) = \sum_{i \in Q_o}^{\oplus} b_i \qquad \text{in the case } (5\beta) \text{ and } Q_o \neq \emptyset,$$

$$\rho(X) = 0 \qquad \text{if } Q_o = \emptyset.$$

<u>Lemma 3:</u> Let $X = (x_1, \ldots, x_n)^T \in M$.

a) If (5α) is satisfied then $\rho(X) \leqslant 1$.

b) If (5β) is satisfied then $\rho(X) \leqslant \sum^{\oplus} x_j$.

Proof: The a) follows immediately from (3) and (14).

b) Let $\sum_{i\in Q_0}^{\oplus} b_i = b_k$. Then according to (5) and (11) we have

$$\rho(X) = b_k \leqslant A_k^{(1)} \circledcirc X \leqslant \sum_{j\in J}^{\oplus} a_{kj}^{(1)} \circ \sum_{j\in J}^{\oplus} x_j \leqslant \sum_{j\in J}^{\oplus} x_j .$$

Lemma 4: Let $X \in M$. Then

a) $red(X) \leqslant X$

b) $red(X) \in M$

c) $0 \neq X = red(X) \Rightarrow (\exists k \in Q_0)(A_k^{(1)} \circledcirc X = b_k)$.

Proof: The first assertion follows easily from Lemma 3 and (5). We show now the b). Consider only the case when $Q_0 \neq \emptyset$ i.e. $\rho(X) > 0$. It is to be shown

$$A_i^{(1)} \circledcirc red(X) \oplus b_i^{(1)} = A_i^{(2)} \circledcirc red(X) \oplus b_i^{(2)} \tag{15}$$

for all $i \in Q$.

First suppose $i \in Q_0$.

i) In the case (5α) $\rho(X) = 1$ implies the assertion immediately. If $\rho(X) < 1$ then

$$A_i^{(1)} \circledcirc X > b_i \quad \text{hence } A_i^{(1)} \circledcirc X = A_i^{(2)} \circledcirc X \quad \text{and}$$

$$A_i^{(1)} \circledcirc red(X) = A_i^{(2)} \circledcirc red(X). \tag{16}$$

At the same time $A_i^{(1)} \circledcirc red(X) = \sum_{j\in Q_0}^{\oplus} b_j \circ (A_j^{(1)} \circledcirc X)^{-1} \circ (A_i^{(1)} \circledcirc X) \geqslant b_i \circ$ $\circ (A_i^{(1)} \circledcirc X)^{-1} \circ (A_i^{(1)} \circledcirc X) = b_i$. This yields that (16) is in fact the same as (15).

ii) In the case (5β) there is $\rho(X) = \sum_{j\in Q_0}^{\oplus} b_j$ and

$$A_i^{(1)} \circledcirc red(X) = \rho(X) \circ (A_i^{(1)} \circledcirc X) = \rho(X) \circ (A_i^{(2)} \circledcirc X \oplus b_i) =$$

$$= A_i^{(2)} \circledcirc red(X) \oplus b_i \circ \sum_{j\in Q_0}^{\oplus} b_j = A_i^{(2)} \circledcirc red(X) \oplus b_i$$

(recall (6)).

Now suppose $i \in Q \setminus Q_0$. If $A_i^{(1)} \circledcirc X = A_i^{(2)} \circledcirc X > b_i$ then (15) follows immediately and in the case when $A_i^{(1)} \circledcirc X \leqslant b_i$, $A_i^{(2)} \circledcirc X \leqslant b_i$ we get (15) applying (9) to the inequality in a).

c) Recall that $0 \neq X = red(X)$ implies $Q_0 \neq \emptyset$.

$$1 = \rho(X) = \sum_{i\in Q_0}^{\oplus} b_i \otimes (A_i \otimes X)^{-1} = b_k \otimes (A_k \otimes X)^{-1} \text{ for some } k \in Q_0.$$

ii) In the case (5β)

$X = \mathrm{red}(X)$ gives for all $t \in J$ using (5) and (11)

$$x_t = \rho(X) \otimes x_t \leq \rho(X) = \sum_{i\in Q_0}^{\oplus} b_i = b_k \leq A_k^{(1)} \otimes X \leq \sum_{j\in J}^{\oplus} a_{kj}^{(1)} \otimes$$

$$\otimes \sum_{j\in J}^{\oplus} x_j \leq \sum_{j\in J}^{\oplus} x_j = x_m.$$

Specially, for $t = m$ we get

$$x_m \leq \rho(X) \leq A_k^{(1)} \otimes X \leq x_m$$

and hence $\rho(X) = A_k^{(1)} \otimes X$, QED.

Lemma 5: If $0 \neq X \in M$ and $Y = \mathrm{red}(X) = (y_1,\ldots,y_n)^T$ then there exists $k \in Q_0$ and $t \in J$ satisfying the following conditions:

i) $$\bigcup_{j\in J} S_{kj} = Q_0, \tag{17}$$

$$y_t = b_k \otimes (a_{kt}^{(1)})^{-1}, \tag{18}$$

in the case (5α),

ii) $$\bigcup_{j\in J} S_j = Q_0, \tag{19}$$

$$y_t = b_k = \sum_{i\in Q_0}^{\oplus} b_i, \tag{20}$$

in the case (5β).

Proof: i) The existence of y_t satisfying (18) follows immediately from Lemma 4 because $\mathrm{red}(\mathrm{red}(X)) = \mathrm{red}(X)$.

It remains to show (17). Let k be the index from Lemma 4, c). Then for all $j \in J$ there is

$$a_{kj}^{(1)} \otimes y_j \leq A_k^{(1)} \otimes Y = b_k \tag{21}$$

and hence those $j \in J$ for which $a_{kj} > 0$ satisfy the inequality

$$y_j \leq (a_{kj}^{(1)})^{-1} \otimes b_k.$$

Take an arbitrary $i \in Q_0$. Then

$$0 < b_i \leq A_i^{(1)} \otimes Y = a_{ih}^{(1)} \otimes y_h$$

$$a_{ih} > 0.$$

Moreover, supposing $a_{kh} > 0$ we get using (21)

$$b_i \circ (a_{ih}^{(1)})^{-1} \leq y_h \leq (a_{kh}^{(1)})^{-1} \circ b_k,$$

i.e.

$$a_{ih}^{(1)} \circ b_i^{-1} \geq a_{kh}^{(1)} \circ b_k^{-1} \text{ and thus } i \in S_{kh}.$$

ii) The existence of y_t satisfying (20) follows from the fact that relations

$$b_k \leq b_k \oplus A_k^{(2)} \circ X = A_k^{(1)} \circ X = a_{kh}^{(1)} \circ x_h$$

hold and imply $x_h \geq b_k$. Hence $y_h = b_k \circ x_h = b_k$.

At last we show (19). Let $i \in Q_o$. Then there exists an index $h \in J$ satisfying

$$a_{ih}^{(1)} \circ y_h \geq b_i$$

and thus $a_{ih}^{(1)} \geq b_i$, i.e. $i \in S_h$, QED.

The proof of Theorem 1 follows immediately from the following Lemma being in fact a corollary of Lemma 5.

Lemma 6: For every $X \in M$ there exists a vector $Y \in \mathcal{B}(M)$ satisfying the inequality:

$$Y \leq X. \tag{22}$$

Proof: First of all notice that $\mathcal{B}(\{0\}) = \{0\}$.

Let $0 \neq X \in M$ and $\text{red}(X) = X^{(1)} = (x_1^{(1)},\dots,x_n^{(1)})^T$. According to Lemma 5 there exists $k \in J$ with property

$$x_k^{(1)} \in R_k.$$

Assume for simplicity $k = n$. This yields

$$\tilde{X}^{(1)} = (x_1^{(1)},\dots,x_{n-1}^{(1)})^T \in M(x_n = x_n^{(1)}).$$

Denote $y_n = x_n^{(1)}$ and $X^{(2)} = \text{red}(\tilde{X}^{(1)}) = (x_1^{(2)},\dots,x_{n-1}^{(2)})^T$.

It follows again from Lemma 5 that there exists $k \in J$ with property $x_k^{(2)} \in R_k(x_n = x_n^{(1)})$ and suppose now $k = n - 1$. Thus

$$\tilde{X}^{(2)} = (x_1^{(2)},\dots,x_{n-2}^{(2)})^T \in M(x_n = x_n^{(1)},\ x_{n-1} = x_{n-1}^{(2)}).$$

Take

$$y_{n-1} = x_{n-1}^{(2)} \text{ and } X^{(3)} = \text{red}(\tilde{X}^{(2)}).$$

satisfying (22) due to the assertions a) and b) of Lemma 4, QED.

The following numerical example will illustrate the reduction process used in the proof of Lemma 6.

Example 1: Consider $E = [R^+, \max, .]$ where R^+ is the set of non-negative reals. The vector $X = (3,5,4)^T$ is a solution of the system (written as well as all other in the standard form)

$$\begin{pmatrix} 2 & 3 & 1 \\ 2 & 1 & 0 \\ 2 & 2 & 1 \\ 3 & 0 & 3 \end{pmatrix} \otimes \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \oplus \begin{pmatrix} 0 \\ 6 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 3 \\ 0 & 1 & 1 \\ 3 & 2 & 2 \\ 4 & 2 & 1 \end{pmatrix} \otimes \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \oplus \begin{pmatrix} 5 \\ 6 \\ 1 \\ 3 \end{pmatrix}$$

Here $J = \{1,2,3\}$, $Q = \{1,2,3,4\}$, $Q_0 = \{1,3\}$ and $\{A_i^{(1)} \otimes X \mid i \in Q_0\} = \{15,10\}$. Therefore $\rho(X) = \frac{5}{15} \oplus \frac{1}{10} = \frac{1}{3}$, $X^{(1)} = \text{red}(X) = (1, \frac{5}{3}, \frac{4}{3})^T$. One can verify immediately from definitions that $\frac{5}{3} \in R_2$ and thus $j_1 = 2$, $y_{j_1} = \frac{5}{3}$. Consequently $\tilde{X}^{(1)} = (1, \frac{4}{3})^T$ is a solution of the system arising by putting $x_2 = \frac{5}{3}$:

$$\begin{pmatrix} 2 & 1 \\ 2 & 0 \\ 2 & 1 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_3 \end{pmatrix} \oplus \begin{pmatrix} 5 \\ 6 \\ 10/3 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 0 & 1 \\ 3 & 2 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_3 \end{pmatrix} \oplus \begin{pmatrix} 5 \\ 6 \\ 10/3 \\ 10/3 \end{pmatrix}$$

Now, $Q_0(x_2 = 5/3) = \{4\}$, $\rho(X^{(1)}) = (10/3)/4 = 5/6$,

$X^{(2)} = \text{red}(\tilde{X}^{(1)}) = (5/6, 10/9)^T$; $10/9 \in R_3$ $(x_2 = 5/3)$, $j_2 = 3$, $y_{j_2} = \frac{10}{9}$ and the vector $\tilde{X}^{(2)} = (5/6)$ is a solution of the system

$$\begin{pmatrix} 2 \\ 2 \\ 2 \\ 3 \end{pmatrix} \otimes (x_1) + \begin{pmatrix} 5 \\ 6 \\ 10/3 \\ 10/3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 3 \\ 4 \end{pmatrix} \otimes (x_1) + \begin{pmatrix} 5 \\ 6 \\ 10/3 \\ 10/3 \end{pmatrix}$$

Now, $Q_0(x_2 = 5/3, x_3 = 10/9) = \emptyset$, $\rho(X^{(2)}) = 0$, $X^{(3)} = \text{red}(\tilde{X}^{(2)}) = (0)$, $j_3 = 1$, $y_{j_3} = 0 \in R_1(x_2 = \frac{5}{3}, x_3 = \frac{10}{9})$. As a result, $Y = (0, \frac{5}{3}, \frac{10}{9})^T \in B(M)$.

OPTIMIZATION

Definition: A function $f\colon E_n \to E$ is said to be isotone if

As a corollary of Lemma 6 we have that for every isotone function $f\colon E_n \to E$ there is

$$\inf_{X\in M} f(X) = \inf_{X\in \mathcal{B}(M)} f(X) = \min_{X\in \mathcal{B}(M)} f(X).$$

We summarize all results in

Theorem 2: Let $f\colon E_n \to E$ be an isotone function and let the solution set M of the system (12) be nonempty. Then a) there exists $\min_{X\in M} f(X)$, and b) $\min_{X\in M} f(X) = \min_{X\in \mathcal{B}(M)} f(X)$.

The set $\mathcal{B}(M)$ can be helpful in solving extremally linear programs because the function

$$f(X) = C^T \bar{\circ}\, X = \sum_{j\in J}^{\oplus} c_j \circ x_j, \quad C = (c_1,\dots,c_n)^T \in E_n$$

is isotone due to (9).

Thus, Theorem 2 enables us to use the following procedure in order to solve extremally linear programs:

To find out for every permutation $(j_1,j_2,\dots,j_n)$ of the set J whether some of vectors $(\bar{x}_1,\bar{x}_2,\dots,\bar{x}_n)^T$ satisfying (13) are at the same time elements of M.

i) If no (or if even does not exist any vector satisfying (13)) then according to Theorem 1 there is $M = \emptyset$.

ii) If yes then compile $\mathcal{B}(M)$.

According to Theorem 2 it remains to find the optimal value on the finite set $\mathcal{B}(M)$.

This procedure is used in the following example.

Example 2: Let E be the same as in Example 1. Consider the system of equations

$$\begin{pmatrix} 1 & 1 & 2 \\ 3 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} \bar{\circ} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \oplus \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 0 \end{pmatrix} \bar{\circ} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \oplus \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \tag{23}$$

This system is, as well as all other systems of equations in this example, in the standard form.

Here $J = Q = \{1,2,3\}$, $Q_o = \{1,2\}$.

R_1, $R_3(x_1 = \bar{x}_1)$ for all $\bar{x}_1 \in R_1$ and $R_2(x_1 = \bar{x}_1, x_3 = \bar{x}_3)$ for all $\bar{x}_1 \in R_1$ and $\bar{x}_3 \in R_3(x_1 = \bar{x}_1)$. It follows immediately from the definitions that $R_1 = \{2, \frac{1}{3}\}$.

a) Putting $x_1 = 2$ we get from (23):

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} \oplus \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} \oplus \begin{pmatrix} 6 \\ 6 \\ 3 \end{pmatrix}$$

Here $Q_0(x_1 = 2) = \{1,2\}$. One can now easily verify that $R_3(x_1 = 2) = \{6,3\}$.

a$_1$) Putting $x_3 = 6$ we have

$$\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \otimes (x_2) \oplus \begin{pmatrix} 0 \\ 6 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \otimes (x_2) \oplus \begin{pmatrix} 12 \\ 6 \\ 6 \end{pmatrix}$$

and thus $Q_0(x_1 = 2, x_3 = 6) = \{1,3\}$, $R_2(x_1 = 2, x_3 = 6) = \{12\}$. It remains to verify that $(2,12,6)^T \notin M$.

a$_2$) Putting $x_3 = 3$ we have

$$\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \otimes (x_2) \oplus \begin{pmatrix} 6 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \otimes (x_2) \oplus \begin{pmatrix} 6 \\ 6 \\ 3 \end{pmatrix}$$

and thus $Q_0(x_1 = 2, x_3 = 3) = \{2\}$, $R_2(x_1 = 2, x_3 = 3) = \{3\}$. We see that $(2,3,3)^T$ is not only a vector satisfying (13) but also an element of M and therefore $(2,3,3)^T \in B(M)$.

b) Putting $x_1 = \frac{1}{3}$ we get

$$\begin{pmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} \oplus \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} \oplus \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$$

Here $Q_0(x_1 = \frac{1}{3}) = \{1\}$, $R_3(x_1 = \frac{1}{3}) = \{1\}$.

Putting $x_3 = 1$ we have

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \otimes (x_2) \oplus \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \otimes (x_2) \oplus \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} .$$

and is an element of M, therefore $(\frac{1}{3},0,1)^T \in B(M)$.

Let us take now the permutation (2,3,1). We find out that $R_2 = \{2\}$ and putting $x_2 = 2$ we get from (23):

$$\begin{pmatrix}1 & 2\\ 3 & 0\\ 1 & 1\end{pmatrix} \otimes \begin{pmatrix}x_1\\ x_2\end{pmatrix} \oplus \begin{pmatrix}2\\ 0\\ 3\end{pmatrix} = \begin{pmatrix}3 & 0\\ 0 & 1\\ 0 & 0\end{pmatrix} \otimes \begin{pmatrix}x_1\\ x_3\end{pmatrix} \oplus \begin{pmatrix}2\\ 4\\ 3\end{pmatrix}$$

and thus $Q_0(x_2 = 2) = 2$, $R_3(x_2 = 2) = \emptyset$.

For other permutations the following results can be obtained: Permutation (1,2,3):

$R_1 = \{2, \frac{1}{3}\}$, $R_2(x_1 = 2) = \{6,3\}$, $R_3(x_1 = 2, x_2 = 6) = \emptyset$,

$R_3(x_1 = 2, x_2 = 3) = \{3\}$, $(2,3,3)^T \in M$,

$R_2(x_1 = \frac{1}{3}) = \{2\}$, $R_3(x_1 = \frac{1}{3}, x_2 = 2) = \emptyset$.

Permutation (2,1,3):

$R_2 = \{2\}$, $R_1(x_2 = 2) = \{\frac{4}{3}\}$, $R_3(x_2 = 2, x_1 = \frac{4}{3}) = \{2\}$, $(\frac{4}{3},2,2)^T \in M$.

Permutation (3,1,2):

$R_3 = \{1\}$, $R_1(x_3 = 1) = \{\frac{1}{3}\}$, $R_2(x_3 = 1, x_1 = \frac{1}{3}) = \{0\}$, $(\frac{1}{3},0,1)^T \in M$.

Permutation (3,2,1):

$R_2(x_3 = 1) = \emptyset$.

Hence we deduce that

$$B(M) = \{(2,3,3)^T, (\tfrac{1}{3},0,1)^T, (\tfrac{4}{3},2,2)^T\}$$

and since this set has a minimum we may assert that even for every isotone function $f: R_3^+ \to R^+$ there is

$$\min_{X \in M} f(X) = f(\tfrac{1}{3},0,1),$$

where M is the solution set of the system (23).

CONCLUSIONS

The procedure for solving two sided extremally linear systems provided by the just presented theory has to be considered as one of the first attempts to overcome the problem in a general case. Future research would be perhaps useful in one of the following directions:

2. To investigate properties of systems mentioned above by means of the theory of matroids.
3. To determine connections with polymatroids.
4. To transform (at least in special cases) two sided systems onto one sided ones using results described in [4].
5. To try to build a theory analogical to that of classical linear programming. Some attempts (using extremally convex sets and their "extreme points") have been made in [1]. See also [10]. In particular, under which additional assumptions would hold an analogy with the first assertion of Theorem 2 for an arbitrary extremally convex set M? This would be one but not the only generalization of the results presented in this paper.

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