

ON THE COEFFICIENTS OF THE MAX-ALGEBRAIC
CHARACTERISTIC POLYNOMIAL AND EQUATION

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No polynomial algorithms are known for finding the coefficients of the characteristic polynomial and characteristic equation of a matrix in max-algebra. The following are proved: (1) The task of finding the max-algebraic characteristic polynomial for permutation matrices encoded using the lengths of their constituent cycles is *NP*-complete. (2) The task of finding the lowest order finite term of the max-algebraic characteristic polynomial for a $\{0, -\infty\}$ matrix can be converted to the assignment problem. (3) The task of finding the max-algebraic characteristic equation of a $\{0, -\infty\}$ matrix can be converted to that of finding the conventional characteristic equation for a $\{0, 1\}$ matrix and thus it is solvable in polynomial time.

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1. DEFINITIONS AND KNOWN RESULTS

If we replace the operations of addition and multiplication in the real numbers by taking the maximum of two numbers and by adding two numbers, we obtain the so-called *max-algebra* which offers an attractive language to deal with problems in automata theory, scheduling theory, and discrete event systems, see e.g. the monographs of Baccelli, Cohen, Olsder and Quadrat [2], Cuninghame–Green [4] and Zimmermann [9]. Significant effort has been devoted to building up a theory similar to that of linear algebra, for instance to study systems of linear equations, eigenvalue problems, independence, rank and dimension.

In this paper we deal with the max-algebraic characteristic polynomial (or, briefly characteristic maxpolynomial) of a square matrix as defined in Cuninghame–Green [5] and with some aspects of the max-algebraic characteristic equation as defined in [2]. Both these concepts are related to the minimal-dimensional realisation problem for discrete-event dynamic systems [2] and the first one has also some interesting operational research interpretation (see the job rotation problem below). Since to our knowledge no efficient method for finding the max-algebraic characteristic polynomial and equation exists we study in this paper some questions related to these two problems which can be solved in polynomial time.

Let us denote $a \oplus b = \max(a, b)$ and $a \otimes b = a + b$ for $a, b \in \overline{\mathbf{R}}$ where $\overline{\mathbf{R}} = \mathbf{R} \cup \{-\infty\}$. The iterated product $a \otimes a \otimes \dots \otimes a$, in which the letter a appears k -times will be denoted by $a^{(k)}$. Let us extend the pair of operations (\oplus, \otimes) to matrices and vectors in the same way as in conventional linear algebra. That is if $\mathbf{A} = (a_{ij})$, $\mathbf{B} = (b_{ij})$ and $\mathbf{C} = (c_{ij})$ are matrices or vectors over $\overline{\mathbf{R}}$ of compatible sizes then $\mathbf{C} = \mathbf{A} \oplus \mathbf{B}$ if $c_{ij} = a_{ij} \oplus b_{ij}$ for all i, j and $\mathbf{C} = \mathbf{A} \otimes \mathbf{B}$ if $c_{ij} = \sum_k^{\oplus} a_{ik} \otimes b_{kj}$ for all i, j .

For any set X and a positive integer n the symbol $X(n, n)$ will denote the set of all $n \times n$ matrices over X . The letter \mathbf{I} stands for a square matrix of an appropriate order whose diagonal entries are 0 and off-diagonal ones are $-\infty$. By a principal submatrix of $\mathbf{A} = (a_{ij}) \in \overline{\mathbf{R}}(n, n)$ we understand as usual any matrix of the form

$$\begin{pmatrix} a_{i_1 i_1} & a_{i_1 i_2} & \dots & a_{i_1 i_k} \\ a_{i_2 i_1} & a_{i_2 i_2} & \dots & a_{i_2 i_k} \\ \vdots & \vdots & & \vdots \\ a_{i_k i_1} & a_{i_k i_2} & \dots & a_{i_k i_k} \end{pmatrix}$$

where $1 \leq i_1 < i_2 < \dots < i_k \leq n$. This matrix will be denoted by $\mathbf{A}(i_1, i_2, \dots, i_k)$.

The assignment problem for an $n \times n$ matrix \mathbf{A} is the task of selecting n entries of the matrix, one from each row and from each column so that their sum is maximal. The selection of n such entries is fully described by a permutation, say π , of the set $N = \{1, \dots, n\}$. Let us denote by P_n the set of all permutations of N . Then the assignment problem is the task of finding a permutation $\pi \in P_n$ which maximises $\sum_{i=1}^n a_{i, \pi(i)}$. The quantity

$$\max_{\pi \in P_n} \sum_{i=1}^n a_{i, \pi(i)} \quad (1)$$

will be called the optimal assignment problem value (for \mathbf{A}).

The max-algebraic permanent of $\mathbf{A} = (a_{ij}) \in \overline{\mathbf{R}}(n, n)$ is defined as an analogue of the classical one:

$$\text{maper}(\mathbf{A}) = \sum_{\pi \in P_n}^{\oplus} w(\mathbf{A}, \pi) \quad (2)$$

where

$$w(\mathbf{A}, \pi) = \prod_{i \in N}^{\otimes} a_{i, \pi(i)}.$$

In conventional notation $\text{maper}(\mathbf{A}) = \max_{\pi \in P_n} \sum_{i \in N} a_{i, \pi(i)}$, thus the max-algebraic permanent of \mathbf{A} is the optimal assignment problem value for \mathbf{A} . There are a number of efficient solution methods for finding this value, one of the best known is the Hungarian method of computational complexity $O(n^3)$, see for instance Ahuja et al [1].

The max-algebraic characteristic polynomial of the matrix $\mathbf{A} = (a_{ij}) \in \overline{\mathbf{R}}(n, n)$ has been defined in Cuninghame-Green [5] as

$$\chi_{\mathbf{A}}(x) = \text{maper}(\mathbf{A} \oplus x \otimes \mathbf{I}),$$

that is the max-algebraic permanent of the matrix

$$\begin{pmatrix} a_{11} \oplus x & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} \oplus x & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \oplus x. \end{pmatrix}.$$

It follows from this definition that $\chi_A(x)$ is of the form

$$\delta_0 \oplus \delta_1 \otimes x \oplus \dots \oplus \delta_{n-1} \otimes x^{(n-1)} \oplus x^{(n)} \tag{3}$$

or, briefly $\sum_{i=1, \dots, n}^{\oplus} \delta_i \otimes x^{(i)}$, where $\delta_n = 0$ and, by convention, $x^{(0)} = 0$. It has been proved in Cuninghame-Green [5] that

$$\delta_k = \sum_{B \in A_k}^{\oplus} \text{maper}(B) \tag{4}$$

where A_k is the set of all principal submatrices of A of order $n - k$. Hence $\delta_0 = \text{maper}(A)$ and $\delta_{n-1} = \max(a_{11}, a_{22}, \dots, a_{nn})$. Obviously, $\delta_k = -\infty$ if all $B \in A_k$ have $\text{maper}(B) = -\infty$ in which case the term $\delta_k \otimes x^{(k)}$ is omitted from $\chi_A(x)$ by convention. Note that $\chi_A(x)$ may reduce to just $x^{(n)}$, and that $\chi_A(x)$ is not affected by a simultaneous permutation of the rows and columns of A . More details can be found in Butkovič and Murfitt [3].

If for some $k \in \{0, \dots, n\}$

$$\delta_k \otimes x^{(k)} \leq \sum_{i \neq k}^{\oplus} \delta_i \otimes x^{(i)}$$

holds for all $x \in \mathbf{R}$ then the term $\delta_k \otimes x^{(k)}$ is called inessential, otherwise it is called essential. Hence

$$\chi_A(x) = \sum_{i \neq k}^{\oplus} \delta_i \otimes x^{(i)}$$

holds for all $x \in \mathbf{R}$ if $\delta_k \otimes x^{(k)}$ is inessential, and therefore inessential terms may be ignored if $\chi_A(x)$ is considered as a function.

Note that although $\text{maper}(B)$ can easily be found for any matrix B , δ_k cannot be computed from (4) efficiently since the number of matrices in A_k is $\binom{n}{k}$.

One of the motivations for investigating the max-algebraic characteristic equation and characteristic polynomial is related to the following two combinatorial optimisation problems:

OPTIMAL AP-SUBMATRIX (OAPSM). Given a matrix $A \in \overline{\mathbf{R}}(n, n)$ and $k \in \{1, \dots, n\}$, find the biggest optimal assignment problem value of a $k \times k$ submatrix of A .

OPTIMAL AP-PRINCIPAL SUBMATRIX (PRINCIPAL OAPSM).

Given $A \in \overline{\mathbf{R}}(n, n)$ and $k \in \{1, \dots, n\}$, find the biggest optimal assignment problem value of a $k \times k$ principal submatrix of A .

Although it is not difficult to solve OAPSM, to the author's knowledge no polynomial method is known for PRINCIPAL OAPSM. In Butkovič and Murfitt [3] a polynomial method for finding all essential terms of a max-algebraic characteristic

polynomial was presented. It follows from (4) that in the case when all terms of the max-algebraic characteristic polynomial are essential this method also solves the PRINCIPAL OAPSM.

There is also a practical managerial motivation for the study of OAPSM which we call the *job rotation problem*: Suppose that a company with n employees requires these workers to swap their jobs (possibly on a regular basis) in order to avoid exposure to monotonous tasks (for instance manual workers at an assembly line or ride operators in a theme park). It may also be required that to maintain stability of service only a certain number of employees, say k ($k < n$), actually swap their jobs. With each pair old job - new job a coefficient may be associated expressing the cost (for instance for an additional training) or the preference of the worker to this particular change. Let us denote the matrix of these coefficients as \mathbf{A} . So the aim is to select k employees and to suggest a plan of the job changes between them so that the sum of the coefficients corresponding to these changes is maximal. This task leads to finding a $k \times k$ principal submatrix of \mathbf{A} for which the optimal assignment problem value is maximal (the diagonal entries can be set to $+\infty$ or $-\infty$ to avoid an assignment to the same job).

The definition of the max-algebraic characteristic equation is motivated by that in Baccelli et al [2]: Let $\pi \in P_n$ and $\nu \in \bar{\mathbf{R}}$. Then P_n^+ [P_n^-] stands for the set of even [odd] permutations of the set P_n and

$$\begin{aligned} p^+(\mathbf{A}, \nu) &= \|\{\pi \in P_n^+; w(\mathbf{A}, \pi) = \nu\}\|, \\ p^-(\mathbf{A}, \nu) &= \|\{\pi \in P_n^-; w(\mathbf{A}, \pi) = \nu\}\|. \end{aligned}$$

The max-algebraic characteristic equation of the matrix \mathbf{A} is

$$\lambda^{(n)} \oplus \sum_{k \in J} c_{n-k} \otimes \lambda^{(n-k)} = c_{n-1} \otimes \lambda^{(n-1)} \oplus \sum_{k \in \bar{J}} c_{n-k} \otimes \lambda^{(n-k)}$$

where for $k = 1, \dots, n$

$$\begin{aligned} c_{n-k} &= \max \left\{ \nu; \sum_{\mathbf{B} \in A_k} p^+(\mathbf{B}, \nu) \neq \sum_{\mathbf{B} \in A_k} p^-(\mathbf{B}, \nu) \right\}, \\ J &= \{j; d_{n-j} > 0\}, \bar{J} = \{j; d_{n-j} < 0\} \\ d_{n-k} &= (-1)^k \left(\sum_{\mathbf{B} \in A_k} p^+(\mathbf{B}, c_{n-k}) - \sum_{\mathbf{B} \in A_k} p^-(\mathbf{B}, c_{n-k}) \right). \end{aligned}$$

Note that if $k \in \{1, \dots, n\}$ and $\text{maper}(\mathbf{B}) = -\infty$ for all $\mathbf{B} \in A_k$ then the term $c_{n-k} \otimes \lambda^{(n-k)}$ does not appear on either side of the equation.

If $\mathbf{A} = (a_{ij}) \in \bar{\mathbf{R}}(n, n)$ then $D(\mathbf{A})$ will denote the digraph with the node set $\{1, \dots, n\}$ and arc set $\{(i, j); a_{ij} \text{ is finite}\}$. Note that the max-algebraic characteristic equation plays a crucial role in investigating max-algebraic discrete-event dynamic systems [2].

2. NEW RESULTS

The task of finding the lowest order term in the max-algebraic characteristic polynomial of a matrix A is equivalent to the task of finding the maximal value of k for which there is a $k \times k$ principal submatrix B of A with finite $m\text{aper}(B)$. It is easily seen that this is equivalent to each of the following combinatorial problems:

- (A) (By replacing $-\infty$ by 1 and finite elements by 0). Given a 0-1 matrix A , find the maximal value of k for which A contains a $k \times k$ principal submatrix with k independent zeros (that is k zeros no two of which are taken either from the same row or the same column).
- (B) (After swapping 0's and 1's in (A)) Given a 0-1 matrix A , find the maximal value of k for which A contains a $k \times k$ principal submatrix whose conventional permanent is non-zero.

In what follows the letter T stands for the set $\{-\infty, 0\}$. Obviously, every (finite) coefficient of a max-algebraic characteristic polynomial of a matrix from $T(n, n)$ is 0.

Theorem 1. If $A \in T(n, n)$ is a permutation matrix given by the list of lengths of constituent cycles and $k \in \{1, \dots, n\}$ then the task of deciding whether δ_k is 0 or $-\infty$ is *NP*-complete. Hence the task of finding the max-algebraic characteristic polynomial for permutation matrices encoded using the lengths of their constituent cycles is *NP*-complete.

Proof. Suppose that $\{\ell_1, \ell_2, \dots, \ell_s\}$ is the list of the lengths of constituent cycles of the matrix A . Thus A is the permutation matrix corresponding to the permutation

$$\pi = \pi_1 \circ \pi_2 \circ \dots \circ \pi_s$$

where $\pi_1, \pi_2, \dots, \pi_s$ are cycles of the lengths $\ell_1, \ell_2, \dots, \ell_s$. Since the max-algebraic characteristic polynomial is not affected by simultaneous permutations of the rows and columns, it can be assumed that $A = \text{blockdiag}(A_1, A_2, \dots, A_s)$ where each of A_j is of the form

$$A_j = \begin{pmatrix} \cdot & 0 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 0 & \cdot & -\infty & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -\infty & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}.$$

Clearly, B is a principal submatrix of A with $m\text{aper}(B) = 0$ if and only if

$$B = \text{blockdiag}(A_{i_1}, A_{i_2}, \dots, A_{i_r})$$

where $\{i_1, i_2, \dots, i_r\} \subseteq \{1, \dots, s\}$.

Let $k \in \{1, \dots, n\}$. Then $\delta_{n-k} = 0 \Leftrightarrow$ there is a $k \times k$ principal submatrix B of A with $m\text{aper}(B) = 0 \Leftrightarrow k = \ell_{i_1} + \ell_{i_2} + \dots + \ell_{i_r}$ for some $\{i_1, i_2, \dots, i_r\} \subseteq \{1, \dots, s\}$.

Hence if there is a polynomial-time algorithm for deciding whether $\delta_{n-k} = 0$ for $k \in \{1, \dots, n\}$ then there is also a polynomial-time algorithm for solving the problem:

Given k and $\ell_i, i \in S = \{1, \dots, s\}$, is there a set $R \subseteq S$ such that $k = \sum_{i \in R} \ell_i$? This problem is the 0-1 KNAPSACK which is *NP*-complete [7]. This completes the proof. \square

Remark. (Klinz [6]) It is well known that 0-1 KNAPSACK is solvable by pseudopolynomial algorithms, that is algorithms polynomial in $L = \sum_{i \in S} \ell_i$. It is therefore clear that if the constituent cycles are given as sequences of nodes (and hence the input of \mathbf{A} has size $L = n$) then any such algorithm is polynomial in n and decides whether $\delta_{n-k} = 0$ for a $k \in \{1, \dots, n\}$.

It may be useful to see that the problem of finding the max-algebraic characteristic polynomial has a very natural graph-theoretic interpretation:

Theorem 2. If $\mathbf{A} \in T(n, n)$ then $\delta_{n-k} = 0$ if and only if the following holds: In the digraph $D(\mathbf{A})$ there is a set of k nodes which is the union of the sets of nodes of node-disjoint cycles in $D(\mathbf{A})$.

Proof. $\delta_{n-k} = 0 \Leftrightarrow$ there is a matrix $\mathbf{B} = \mathbf{A}(i_1, i_2, \dots, i_k)$ with $\text{maper}(\mathbf{B}) = 0 \Leftrightarrow$ there exists a permutation π of the set $\{i_1, i_2, \dots, i_k\}$ such that

$$a_{i_p \pi(i_p)} = 0 \quad (p = 1, \dots, k).$$

Let $\pi = \pi_1 \circ \pi_2 \circ \dots \circ \pi_q$ be a decomposition of π to cyclic permutations. Each of $\pi_1, \pi_2, \dots, \pi_q$ determines a cycle in $D(\mathbf{A})$ and all these cycles are pairwise disjoint.

Conversely, if the set of nodes $I = \{i_1, \dots, i_k\}$ is a union of q pairwise node-disjoint cycles then these cycles define cyclic permutations $\pi_1, \pi_2, \dots, \pi_q$ and $\pi = \pi_1 \circ \pi_2 \circ \dots \circ \pi_q$ is a permutation of the set I satisfying $a_{i_p \pi(i_p)} = 0$ ($p = 1, \dots, k$). Hence if $\mathbf{B} = \mathbf{A}(i_1, i_2, \dots, i_k)$ then $\text{maper}(\mathbf{B}) = 0$ and so $\delta_{n-k} = 0$. The theorem statement follows. \square

Theorem 3. If $\mathbf{A} \in T(n, n)$ then the biggest value of k for which $\delta_{n-k} = 0$ is $n + \text{maper}(\mathbf{A} \oplus (-1) \otimes \mathbf{I})$ and thus it can be determined using $O(n^3)$ operations by solving the assignment problem for the matrix $\mathbf{A} \oplus (-1) \otimes \mathbf{I}$.

Proof. Let us denote $K = \{k; \delta_{n-k} = 0\}$ and $k_0 = \max K$. If $\mathbf{A} \in T(n, n)$ then all finite δ_{n-k} are 0, thus in the conventional notation $\chi_{\mathbf{A}}(x) = \max_{k \in K} (n - k)x$. Therefore, for $x < 0$ there is

$$\chi_{\mathbf{A}}(x) = x \cdot \min_{k \in K} (n - k) = (n - k_0)x$$

and thus $\chi_{\mathbf{A}}(-1) = k_0 - n$ from which the result follows. \square

Corollary. The problems (A) and (B) above are solvable using $O(n^3)$ operations.

Theorem 4. If $\mathbf{A} \in T(n, n)$ then the max-algebraic characteristic equation for \mathbf{A} can be found in polynomial time.

Proof. If $\mathbf{A} \in T(n, n)$ then all finite c_{n-k} are 0. Note that if $k \in \{1, \dots, n\}$ and $\text{maper}(\mathbf{B}) = -\infty$ for all $\mathbf{B} \in A_k$ then the term $c_{n-k} \otimes \lambda^{(n-k)}$ does not appear on either side of the equation. If $\mathbf{B} = (b_{ij}) \in T(n, n)$ then $p^+(\mathbf{B}, 0)$ is the number of even permutations which select only zeros from \mathbf{B} . Let us denote the matrix $(e^{b_{ij}})$ as $e^{\mathbf{B}}$. This matrix is zero-one, zeros corresponding to $-\infty$ in \mathbf{B} and ones corresponding to zeros in \mathbf{B} . So $p^+(\mathbf{B}, 0)$ is the number of even permutations which select only ones from $e^{\mathbf{B}}$. Similarly for $p^-(\mathbf{B}, 0)$. Since $e^{\mathbf{B}}$ is zero-one, all terms in the standard determinant expansion of $e^{\mathbf{B}}$ are either 1 (if the corresponding permutation is even and selects only ones) or -1 (if the corresponding permutation is odd and selects only ones) or 0 (otherwise). Hence $\det e^{\mathbf{B}} = p^+(\mathbf{B}, 0) - p^-(\mathbf{B}, 0)$. Since d_{n-k} in the definition of the max-algebraic characteristic equation can be written as

$$d_{n-k} = (-1)^k \sum_{\mathbf{B} \in A_k} (p^+(\mathbf{B}, c_{n-k}) - p^-(\mathbf{B}, c_{n-k})),$$

it follows that $d_{n-k} = (-1)^k (\sum_{\mathbf{B} \in A_k} \det e^{\mathbf{B}})$ which is the coefficient at λ^k of the conventional characteristic polynomial of the matrix $e^{\mathbf{A}}$. This characteristic polynomial can be found in polynomial time (van Leeuwen [7]).

A matrix $\mathbf{A} = (a_{ij})$ is called diagonally dominant [strictly diagonally dominant] if the identity is an optimal permutation [unique optimal permutation] to the assignment problem for \mathbf{A} and it is called bi-diagonal if a_{ij} is finite if and only if $|i - j| = 1$.

For a given sequence $\{g_r; r = 1, \dots, 2n - 1\}$, $n \geq 2$, the Hankel matrix is the $n \times n$ matrix $\mathbf{H} = (h_{ij})$ having $h_{ij} = g_{i+j-1}$:

$$\mathbf{H} = \begin{bmatrix} g_1 & g_2 & g_3 & \dots & g_n \\ g_2 & g_3 & \dots & g_{n+1} & \\ g_3 & \dots & g_{n+2} & & \\ \vdots & & \dots & \vdots & \\ g_n & & \dots & g_{2n-1} & \end{bmatrix}.$$

Note that Hankel matrices play an essential role in investigating the max-algebraic discrete-event dynamic systems.

In the light of the previous results it may also be interesting to mention the following properties proved by Murfitt [8]. \square

Theorem 5. The max-algebraic characteristic equation [characteristic polynomial] for the matrix $\mathbf{A} \in \bar{\mathbf{R}}(n, n)$ can be found in polynomial time if \mathbf{A} is of one of the following types:

- Strictly diagonally dominant [diagonally dominant],
- Hankel-matrix corresponding to a strictly concave sequence;

– Bi-diagonal.

Note that [strictly] diagonally dominant matrices include [strictly] Monge matrices which in turn include Hankel matrices corresponding to [strictly] convex sequences.

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