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Finding all essential terms of a characteristic maxpolynomial☆

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Dedicated to Professor Raymond A. Cuninghame-Green on the occasion of his 70th birthday

Abstract

Let us denote $a \oplus b = \max(a, b)$ and $a \otimes b = a + b$ for $a, b \in \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\}$ and extend this pair of operations to matrices and vectors in the same way as in linear algebra. We present an $O(n^2(m + n \log n))$ algorithm for finding all essential terms of the max-algebraic characteristic polynomial of an $n \times n$ matrix over $\overline{\mathbb{R}}$ with m finite elements. In the cases when all terms are essential, this algorithm also solves the following problem: Given an $n \times n$ matrix A and $k \in \{1, \dots, n\}$, find a $k \times k$ principal submatrix of A whose assignment problem value is maximum.

MSC: 90C27; 15A15

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1. Introduction

If we replace the operations of addition and multiplication in the real numbers by taking the maximum of two numbers and by adding two numbers, we obtain the so-called *max-algebra* which offers an attractive language to deal with problems in

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automata theory, scheduling theory, and discrete event systems, see e.g. the monographs of Baccelli et al. [2], Cuninghame-Green [5] and Zimmermann [14]. Further important papers in this direction are of Cuninghame-Green [7], Gaubert [9] and Gondran and Minoux [12]. Specifically, significant effort has been devoted to build up a theory similar to that of linear algebra, for instance [5], to study systems of linear equations, eigenvalue problems, independence, rank and dimension.

In this paper, we deal with the max-algebraic characteristic polynomial (or, briefly *characteristic maxpolynomial*) of a square matrix as defined in Cuninghame-Green [6]. This concept is closely related to that of the max-algebraic characteristic equation of a matrix (although it should be mentioned that this relation is not as strong as in linear algebra—see Section 2) which plays a crucial role for analysing and solving max-algebraic discrete-event dynamic systems, see e.g. [2,3,10].

To the authors' knowledge, there is no polynomial method for finding all coefficients of a characteristic maxpolynomial. However, because of the absorbing effect of the operation $\oplus$, some terms of a characteristic maxpolynomial may be omitted without changing it as a function. All the remaining terms are called *essential* (for an exact definition see below). An $O(n^5)$ method for finding all essential terms of the characteristic maxpolynomial of an $n \times n$ matrix with entries from $\mathbb{Q} \cup \{-\infty\}$ was presented in [4]. In that method, the rationality of the matrix entries has explicitly been exploited.

After the necessary definitions, we present first an $O(n^4)$ algorithm for finding all essential terms of the characteristic maxpolynomial of a *real* $n \times n$ matrix and then we show how this method can easily be modified to solve the same problem for matrices which admit $-\infty$ as an entry. In this case, the complexity can be reduced to $O(n^2(m+n\log n))$, where m is the number of finite entries of A. We also discuss an OR interpretation of the coefficients of a characteristic maxpolynomial and an interpretation in combinatorial terms. In particular, it will be explained how this method enables us to readily identify a $k \times k$ principal submatrix of an $n \times n$ matrix A with maximum assignment problem value provided that $n-k$ corresponds to an essential term of the characteristic maxpolynomial of A.

2. Definitions

Let us denote $a \oplus b = \max(a, b)$ and $a \otimes b = a + b$ for $a, b \in \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\}$. The iterated product $a \otimes a \otimes \cdots \otimes a$ in which the letter a appears k-times will be denoted by $a^{(k)}$. Let us extend the pair of operations $(\oplus, \otimes)$ to matrices and vectors in the same way as in conventional linear algebra. That is, if $A = (a_{ij}), B = (b_{ij})$ are matrices or vectors over $\overline{\mathbb{R}}$ of compatible sizes then we write $C = A \oplus B$ if $c_{ij} = a_{ij} \oplus b_{ij}$ for all i, j and $C = A \otimes B$ if $c_{ij} = \bigoplus_k a_{ik} \otimes b_{kj}$ for all i, j.

For any set X and positive integer n, the symbol $X^{n \times n}$ will denote the set of all $n \times n$ matrices over X. The letter I stands for a square matrix of an appropriate order with diagonal entries 0 and off-diagonal entries $-\infty$.

A *principal submatrix* of $A=(a_{ij})\in\overline{\mathbb{R}}^{n\times n}$ is as usual any matrix of the form

$$\begin{pmatrix} a_{i_1i_1} & a_{i_1i_2} & \dots & a_{i_1i_k} \\ a_{i_2i_1} & a_{i_2i_2} & \dots & a_{i_2i_k} \\ \vdots & \vdots & & \vdots \\ a_{i_ki_1} & a_{i_ki_2} & \dots & a_{i_ki_k} \end{pmatrix},$$

where $1\leqslant i_1<\cdots<i_k\leqslant n$. This matrix will be denoted by $A(i_1,i_2,\dots,i_k)$. The max-algebraic *permanent* of A is defined as an analogue of the classical one:

$$maper(A)=\bigoplus_{\pi\in P_n}\bigotimes_{i\in N} a_{i,\pi(i)},$$

where P_n stands for the set of all permutations of the set $N=\{1,\dots,n\}$. In the conventional notation

$$maper(A)=\max_{\pi\in P_n}\sum_{i\in N} a_{i,\pi(i)},$$

which is the optimal value of the assignment problem for the matrix A. There are a number of efficient solution methods [1] for finding $maper(A)$, one of the best known is the Hungarian method of computational complexity $O(n^3)$. The set of all optimal permutations will be denoted by $ap(A)$, that is,

$$ap(A)=\left\{\pi\in P_n; maper(A)=\sum_{i\in N} a_{i,\pi(i)}\right\}.$$

A matrix A will be called *diagonally dominant* if $id\in ap(A)$. (Note that throughout the paper id stands for the identity permutation.)

The *characteristic maxpolynomial* of A has been defined in [6] as

$$\chi_A(x)=maper(A\oplus x\otimes I)=maper\begin{pmatrix} a_{11}\oplus x & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22}\oplus x & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn}\oplus x \end{pmatrix}.$$

It follows immediately from this definition that $\chi_A(x)$ is of the form

$$\delta_0\oplus(\delta_1\otimes x)\oplus\cdots\oplus(\delta_{n-1}\otimes x^{(n-1)})\oplus x^{(n)}$$

or briefly $\bigoplus_{i=0}^{n}\delta_i\otimes x^{(i)}$ where $\delta_n=0$ and, by convention, $x^{(0)}=0$.

Example 1. If

$$A=\begin{pmatrix} 1 & 3 & 2 \\ 0 & 4 & 1 \\ 2 & 5 & 0 \end{pmatrix},$$

then

$$\chi_A(x) = maper \begin{pmatrix} 1 \oplus x & 3 & 2 \\ 0 & 4 \oplus x & 1 \\ 2 & 5 & 0 \oplus x \end{pmatrix}$$

$$= (1 \oplus x) \otimes (4 \oplus x) \otimes (0 \oplus x) \oplus (3 \otimes 1 \otimes 2) \oplus (2 \otimes 0 \otimes 5)$$

$$\oplus (2 \otimes (4 \oplus x) \otimes 2) \oplus$$

$$\oplus ((1 \oplus x) \otimes 1 \otimes 5) \oplus (3 \otimes 0 \otimes (0 \oplus x))$$

$$= x^{(3)} \oplus 4 \otimes x^{(2)} \oplus 6 \otimes x \oplus 8.$$

In other words,

$$\chi_A(x) = \max(3x, 4 + 2x, 6 + x, 8).$$

It has been proved in [6] that for $k = 0, 1, \ldots, n-1$

$$\delta_k = \bigoplus_{B \in P_k(A)} maper(B),$$

where $P_k(A)$ is the set of all principal submatrices of A of order $n-k$. Hence, we can readily compute $\delta_0 = maper(A)$ and $\delta_{n-1} = \max(a_{11}, a_{22}, \ldots, a_{nn})$, but other coefficients cannot be found efficiently from this relation as the number of matrices in $P_k(A)$ is $\binom{n}{k}$. Note that $\delta_k = -\infty$ if $maper(B) = -\infty$ for all $B \in P_k(A)$ in which case the term $\delta_k \otimes x^{(k)}$ may be omitted. Also, $\chi_A(x)$ may reduce to just $x^{(n)}$, for instance if $a_{ij} = -\infty$ for all i, j $(i \geqslant j)$. We will discuss this and related questions towards the end of the paper.

The coefficients δ_k are closely related to the following combinatorial problem:

Best Principal Submatrix Problem (*BPSM*(k)): *Given an $n \times n$ matrix A and $k \leqslant n$, find a $k \times k$ principal submatrix of A whose optimal assignment problem value is maximal.*

Obviously, δ_{n-k} is the optimal assignment problem value for the principal submatrix which solves BPSM(k).

Note that to the authors' knowledge, no polynomial method for solving BPSM(k) exists although its modification arising after removing the word "principal" is well known in the literature (see e.g. [8]) and is polynomially solvable. This can also be seen from the following simple observation: Let $\tilde{A}$ be the $(2n-k) \times (2n-k)$ matrix arising from an $n \times n$ matrix A by adding $n-k$ rows and $n-k$ columns $(k < n)$ so that the entries in the intersection of these columns and rows are $-\infty$ and the remaining new entries are zero. If the assignment problem is solved for $\tilde{A}$ then every permutation selects $2n-k$ entries from $\tilde{A}$. If A is finite then any optimal (maximising) permutation avoids selecting entries from the intersection of the new columns and rows. But as it selects some other $n-k$ elements from the new rows and $n-k$ different elements from the new columns, it will select exactly $2n-k-2(n-k) = k$ elements from A. No two of these k elements are from the same row or from the same column and so they represent a selection of k independent entries from a $k \times k$ submatrix of A. Their sum is maximum as the only elements taken from outside A are zero. So the best $k \times k$ submatrix problem can readily be solved as the classical assignment problem for a special matrix of order $2n - k$.

The characteristic maxpolynomial of A written using usual algebra is

$$\chi_A(x) = \max(\delta_0, \delta_1 + x, \delta_2 + 2x, \ldots, \delta_{n-1} + (n-1)x, nx).$$

Hence, $\chi_A(x)$ is the upper envelope of $n+1$ affine-linear functions and thus a piecewise linear and convex function. If for some $k \in \{0, \ldots, n\}$ the inequality

$$\delta_k \otimes x^{(k)} \leqslant \bigoplus_{i \neq k} \delta_i \otimes x^{(i)}$$

holds for every real x then the term $\delta_k \otimes x^{(k)}$ is called *inessential*, otherwise it is called *essential*. Hence,

$$\chi_A(x) = \bigoplus_{i \neq k} \delta_i \otimes x^{(i)}$$

holds for all $x \in \mathbb{R}$ if $\delta_k \otimes x^{(k)}$ is inessential, and therefore all inessential terms may be ignored if $\chi_A(x)$ is considered as a function. As already mentioned, the aim of this paper is to present an $O(n^2(m + n \log n))$ method for finding all essential terms of a characteristic maxpolynomial for a matrix with m finite entries. It then follows that when all terms are essential then this method solves BPSM(k) for all $k = 1, \ldots, n$.

There is an operations research interpretation of the coefficients of the characteristic maxpolynomial which we may call the *job rotation problem*: Suppose that a company with n employees requires these workers to swap their jobs (possibly on a regular basis) in order to avoid exposure to monotonous tasks (for instance manual workers at an assembly line or ride operators in a theme park). It may also be required that to maintain stability of service only a certain number of employees, say k ($k < n$), actually swap their jobs. With each pair old job–new job, a coefficient may be associated expressing the cost (for instance for an additional training) or the preference of the worker to this particular change. So the aim may be to select k employees and to suggest a plan of the job changes between them so that the sum of the coefficients corresponding to these changes is minimum or maximum. This task leads to finding a $k \times k$ principal submatrix of A for which the assignment problem value is minimum or maximum (the diagonal entries can be set to $+\infty$ or $-\infty$ to avoid an assignment to the same job).

There are alternative ways of defining the characteristic polynomial in max-algebra, see [13] as well as [11]. The concept of a characteristic equation introduced in [13] enabled the authors to prove an analogue of the Cayley–Hamilton theorem. Let A be an $n \times n$ matrix. Let us denote by P_n^+, P_n^- the set of even and odd permutations in P_n, respectively, and $p^+(A, v) = |\{\pi \in P_n^+; w(A, \pi) = v\}|$, $p^-(A, v) = |\{\pi \in P_n^-; w(A, \pi) = v\}|$ where $w(A, \pi)$ stands for $\sum_{i \in N} a_{i,\pi(i)}$. Then A satisfies the following equation called the *max-algebraic characteristic equation* of A:

$$\lambda^{(n)} \oplus \bigoplus_{k \in J} c_{n-k} \otimes \lambda^{(n-k)} = c_{n-1} \otimes \lambda^{(n)} \oplus \bigoplus_{k \in \overline{J}} c_{n-k} \otimes \lambda^{(n-k)},$$

where

$$c_{n-k} = \max\left\{v; \sum_{B\in P_k(A)} p^+(B,v) \neq \sum_{B\in P_k(A)} p^-(B,v)\right\} \quad (k=1,\ldots,n),$$

$$J = \{j; d_{n-j} > 0\}, \overline{J} = \{j; d_{n-j} < 0\}$$

and

$$d_{n-k} = (-1)^k \left(\sum_{B\in P_k(A)} p^+(B,c_{n-k}) - \sum_{B\in P_k(A)} p^-(B,c_{n-k})\right) \quad (k=1,\ldots,n).$$

It follows immediately that $\delta_k = c_k$ if the numbers of odd and even permutations for which the value of the coefficient δ_k is attained are different.

In [11], a related concept of the *characteristic bi-polynomial* of A was defined. Let us denote

$$maper^+(A) = \bigoplus_{\pi\in P_n^+} \bigotimes_{i\in N} a_{i,\pi(i)},$$

$$maper^-(A) = \bigoplus_{\pi\in P_n^-} \bigotimes_{i\in N} a_{i,\pi(i)}.$$

The characteristic bi-polynomial of A is defined as $(P^+(\lambda), P^-(\lambda))$, where

$$P^+(\lambda) = maper^+(A_\lambda),$$

$$P^-(\lambda) = maper^-(A_\lambda)$$

and A_λ stands for the $2n\times 2n$ matrix

$$\begin{pmatrix} A & \lambda\otimes I \\ I & I \end{pmatrix}.$$

It is not difficult to prove that $P^+(\lambda)\oplus P^-(\lambda) = \chi_A(x)$. However, no polynomial algorithm for finding $P^+(\lambda)$ and $P^-(\lambda)$ was given in [11]. An important feature of the bi-polynomial proved in [11] is that $P^+(\lambda) = P^-(\lambda)$ is satisfied by the eigenvalue of A.

3. Finding the essential terms of a characteristic maxpolynomial

Let a real matrix $A = (a_{ij})$ with finite entries be given. Finding the essential terms of the characteristic maxpolynomial of a matrix A is equivalent to determining the function $z(x)$, $x\in\mathbb{R}$, where

$$z(x) := \max_{\varphi} \sum_{i=1}^{n} a(x)_{i\varphi(i)} \tag{1}$$

and

$$a(x)_{ij} := \begin{cases} \max(x, a_{ii}) & \text{for } i = j, \\ a_{ij} & \text{for } i \neq j. \end{cases}$$

Thus, $z(x)$ as a function coincides with $\chi_A(x)$. Since $\chi_A(x)$ is piecewise linear and convex and all its linear pieces are of the form $z_k(x) := kx + c_k$ for $k = 0, 1, \ldots, n$ and certain constants c_k, the function $z(x)$ has at most n breakpoints. The main idea of the method for finding all linear pieces of $z(x)$ is based on the fact that it is easy to evaluate $z(x)$ for any real x as this is simply $maper(A \oplus x \otimes I)$. By a suitable choice of $O(n)$ values of x we will be able to identify all linear pieces of $z(x)$.

Let $\varphi \in ap(A(x)) = ap((a(x)_{ij}))$ for a particular real value of x (recall that $ap(A)$ denotes the set of optimal permutations to the assignment problem for a square matrix A). We call a diagonal entry $a(x)_{ii}$ of the matrix $A(x)$ *active*, if $x \geqslant a_{ii}$ and if this diagonal position is selected by φ, that is $\varphi(i) = i$. If there are exactly k active values for a certain x and permutation φ then this means that $z(x) = kx + c_k = x^{(k)} \otimes c_k$, that is, the value of $z(x)$ is determined by the linear piece with the slope k. Here, c_k is the sum of $n-k$ non-active entries of $A(x)$ selected by φ. No two of these non-active entries can be from the same row or column and they are all in the submatrix, say B, arising by removing the rows and columns of all active elements. Since all active elements are on the diagonal, B is principal and the $n-k$ non-active elements form a feasible solution to the assignment problem for B. This solution is also optimal by optimality of φ. This yields the following:

Proposition 3.1. *Let $x \in \mathbb{R}$ and $\varphi \in P_n$. If $z(x) = maper(A(x)) = \sum_{i=1}^{n} a(x)_{i\varphi(i)}$, $i_1, i_2, \ldots, i_k$ are indices of all active entries and $\{j_1, \ldots, j_{n-k}\} = N - \{i_1, i_2, \ldots, i_k\}$ then $A(j_1, \ldots, j_{n-k})$ is a solution to BPSM$(n-k)$ for A.*

There may, of course, be several optimal permutations for the same value of x selecting different numbers of active elements which means that the value of $z(x)$ may be equal to the function value of several linear pieces with different slopes at x. We will pay special attention to this question in Proposition 3.8.

Proposition 3.2. *If $z(\bar{x}) = z_r(\bar{x}) = z_s(\bar{x})$ for some real number $\bar{x}$ and some integers $r < s$ then there are no essential terms with the slope $k \in (r, s)$ and $\bar{x}$ is a breakpoint of $z(x)$.*

Proof. Since $z_r(\bar{x}) = \delta_r + r\bar{x} = z(\bar{x}) \geqslant \delta_k + k\bar{x}$ for every k, we have $z_r(x) = \delta_r + rx \geqslant \delta_k + kx = z_k(x)$ for every $x < \bar{x}$ and $k > r$, thus $z(x) \geqslant z_r(x) \geqslant z_k(x)$ for every $x < \bar{x}$ and for every $k > r$.

Similarly, $z(x) \geqslant z_s(x) \geqslant z_k(x)$ for every $x > \bar{x}$ and for every $k < s$. Hence, $z(x) \geqslant z_k(x)$ for every x and for every integer slope k with $r + 1 \leqslant k \leqslant s - 1$. □

For $x \leqslant \tilde{a} = \min(a_{11}, a_{22}, \ldots, a_{nn})$, $z(x)$ is given by $\max_{\varphi} \sum_{i=1}^{n} a_{i\varphi(i)} = maper(A)$. Let us denote this value by c_0. We define the function $z_0(x) := c_0$ and then obviously, $z(x) = z_0(x)$ for $x \leqslant \tilde{a}$.

Now, let $\alpha^* := \max_{i,j} a_{ij}$ and let E be the matrix whose entries are all equal to 1. For $x \geqslant \alpha^*$ the matrix $A(x) - \alpha^* \cdot E$ has only non-negative elements on its main diagonal. All off-diagonal elements are negative. Therefore we get $z(x) = nx$ for $x \geqslant \alpha^*$. We define $z_n(x) := nx$. Note that for finding $z(x)$ there is no need to determine α^*.

The intersection point of $z_0(x)$ with $z_n(x)$ is $x_1 = c_0/n$. We find $z(x_1)$ by solving the assignment problem $\max_\varphi \sum_{i=1}^{n} a(x_1)_{i\varphi(i)}$.

Corollary 3.3. *If* $z(x_1) = z_0(x_1)$ *then* $z(x) = \max(z_0(x), z_n(x))$.

Thus, if $z(x_1) = z_0(x_1)$, we are done and the function $z(x)$ has the form

$$z(x) = \begin{cases} z_0(x) & \text{for } x \leqslant x_1, \\ z_n(x) & \text{for } x \geqslant x_1. \end{cases} \tag{2}$$

Otherwise we have found a new linear piece of $z(x)$. Let us call it $z_k(x) := kx + c_k$, where k is the number of *active* elements in the corresponding optimal solution and c_k is given by $c_k := z(x_1) - kx_1$.

Next, we intersect $z_k(x)$ with $z_0(x)$ and with $z_n(x)$. Let x_2 and x_3, respectively, be the corresponding intersection points. We generate a list $L := (x_2, x_3)$. Now we choose an element from the list, say x_2, and determine $z(x_2)$. If $z(x_2) = z_0(x_2)$, then x_2 is a breakpoint of $z(x)$. By Proposition 3.2, this means that there are no essential terms of the characteristic maxpolynomial with slopes between 0 and k. We delete x_2 from our list and process a next point from L. For every point of the list, we either find a new slope which leads to two new points in the list or we detect that the currently investigated point is a breakpoint of L. In this case, this point will be deleted and no new points are generated. If the list L is empty, we are done and we have already found the correct function $z(x)$. Since every point of the list leads either to a new slope (and therefore to two new points in L) or it is a breakpoint of the graph of $z(x)$ in which case this point is deleted from L, the list has no more than $O(n)$ entries. This means the procedure stops after investigating at most $O(n)$ linear assignment problems. Thus we have shown:

Theorem 3.4. *All essential terms of the characteristic max-polynomial of a real matrix A can be found in* $O(n^4)$ *steps.*

Proposition 3.5. *If* $A = (a_{ij})$ *is diagonally dominant then so are all principal submatrices of A and all coefficients of the characteristic maxpolynomial can be found by the formula*

$$\delta_{n-k} = a_{i_1 i_1} + a_{i_2 i_2} + \cdots + a_{i_k i_k}, \quad (k = 0, 1, \ldots, n-1),$$

where $a_{i_1 i_1} \geqslant a_{i_2 i_2} \geqslant \cdots \geqslant a_{i_n i_n}$.

Proof. Let A be a diagonally dominant matrix, $B = A(i_1, i_2, \ldots, i_k)$ for some indices $i_1, i_2, \ldots, i_k$ and suppose that $id \notin ap(B)$. Take any $\pi \in ap(B)$ and extend π to a permutation σ of the set N by setting $\sigma(i) = i$ for every $i \notin \{i_1, i_2, \ldots, i_k\}$. Then obviously σ is a permutation of a bigger weight than that of $id \in P_n$, a contradiction. The formula follows in a straightforward way. □

The proof of the following statement is straightforward.

Proposition 3.6. *Let* $A=(a_{ij}),B=(b_{ij})\in\mathbb{R}^{n\times n}$, $r,s\in N$, $a_{rs}\leqslant b_{rs}, a_{ij}=b_{ij}$ *for all* i,j $(i\neq r, j\neq s)$. *If* $\pi\in ap(A)$ *satisfies* $\pi(r)=s$ *then* $\pi\in ap(B)$.

Corollary 3.7. *If* $id\in ap(A(\bar{x}))$ *then* $id\in ap(A(x))$ *for all* $x\geqslant\bar{x}$.

Remarks.

1. A diagonal element of $A(y)$ may not be active for some y with $y>x$ even if it is active in $A(x)$. For instance, consider the following (4×4)-matrix A:

$$\begin{pmatrix} 0 & 0 & 0 & 29 \\ 0 & 8 & 20 & 0 \\ 0 & 0 & 12 & 28 \\ 29 & 28 & 0 & 16 \end{pmatrix}.$$

   For $x=4$, the unique optimal permutation is $\varphi=(1)\ (2,3,4)$ of value 80, in which the first diagonal element is active. For $y=20$, the unique optimal permutation is $\varphi=(1,4)(2)(3)$ of value 98, in which the second and third, but not the first diagonal element of the matrix are active.
2. If an intersection point x is found by intersecting two linear functions with the slopes k and $k+1$, respectively, this point is immediately deleted from the list L since it cannot lead to a new essential term (as there is no slope strictly between k and $k+1$).
3. If at an intersection point y, the slope of $z(x)$ changes from k to l with $l-k\geqslant 2$, then an upper bound for an inessential term c_r related to the polynomial $rx+c_r$, $k<r<l$, can be obtained by $z(y)-ry$. Due to the convexity of the function $z(x)$, this is the best upper bound on c_r which can be obtained by using the values of $z(x)$.

Taking into account our previous discussion, we arrive at the following algorithm. The values x which have to be investigated are stored as triples $(x,k(l),k(r))$ in a list L. Such a triple tells us that x has been found as the intersection point of two linear functions with the slopes $k(l)$ and $k(r)$, $k(l)<k(r)$.

The Algorithm.

Input: $A=(a_{ij})\in\mathbb{R}^{n\times n}$.

Output: All essential terms of the characteristic maxpolynomial of A, in the form $kx+c_k$.

1. Solve the assignment problem with the cost matrix A.
   If the identical permutation $id\in ap(A)$, stop (A is diagonally dominant and all terms of the characteristic maxpolynomial are essential with coefficients determined by the formula of Proposition 3.5).
   Otherwise, let $c_0=maper(A)$. Define $z_0(x):=c_0$.
2. Determine x_1 as the intersection point of $z_0(x)$ and $z_n(x):=nx$.

3. Let $L := \{(x_1, 0, n)\}$.
4. If $L = \emptyset$, stop. The function $z(x)$ has been found. Otherwise choose an arbitrary element $(x_i, k_i(l), k_i(r))$ from L and remove it from L.
5. If $k_i(r) = k_i(l) + 1$, then (see Remark 3 above) go to Step 4. (x_i is a breakpoint of $z(x)$; for x close to x_i the function $z(x)$ has slope $k_i(l)$ for $x < x_i$, and $k_i(r)$ for $x > x_i$.)
6. Find $z(x_i) = maper(A(x_i))$. Take an arbitrary optimal permutation to the assignment problem for the matrix $A(x_i)$ and let k_i be the number of active elements in this solution. Set $c_{k_i} := z(x_i) - k_i x_i$.
7. If $id \in ap(A(x_i))$ then (see Proposition 3.6) remove all triples $(y, ., .)$ from L with $y > x_i$. If $a_{s_1 s_1} \leqslant a_{s_2 s_2} \leqslant \cdots \leqslant a_{s_n s_n}$ then the function $z(x)$ is (after setting $k = k_i$) given by

$$z(x) = \begin{cases} kx + c_k & \text{for } a_{s_k s_k} \leqslant x \leqslant a_{s_{k+1} s_{k+1}}, \\ \ldots & \text{for } \ldots, \\ nx & \text{for } x \geqslant a_{s_n s_n}. \end{cases}$$

   Go to Step 4.
   Otherwise define $z_i(x) := k_i x + c_{k_i}$.
8. Intersect $z_i(x)$ with the lines with the slopes $k_i(l)$ and $k_i(r)$. Let y_1 and y_2 be the intersection points, respectively. Add the triples $(y_1, k_i(l), k_i)$ and $(y_2, k_i, k_i(r))$ to the list L and go to Step 4.

Example 2. Let the matrix

$$A := \begin{pmatrix} 0 & 4 & -2 & 3 \\ 2 & 1 & 3 & -1 \\ -2 & -3 & 1 & 0 \\ 7 & -2 & 8 & 4 \end{pmatrix}$$

be given. Solving the corresponding assignment problem yields

$$\begin{pmatrix} -4 & 0 & -6 & -1 \\ -1 & -2 & 0 & -4 \\ -3 & -4 & 0 & -1 \\ -1 & -10 & 0 & -4 \end{pmatrix},$$

$$\begin{pmatrix} -3 & 0^* & -6 & 0 \\ 0^* & -2 & 0 & -3 \\ -2 & -4 & 0 & 0^* \\ 0 & -10 & 0^* & -3 \end{pmatrix}.$$

Thus $z_0(x) = 14$.

Now, we solve $14 = 4x$ and we get $x_1 = 3.5$. By solving the assignment problem for $x_1 = 3.5$ we get

$$\begin{pmatrix} 3.5 & 4 & -2 & 3 \\ 2 & 3.5 & 3 & -1 \\ -2 & -3 & 3.5 & 0 \\ 7 & -2 & 8 & 4 \end{pmatrix},$$

$$\begin{pmatrix} -0.5 & 0 & -6 & -1 \\ -1.5 & 0 & -0.5 & -4.5 \\ -5.5 & -6.5 & 0 & -3.5 \\ -1 & -10 & 0 & -4 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & -6 & 0 \\ -1 & 0 & -0.5 & -3.5 \\ -5 & -6.5 & 0 & -2.5 \\ -0.5 & -10 & 0 & -3 \end{pmatrix},$$

$$\begin{pmatrix} 0 & -0.5 & -6.5 & 0^* \\ -0.5 & 0^* & -0.5 & -3 \\ -4.5 & -6.5 & 0^* & -2 \\ 0^* & -10 & 0 & -2.5 \end{pmatrix}.$$

Thus $z_2(3.5) = 17$ and we get $z_2(x) := 2x + 10$. Intersecting this function with $z_0(x)$ and $z_4(x)$ yields the two new points $x_2 := 2$ (solving $14 = 2x + 10$) and $x_3 := 5$ (solving $2x + 10 = 4x$). Investigating $x = 2$ shows that the slope changes at this point from 0 to 2. Thus we have here a breakpoint of $z(x)$. Finding the value $z(5)$ amounts in solving the assignment problem with the cost matrix

$$\begin{pmatrix} 5 & 4 & -2 & 3 \\ 2 & 5 & 3 & -1 \\ -2 & -3 & 5 & 0 \\ 7 & -2 & 8 & 5 \end{pmatrix}.$$

This assignment problem yields the solution $z(5) = 20 = z_4(5)$. Thus, no new essential term has been found and we have $z(x)$ completely determined as

$$z(x) = \begin{cases} 14 & \text{for } 0 \leqslant x \leqslant 2, \\ 2x + 10 & \text{for } 2 \leqslant x \leqslant 5, \\ 4x & \text{for } x \geqslant 5. \end{cases}$$

We say that $A=(a_{ij})\in\overline{\mathbb{R}}^{n\times n}$ is in *normal form* if

(i) $a_{ij}\leqslant 0$ for all $i,\ j=1,\ldots,n$ and
(ii) there exists $\pi\in P_n$ such that $a_{i,\pi(i)}=0$ for all $i\in N$.

The Hungarian method for solving the assignment problem transforms every matrix, say, $A=(a_{ij})\in\overline{\mathbb{R}}^{n\times n}$, with at least one finite permutation to a matrix B in normal form such that $ap(A)=ap(B)$. This enables a very efficient description of the set $ap(A)$ since obviously $\pi\in ap(B)$ if and only if $b_{i,\pi(i)}=0$ for all $i\in N$. Using the normal form, we may find an optimal solution to the assignment problem which satisfies additional requirements or, we may also optimise (with respect to some criterion) over the set of optimal solutions as it is described in the following statement. Note that in the assignment problem we may wish to minimise rather than maximise and that the minimisation problem for a matrix A can simply be converted into a maximisation by taking the matrix $-A$ instead of A.

Proposition 3.8. *Let $\bar{x}$ be a fixed real value and let $B=(b_{ij})$ be a normal form of the matrix $A(\bar{x})$. Let $C=(c_{ij})$ be a matrix arising from B by the formula*

$$c_{ij}=\begin{cases}0 & if\ b_{ij}=0\ and\ (i,j)\ is\ not\ active,\\ 1 & if\ (i,j)\ is\ active,\\ -\infty & otherwise.\end{cases}$$

Then every $\pi\in ap(C)$ [$\pi\in ap(-C)$] is an optimal solution to the assignment problem for $A(\bar{x})$ with maximal [minimal] number of active elements. (Note that $b_{ii}=0$ if (i,i) is an active entry of $A(\bar{x})$.)

Proof. Statement immediately follows from the previous discussion. □

If for some value of $\bar{x}$ there are two or more optimal solutions to the assignment problem for $A(\bar{x})$ with different numbers of active elements, then using Proposition 3.8 we can find an optimal solution with the smallest number and another one with the biggest number of active elements. This enables us in Step 6 of the Algorithm to find two new lines (rather than one):

(a) $z_k(x):=kx+c_k$, where k is the minimal number of *active* elements of an optimal solution to the assignment problem for $A(\bar{x})$ and c_k is given by $c_k:=z(\bar{x})-k\bar{x}$ and

(b) $z_{k'}(x):=k'x+c_{k'}$, where k' is the maximal number of *active* elements of an optimal solution to the assignment problem for $A(\bar{x})$ and $c_{k'}$ is given by $c_{k'}:=z(\bar{x})-k'\bar{x}$.

In Step 8 of the Algorithm, we then intersect $z_k(x)$ with the line having the slope $k_i(l)$ and $z_{k'}(x)$ with the line having slope $k_i(r)$.

4. The case of sparse matrices and concluding remarks

If some (but not all) entries of A are $-\infty$, the same algorithm as in the finite case can be used except that the lowest-order finite term has to be found since a number of the first coefficients $\delta_0, \delta_1, \ldots$ of the characteristic maxpolynomial may be $-\infty$. The following theorems, see [4], are useful here.

Theorem 4.1. *$\chi_A(x) = x^{(n)}$ if and only if the digraph $D = (N, E)$ with node set $N = \{1, \ldots, n\}$ and arc set $E = \{(i, j); a_{ij}$ is finite$\}$ is acyclic.*

Theorem 4.2. *If $\delta_k \otimes x^{(k)}$ is the lowest-order finite term of the characteristic maxpolynomial of a matrix A then k is the number of active elements in $A(\bar{x})$ where $\bar{x}$ is any real number satisfying*

$$\bar{x} < \underline{\delta} - \overline{\delta}$$

and $\delta_k = z(\bar{x}) - k\bar{x}$. Here $\underline{\delta} = \min(0, nA_{\min}), \overline{\delta} = \max(0, nA_{\max})$ and $A_{\max}[A_{\min}]$ is the biggest [smallest] finite entry in A.

These theorems enable to identify the trivial case and to modify the beginning of the Algorithm by finding the intersection of the lowest-order term with $x^{(n)}$.

Moreover, instead of considering the classical assignment problem we rather formulate the problem in Step 6 of the algorithm as the maximum weight perfect matching problem in a bipartite graph $(N, N; E)$. This graph has an edge $(i, j) \in E$ if and only if a_{ij} is finite. It is well known (see e.g. [1]) that the maximum weight perfect matching problem in a graph with m edges can be solved by a shortest augmenting path method using Fibonacci heaps in $O(n(m + n \log n))$ time. Since in the worst case $O(n)$ such maximum weight perfect matching problems must be solved, we get the following theorem.

Theorem 4.3. *Provided that the $n \times n$ matrix A has m finite entries, all essential terms of the characteristic maxpolynomial can be found in $O(n^2(m + n \log n))$ time.*

We note that BPSM(k) has interesting combinatorial aspects if considered for matrices whose all finite entries are zero. To see this, let us denote $T = \{-\infty, 0\}$. If $A = (a_{ij}) \in T^{n \times n}$, then $\delta_k = 0$ or $\delta_k = -\infty$ for every $k = 0, 1, \ldots, n-1$. Clearly, $\delta_k = 0$ if and only if there is a $k \times k$ principal submatrix of A with k independent zeros, that is with k zeros selected by a permutation or, equivalently, k zeros no two of which are either from the same row or from the same column.

It is easy to see that if $A = (a_{ij}) \in T^{n \times n}$, then $B = e^A = (e^{a_{ij}}) = (b_{ij})$ is a zero–one matrix. If $\pi \in P_n$, then

$$\prod_{i \in N} b_{i,\pi(i)} = \prod_{i \in N} e^{a_{i,\pi(i)}} = e^{\sum_{i \in N} a_{i,\pi(i)}}.$$

Since $per(B) = \sum_{\pi \in N} \prod_{i \in N} b_{i,\pi(i)}$, we have that $per(B) > 0$ is equivalent to $(\exists \pi \in P_n)(\forall i \in N) b_{i,\pi(i)} = 1$. But this is equivalent to $(\exists \pi \in P_n)(\forall i \in N) a_{i,\pi(i)} = 0$.

Thus, the task of finding the coefficient δ_k of the characteristic maxpolynomial of a square matrix over T is equivalent to the following problem expressed in terms of the classical permanents:

Given an $n \times n$ zero–one matrix A and a positive integer k ($k \leqslant n$), is there a $k \times k$ principal submatrix B of A with positive permanent?

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