On certain properties of the system of linear extremal equations

PETER BUTKOVIČ

1. Introduction

In the article [2] the terms "extremal vector space" and "extremal linear equation" were introduced. Properties of these objects are investigated in [1], [2], [3], [4].

In this article the concept of system of linear extremal equations is extended. Using the results of [1] a possible way to introduce the concept of the independence of vectors with respect to the system of linear extremal equations as well as the concept of the dimension of the extremal vector space with respect to the system of linear extremal equations is shown.

2. Notation and basic concepts

In this article $\boldsymbol{X}$, $\boldsymbol{Y}$, $\boldsymbol{Z}$, $\boldsymbol{W}$ are real column vectors. $\boldsymbol{X}^{\mathrm{T}}$ denotes the transposition of a vector $\boldsymbol{X}$.

Let us denote:

$$
\begin{aligned}
&S = \{1, 2, \ldots, m\}\,, \\
&N = \{1, 2, \ldots, n\}\,, \\
&\mathfrak{M}_n = \{\boldsymbol{X} \mid \boldsymbol{X}^{\mathrm{T}} = [X_1, \ldots, X_n], X_j \geqq 0 \quad \text{for all} \quad j \in N\}\,, \\
&\mathfrak{M}_n^+ = \{\boldsymbol{X} \mid \boldsymbol{X} \in \mathfrak{M}_n, X_j > 0 \quad \text{for all} \quad j \in N\}\,, \\
&\boldsymbol{X}^{\mathrm{T}} \,\bar{\text{o}}\, \boldsymbol{Y} = \max_{j \in N} X_j Y_j \quad \text{for all} \quad \boldsymbol{X}, \boldsymbol{Y} \in \mathfrak{M}_n\,.
\end{aligned}
$$

Supposing $\mathbf{A}$ is a nonnegative matrix of the type (m, n), we denote by $\mathbf{A}_i$ the i-th row vector of the matrix $\mathbf{A}$.

Assume $\boldsymbol{X} \in \mathfrak{M}_n$. We define $\mathbf{A} \,\bar{\text{o}}\, \boldsymbol{X}$ as follows:

$$(\mathbf{A} \,\bar{\text{o}}\, \boldsymbol{X}) \in \mathfrak{M}_m\,, \quad (\mathbf{A} \,\bar{\text{o}}\, \boldsymbol{X})_i = \mathbf{A}_i \,\bar{\text{o}}\, \boldsymbol{X} \quad \text{for} \quad i \in S\,.$$

Assume

$$\boldsymbol{X} = [X_1, \ldots, X_n]^{\mathrm{T}} \in \mathfrak{M}_n\,, \quad \boldsymbol{Y} = [Y_1, \ldots, Y_n]^{\mathrm{T}} \in \mathfrak{M}_n\,.$$

We define $\boldsymbol{X} \oplus \boldsymbol{Y}$ as follows:

$$\boldsymbol{X} \oplus \boldsymbol{Y} \in \mathfrak{M}_n\,, \quad (\boldsymbol{X} \oplus \boldsymbol{Y})_j = \max(X_j, Y_j) \quad \text{for all} \quad j \in N\,.$$

For an arbitrary real number $\alpha \geqq 0$ we denote by $\alpha \circ \mathbf{X}$ the vector $[\alpha X_1, \ldots, \alpha X_n]^T$.

Partial ordering on $\mathfrak{M}_n$ will be defined as usual:

$$\mathbf{X} \leqq \mathbf{Y} \quad \text{if} \quad X_j \leqq Y_j \quad \text{for} \quad j \in N\,.$$

$$\mathbf{X} \geqq \mathbf{Y} \quad \text{if} \quad X_j \geqq Y_j \quad \text{for} \quad j \in N\,.$$

A system of relations of the form:

$$\text{(a)} \qquad \begin{cases} \mathbf{A}^{(11)} \,\bar{\circ}\, \mathbf{X} \oplus \mathbf{B}^{(1)} = \mathbf{A}^{(12)} \,\bar{\circ}\, \mathbf{X} \oplus \mathbf{C}^{(1)}\,, \\ \mathbf{A}^{(21)} \,\bar{\circ}\, \mathbf{X} \oplus \mathbf{B}^{(2)} \leqq \mathbf{A}^{(22)} \,\bar{\circ}\, \mathbf{X} \oplus \mathbf{C}^{(2)}\,, \\ \mathbf{A}^{(31)} \,\bar{\circ}\, \mathbf{X} \oplus \mathbf{B}^{(3)} \geqq \mathbf{A}^{(32)} \,\bar{\circ}\, \mathbf{X} \oplus \mathbf{C}^{(3)}\,. \end{cases}$$

where $\mathbf{A}^{(ij)}$ are nonnegative matrices of the type (m_i, n) for $i = 1, 2, 3$; $j = 1, 2$ and $\mathbf{B}^{(i)}$, $\mathbf{C}^{(i)} \in \mathfrak{M}_{mi}$ for $i = 1, 2, 3$ is called a system of extremal linear equations and inequalities.

If $\mathbf{A}$, $\mathbf{B}$ are real matrices of the type (m, n), $\mathbf{A} = (A_{ij})$, $\mathbf{B} = (B_{ij})$ we denote by $\mathbf{A} * \mathbf{B}$ such a matrix $\mathbf{C} = (C_{ij})$ that

$$C_{ij} = A_{ij} B_{ij} \quad \text{for all} \quad i \in S, j \in N\,.$$

It was shown in [1] that the system (a) can be transformed to the form

$$\text{(b)} \qquad (\mathbf{A} * \mathbf{D}) \,\bar{\circ}\, \mathbf{X} = (\mathbf{A} * \mathbf{D}') \,\bar{\circ}\, \mathbf{X} \oplus \mathbf{B}\,,$$

where $\mathbf{B} \in \mathfrak{M}_m$; $\mathbf{A}$, $\mathbf{D}$, $\mathbf{D}'$ are matrices of the type (m, n); $\mathbf{A}$ is nonnegative; $\mathbf{D} = (D_{ij})$, $\mathbf{D}' = (D'_{ij})$ are zero-one matrices with the property:

$$D_{ij} + D'_{ij} = 1 \quad \text{for all} \quad i \in S, j \in N\,.$$

3. Extremal convex sets

Definition 1.

Let $\mathbf{X}, \mathbf{Y} \in \mathfrak{M}_n$. The set

$$U(\mathbf{X}, \mathbf{Y}) = \{\mathbf{Z} \in \mathfrak{M}_n \mid (\exists \alpha, \beta \in \mathfrak{M}_1) [\alpha \oplus \beta = 1 \;\&\; \mathbf{Z} = \alpha \circ \mathbf{X} \oplus \beta \circ \mathbf{Y}]\}$$

is called an extremal abscissa determined by points $\mathbf{X}$, $\mathbf{Y}$.

Definition 2.

A set $M \subseteq \mathfrak{M}_n$ is said to be extremal convex if $U(\mathbf{X}, \mathbf{Y}) \subseteq M$ for all $\mathbf{X}, \mathbf{Y} \in M$.

Definition 3.

A point $\mathbf{Z}$ of an extremal convex set M is called an extreme point of M if

$$\mathbf{Z} \notin U(\mathbf{X}, \mathbf{Y}) \quad \text{for all} \quad \mathbf{X}, \mathbf{Y} \in M, \mathbf{X} \neq \mathbf{Z} \neq \mathbf{Y}\,.$$

According to these definitions $\mathfrak{M}_n$ and $\emptyset$ are extremal convex sets and $[0, \ldots, 0]^T \in \mathfrak{M}_n$ is the only extreme point of the set $\mathfrak{M}_n$.

Lemma.

Let $\boldsymbol{X}, \boldsymbol{Y} \in \mathfrak{M}_n$ and $\boldsymbol{Z} \in U(\boldsymbol{X}, \boldsymbol{Y})$.
Then either $\boldsymbol{Z} \geqq \boldsymbol{X}$ or $\boldsymbol{Z} \geqq \boldsymbol{Y}$.

Proof:

Let

$$\boldsymbol{Z} = \alpha \circ \boldsymbol{X} \oplus \beta \circ \boldsymbol{Y} ; \quad \alpha, \beta \geqq 0, \alpha \oplus \beta = 1 .$$

Hence either $\alpha = 1$ or $\beta = 1$.

In the case $\alpha = 1$ there is

$$Z_j = \max (X_j, \beta Y_j) \geqq X_j \quad \text{for all} \quad j \in N$$

and therefore $\boldsymbol{Z} \geqq \boldsymbol{X}$.

Analogically for $\beta = 1$. QED.

Supposing M is a convex set we denote by $\xi(M)$ the set of all extreme points of the set M.

Theorem 1.

Suppose $M \subseteq \mathfrak{M}_n$ is an nonempty closed and extremal convex set. Then $\xi(M) \neq \emptyset$.

Proof:

If the set M fulfils the conditions mentioned above, we can define the vector $\boldsymbol{Z} = [Z_1, \ldots, Z_n]^T$ by induction as follows:

i) $$Z_1 = \min \{t \mid [t, t_2, \ldots, t_n]^T \in M\} ,$$

ii) $$Z_j = \min \{t \mid [Z_1, Z_2, \ldots, Z_{j-1}, t, t_{j+1}, \ldots, t_n]^T \in M\} .$$

Vector $\boldsymbol{Z}$ defined in this way is evidently an element of the set M.

Suppose $\boldsymbol{Z} \notin \xi(M)$. Then $\boldsymbol{Z} = \alpha \circ \boldsymbol{X} \oplus \beta \circ \boldsymbol{Y}$; $\alpha, \beta \geqq 0$; $\alpha \oplus \beta = 1$; $\boldsymbol{X}, \boldsymbol{Y} \in M$, $\boldsymbol{X} \neq \boldsymbol{Z} \neq \boldsymbol{Y}$.

Without loss of generality (according to Lemma) we can assume that $\boldsymbol{Z} \geqq \boldsymbol{X}$. Then there exists an index $j \in N$ such that

$$Z_j > X_j .$$

Let

$$k = \min \{j \in N \mid Z_j > X_j\} .$$

Thus for $j < k$, $j \in N$ (if $k > 1$) is $Z_j = X_j$ and therefore $\boldsymbol{X} = [Z_1, \ldots, Z_{k-1}, X_k, \ldots, X_n]^T \in M$.

If $k = 1$, then

$$\boldsymbol{X} = [X_1, \ldots, X_n]^T \in M .$$

But then $X_k < Z_k$ contradicts the definition of vector $\boldsymbol{Z}$ QED.

It was proved in [1] that the set of all solutions $\mathbf{X} \in \mathfrak{M}_n$ of the system (b) is an extremal convex set. Further we shall denote this set by M.

4. Extreme points of the set M

In the following parts of the article we suppose $m \leqq n$.

Let us denote the number of elements of an arbitrary finite set K by $|K|$.

Supposing $\mathbf{A}$ is a nonnegative matrix of the type (m, n) we denote by $A^{(j)}$ the j-th column vector of the matrix $\mathbf{A}$. Let $L = \{j_1, \ldots, j_k\} \subseteq N$ and $j_1 < j_2 < \ldots < j_k$. Supposing $\mathbf{X} = [X_1, \ldots, X_n]^{\mathrm{T}} \in \mathfrak{M}_n$ we denote by $\mathbf{X}(L)$ the vector

$$[X_{j_1}, X_{j_2}, \ldots, X_{j_k}]^{\mathrm{T}} \in \mathfrak{M}_k$$

and

$$\mathbf{A}(L) = (A^{(j_1)}, A^{(j_2)}, \ldots, A^{(j_k)}) .$$

Definition 4.

We say that the system (b) satisfies the non-degeneracy assumption if for every set $L \subseteq N$ such that

$$|L| = m$$

the system of extremal linear equations

$$\text{(b-}L\text{)} \qquad (\mathbf{A}(L) * \mathbf{D}(L)) \mathbin{\bar{\circ}} \mathbf{X}(L) = (\mathbf{A}(L) * \mathbf{D}'(L)) \mathbin{\bar{\circ}} \mathbf{X}(L) \oplus \mathbf{B}$$

has at most one solution.

Supposing $\mathbf{X} \in \mathfrak{M}_n$ we denote

$$P(\mathbf{X}) = \{j \in N \mid X_j > 0\} .$$

The following two theorems were proved in [1].

Theorem 2.

If the system (b) satisfies the non-degeneracy assumption and $\mathbf{W} \in M$ is a vector such that $|P(\mathbf{W})| \leqq m$ then $\mathbf{W} \in \xi(M)$.

Theorem 3.

If $\mathbf{W} \in M$ is a vector such that $|P(\mathbf{W})| > m$ then there exist vectors $\mathbf{Y}, \mathbf{Z} \in M$, $\mathbf{Y} \neq \mathbf{W} \neq \mathbf{Z}$ and real numbers $\alpha, \beta \geqq 0$, $\alpha \oplus \beta = 1$ such that

$$\mathbf{W} = \alpha \circ \mathbf{Y} \oplus \beta \circ \mathbf{Z} \quad (\text{i.e. } \mathbf{W} \notin \xi(M)) .$$

5. The independence of vectors with respect to the system of linear extremal equations

Throughout this section we shall assume that in the system (b) there is

$$\mathbf{B} \in \mathfrak{M}_m^+ .$$

Definition 5.

The system of vectors

$$\text{(S)} \qquad \{\mathbf{A}^{(j)} \in \mathfrak{M}_m \mid j \in L\}\,.$$

$L \subseteq N$ is said to be independent with respect to the system (b) if there exists a set $L' \subseteq N$, $L \cap L' = \emptyset$, $|L'| \geqq 0$ such that for $H = L \cup L'$ the system

$$\text{(b-}H\text{)} \qquad (\mathbf{A}(H) * \mathbf{D}(H))\,\bar{o}\,\mathbf{X}(H) = (\mathbf{A}(H) * \mathbf{D}'(H))\,\bar{o}\,\mathbf{X}(H) \oplus \mathbf{B}$$

has exactly one solution (in $\mathfrak{M}_n$, $h = |H|$).

b) The system of vectors (S) is said to be dependent with respect to (b) if it is not independent in relation to (b).

It is easy to see that every subsystem of the system of vectors independent with respect to (b) is also independent with respect to (b).

Let $\mathbf{Z} \in \mathfrak{M}_n$. In the proof of the following theorem let us write:

$$E(\mathbf{Z}) = (\mathbf{A} * \mathbf{D})\,\bar{o}\,\mathbf{Z}\,,$$
$$E(\mathbf{Z}) = [E_1(\mathbf{Z}), E_2(\mathbf{Z}), \ldots, E_m(\mathbf{Z})]^{\mathrm{T}}\,,$$
$$L_i(\mathbf{Z}) = \{j \in N \mid A_{ij}D_{ij}Z_j = E_i(\mathbf{Z})\} \quad \text{for all} \quad i \in S\,,$$
$$R_i(\mathbf{Z}) = \{j \in N \mid A_{ij}D'_{ij}Z_j = E_i(\mathbf{Z})\} \quad \text{for all} \quad i \in S\,.$$

Theorem 4.

$|L| > m$, $L \subseteq N$ implies that the system of vectors (S) is dependent with respect to (b).

Proof:

Suppose H is an arbitrary set satisfying the condition $L \subseteq H \subseteq N$. It has to be shown that the corresponding system (b-H) has either no solution or at least two solutions. Let $\mathbf{Z} \in \mathfrak{M}_h$ (where $h = |H|$) be a solution of (b). Hence, $h \geqq |L| > m$. There is evidently $E(\mathbf{Z}) \geqq \mathbf{B}$ therefore $E(\mathbf{Z}) \in \mathfrak{M}_m^+$.

Now we shall consider two cases.

a) Assume $P(\mathbf{Z}) \neq H$. Let, for example, $Z_j = 0$. Then

$$(\forall i \in S)\,[j \notin L_i(\mathbf{Z}) \cup R_i(\mathbf{Z})]\,.$$

Writing

$$\lambda = \min_{i \in P(\mathbf{A}^{(j)})} \frac{E_i(\mathbf{Z})}{A_{ij}} > 0$$

we see that the vector

$$[Z_1, \ldots, Z_{j-1}, \lambda, Z_{j+1}, \ldots, Z_n]^{\mathrm{T}}$$

(different from $\mathbf{Z}$) is also the solution of (b).

b) Assume $P(\mathbf{Z}) = H$. Hence $|P(\mathbf{Z})| = H > m$. Let $\mathbf{W} = [W_1, \ldots, W_n]^{\mathrm{T}}$ is such a vector that

$$W_j = Z_j \quad \text{for all} \quad j \in H$$

and

$$W_j = 0 \quad \text{for all} \quad j \in N - H\,.$$

It is easy to see that $\mathbf{W} \in M$. Since $|P(\mathbf{W})| > m$ according to Theorem 3 there exist vectors $\mathbf{X}, \mathbf{Y} \in M$ and $\alpha, \beta \in \mathfrak{M}_1$ such that $\mathbf{X} \neq \mathbf{W} \neq \mathbf{Y}$; $\mathbf{W} = \alpha \circ \mathbf{X} \oplus \beta \circ \mathbf{Y}$; $\alpha \oplus \beta = 1$. If there were $\alpha = 0$ or $\beta = 0$ then $\mathbf{W} = \mathbf{Y}$ or $\mathbf{W} = \mathbf{X}$, respectively. Therefore $\alpha > 0$ and $\beta > 0$. Hence $W_j = 0$ implies $X_j = Y_j = 0$. But it means that $\mathbf{X}(H)$ and $\mathbf{Y}(H)$ are also solutions of (b-H), different from $\mathbf{Z}$. This completes the proof.

Theorem 4 shows that the greatest number of vectors independent with respect to (b) is less than or equal to m.

Definition 6.

The greatest number of vectors independent with respect to (b) is called a dimension of the extremal vector space $\mathfrak{M}_m$ with respect to the system (b). We denote this dimension by $r_b(\mathfrak{M}_m)$.

It was mentioned above that

$$r_b(\mathfrak{M}_m) \leqq m\,.$$

Theorem 5.

Suppose the system (b) satisfies the non-degeneracy assumption. Then $r_b(\mathfrak{M}_m) = m$ if and only if $M \neq \emptyset$.

Proof:

Let $M \neq \emptyset$. Since M is closed and extremal convex subset of $\mathfrak{M}_n$ then according to Theorem 1 there is $\xi(M) \neq \emptyset$. Let, for example, $\mathbf{Z} \in \xi(M)$. It follows from Theorem 2 that $|P(\mathbf{Z})| \leqq m$. We take a set $L \subseteq N$ such that $P(\mathbf{Z}) \subseteq L$ and $|L| = m$. Hence $\mathbf{Z}(L)$ is the solution of the system (b-L). Owing to the non-degeneracy assumption this system has exactly one solution. It means that the system (S) is independent with respect to (b). Hence $r_b(\mathfrak{M}_m) = m$.

Further, if $r_b(\mathfrak{M}_m) = m$ there must exist a set $L \subseteq N$, $|L| = m$ such that the system (S) is independent with respect to (b). It means that the system has exactly one solution. We denote it by $\mathbf{Z}(L)$. Then the vector $\mathbf{Y} = [Y_1, \ldots, Y_n]^{\mathrm{T}}$ where

$$Y_j = Z_j \quad \text{for all} \quad j \in L$$

and

$$Y_j = 0 \quad \text{for all} \quad j \in N - L$$

is the solution of (b) and therefore $M \neq \emptyset$.

This completes the proof.

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RNDr. Peter Butkovič, II. katedra matematiky PF UPJŠ, Komenského 14, 041 54 Košice, ČSSR.

Resumé

NIEKTORÉ VLASTNOSTI SÚSTAV LINEÁRNYCH EXTREMÁLNYCH ROVNÍC

Peter Butkovič

V prácach [2], [3], [4] bol zavedený pojem extremálneho vektorového priestoru a sústavy extremálnych rovníc a nerovností. V tomto článku sa rozširuje pojem sústava extremálnych rovníc a nerovností.

V analógii s metódami lineárneho programovania sa vybudujú niektoré pojmy (extremálne konvexná množina, krajné body extremálne konvexnej množiny, predpoklad nedegenerácie).

Výsledky práce [1] týkajúce sa charakteristiky krajných bodov extremálne konvexnej množiny sa použijú na vytvorenie pojmu nezávislosti vektorov vzhľadom k sústave extremálnych rovníc. Pomocou toho je tiež definovaný rozmer extremálneho vektorového priestoru vzhľadom k sústave extremálnych rovníc.

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