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A compact null set containing a differentiability point of every Lipschitz function

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Abstract We prove that in a Euclidean space of dimension at least two, there exists a compact set of Lebesgue measure zero such that any real-valued Lipschitz function defined on the space is differentiable at some point in the set. Such a set is constructed explicitly.

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1 Introduction

1.1 Background

A theorem of Lebesgue says that any real-valued Lipschitz function on the real line is differentiable almost everywhere. This result is sharp in the sense that for any subset E of the real line with Lebesgue measure zero, there exists a real-valued Lipschitz function not differentiable at any point of E. The exact characterisation of the possible sets of non-differentiability of a Lipschitz function $f : \mathbb{R} \to \mathbb{R}$ is given in [11].

For Lipschitz mappings between Euclidean spaces of higher dimension, the interplay between Lebesgue null sets and sets of points of non-differentiability is less straightforward. By Rademacher's theorem, any real-valued Lipschitz mapping on $\mathbb{R}^n$ is differentiable except on a Lebesgue null set. However, Preiss [8] gave an example of

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a Lebesgue null set E in $\mathbb{R}^n$, for $n \geq 2$, such that E contains a point of differentiability of *every* real-valued Lipschitz function on $\mathbb{R}^n$.

In particular, [8] shows that the latter property holds whenever E is a G_δ-set in $\mathbb{R}^n$—i.e. an intersection of countably many open sets—such that E contains all lines passing through two points with rational coordinates. However, this set is dense in $\mathbb{R}^n$.

In the present paper we construct a much "smaller" set in $\mathbb{R}^n$ for $n \geq 2$—a *compact* Lebesgue null set—that still captures a point of differentiability of every Lipschitz function $f: \mathbb{R}^n \to \mathbb{R}$.

It is important to note that though, setting $n = 2$, any Lipschitz function $f: \mathbb{R}^2 \to \mathbb{R}$ has points of differentiability in such an extremely small set as ours, for any Lebesgue null set E in the plane there is a pair of real-valued Lipschitz functions on $\mathbb{R}^2$ with no common points of differentiability in E [1].

Only a few positive results are known about the case where the codomain is a space of dimension at least two. For $n \geq 3$, there exists a Lebesgue null set in $\mathbb{R}^n$, namely the union of all "rational hyperplanes", such that for all $\varepsilon > 0$ every Lipschitz mapping from $\mathbb{R}^n$ to $\mathbb{R}^{n-1}$ has a point of ε-Fréchet differentiability in that set; see [7].

1.2 Previous research

Let us say a few words about why the method of [8] does not yield a set with the properties we are aiming for. Indeed, [8, Theorem 6.4] says that every Lipschitz function defined on $\mathbb{R}^n$ is differentiable at some point of a G_δ-set E if E satisfies certain conditions, in particular for any two points $u, v \in \mathbb{R}^n$ and any $\eta > 0$, the set E contains a large portion of a path that approximates the line segment $[u, v]$ to within $\eta\|u - v\|$. The closure of such a set E is the whole space $\mathbb{R}^n$.

There is, however, a stronger version of [8, Theorem 6.4] that only requires a local version of this condition for the same conclusion to hold: namely for every $\varepsilon > 0$ and every $x \in E$ there is a neighbourhood of x in which any line segment I can be approximated to within $\varepsilon|I|$ by a curve in E. Let us explain why the closure of any G_δ-set with this property has non-empty interior and hence is of positive measure.

Indeed, by this "local approximation" property there is an open ball B intersecting E and a positive η, such that each open $U \subseteq B$ that intersects E contains a point $x' \in U \cap E$ with the following property: any line segment $I \subseteq B$ through x' of length at most η is pointwise $|I|/2$-close to a curve inside E. It follows that E is dense in B.

Thus in order to construct a closed set of measure zero containing points of differentiability of every Lipschitz function, we introduce crucial new steps, outlined in Subsect. 1.4. Before describing our approach we need some preliminaries.

1.3 Preliminaries

Given real Banach spaces X and Y, a mapping $f: X \to Y$ is called Lipschitz if there exists $L \geq 0$ such that $\|f(x) - f(y)\|_Y \leq L\|x - y\|_X$ for all $x, y \in X$. The smallest such constant L is denoted $\mathrm{Lip}(f)$.

If $f: X \to Y$ is a mapping, then f is said to be Gâteaux differentiable at $x_0 \in X$ if there exists a bounded linear operator $D: X \to Y$ such that for every $u \in X$, the limit

$$\lim_{t \to 0} \frac{f(x_0 + tu) - f(x_0)}{t} \tag{1.1}$$

exists and is equal to $D(u)$. The operator D is called the Gâteaux derivative of f at the point x_0 and is written $f'(x_0)$. If this limit exists for some fixed u we say that f has a directional derivative at x_0 in the direction u and denote the limit by $f'(x_0, u)$.

If f is Gâteaux differentiable at x_0 and the convergence in (1.1) is uniform for u in the unit sphere $S(X)$ of X, we say that f is Fréchet differentiable at x_0 and call $f'(x_0)$ the Fréchet derivative of f.

Equivalently, f is Fréchet differentiable at x_0 if we can find a bounded linear operator $f'(x_0): X \to Y$ such that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for any $h \in X$ with $\|h\| \le \delta$ we have

$$\|f(x_0 + h) - f(x_0) - f'(x_0)(h)\| \le \varepsilon \|h\|.$$

If, on the other hand, we only know this condition for some fixed $\varepsilon > 0$ we say that f is ε-Fréchet differentiable at x_0. Note that f is Fréchet differentiable at x_0 if and only if it is ε-Fréchet differentiable at x_0 for every $\varepsilon > 0$. In [5,6] the notion of ε-Fréchet differentiability is studied in relation to Lipschitz mappings with the emphasis on the infinite dimensional case.

In general, Fréchet differentiability is a strictly stronger property than Gâteaux differentiability. However the two notions coincide for Lipschitz functions defined on a finite dimensional space; see [2].

We now make some comments about the porosity property and its connection with the Fréchet differentiability of Lipschitz functions. Recall first that a subset A of a Banach space X is said to be porous at a point $x \in X$ if there exists $\lambda > 0$ such that for all $\delta > 0$ there exist $r \le \delta$ and $x' \in B(x, \delta)$ such that $r > \lambda \|x - x'\|$ and $B(x', r) \cap A = \emptyset$. Here $B(x, \delta)$ denotes an open ball in the Banach space X with centre at x and radius δ.

A set $A \subseteq X$ is called porous if it is porous at every $x \in A$. A set is said to be σ-porous if it can be written as a countable union of porous sets. The family of σ-porous subsets of X is a σ-ideal. A comprehensive survey on porous and σ-porous sets can be found in [14].

Observe that for a non-empty set A the distance function $f(x) = \operatorname{dist}(x, A)$ is Lipschitz with $\operatorname{Lip}(f) \le 1$ but is not Fréchet differentiable at any porosity point of the set A [2]. Moreover if A is a σ-porous subset of a separable Banach space X we can find a Lipschitz function from X to $\mathbb{R}$ that is not Fréchet differentiable at any point of A. This is proved in [9] for the case in which A is a countable union of closed porous sets and, as per remark in [2, Chap. 6], the proof of [10, Proposition 14] can be used to derive this statement for an arbitrary σ-porous set A.

The set S we are constructing in this paper contains a point of differentiability of every Lipschitz function, so we require S to be non-σ-porous. Such a set should also have plenty of non-porosity points. By the Lebesgue density theorem every σ-porous subset of a finite-dimensional space is of Lebesgue measure zero. We remark that the σ-ideal of σ-porous sets is a proper subset of that of Lebesgue null sets. In order to arrive at an appropriate set that is not σ-porous, has no porosity points and whose closure has measure zero, we use ideas similar to those in [12,13,15].

1.4 Construction

We now outline the method we use to prove that the set S we construct contains a differentiability point of every Lipschitz function.

Given a Lipschitz function $f: \mathbb{R}^n \to \mathbb{R}$, we first find a point $x \in S$ and a direction $e \in S^{n-1}$, the unit sphere of $\mathbb{R}^n$, such that the directional derivative $f'(x, e)$ exists and is locally maximal in the sense that if $\varepsilon > 0$, x' is a nearby point of S, $e' \in S^{n-1}$ is a direction and (x', e') satisfies appropriate constraints, then $f'(x', e') < f'(x, e) + \varepsilon$.

We then prove f is differentiable at x with derivative

$$D(u) = f'(x, e)\langle u, e\rangle.$$

A heuristic outline goes as follows. Assume this is not true. Find $\eta > 0$ and a vector λ with small norm such that $|f(x+\lambda) - f(x) - f'(x, e)\langle\lambda, e\rangle| > \eta\|\lambda\|$. Then construct an auxiliary point $x + h$ lying near the line $x + \mathbb{R}e$ and calculate the ratio

$$\frac{|f(x + \lambda) - f(x + h)|}{\|\lambda - h\|}.$$

We find that this is at least $f'(x, e) + \varepsilon$ for some $\varepsilon > 0$. By using an appropriate mean value theorem [8, Lemma 3.4], it is possible to find a point x' on the line segment $[x+h, x+\lambda]$ and a direction $e' \in S^{n-1}$ such that $f'(x', e') \geq f'(x, e)+\varepsilon$ and (x', e') satisfies the required constraints. This contradicts the local maximality of $f'(x, e)$ and so f is differentiable at x.

Since $f'(x, e)$ is only required to be locally maximal for x in the set S, it is necessary to ensure the above line segment $[x + h, x + \lambda]$ lies in S, if we are to get a contradiction. It is therefore vital to construct S so that it contains lots of line segments.

Crucially, instead of just one set, we introduce a hierarchy of closed null sets M_i, indexed by sequences i of real numbers that are subject to a certain partial ordering. For any point x in M_i the required line segments $[x + h, x + \lambda]$ can be found in every set M_j where j is greater than i in the sense of the partial order. Subsequently we prove in Corollary 5.2 that each set M_i contains a point of differentiability of every Lipschitz function. The desired set S can then be taken equal to the intersection of any of the M_i with a closed ball.

1.5 Structure of the paper

Section 2 is devoted to the description of the partial ordered set and the layers M_i. The existence of line segments close to any point in a previous layer is verified in

Theorem 2.5. In Sect. 5 we will show that this condition is sufficient for any Lipschitz function to have a point of differentiability in each layer.

In Sect. 3 we show in detail how to arrive at a pair (x, e) with "almost maximal" directional derivative $f'(x, e)$. By a modification of the method in [8] we construct a sequence of points x_m and directions $e_m \in S^{n-1}$ such that f has a directional derivative $f'(x_m, e_m)$ that is almost maximal, subject to some constraints. We then argue that (x_m) and (e_m) both converge and that the directional derivative $f'(x, e)$ at $x = \lim_{m\to\infty} x_m$ in the direction $e = \lim_{m\to\infty} e_m$ is locally maximal in the required sense. We eventually show x is a point of differentiability of f.

The convergence of (x_m) is achieved simply by choosing x_{m+1} close to x_m. The convergence of e_m is more subtle; we obtain this by altering the function by an appropriate small linear piece at each stage of the iteration. Then picking (x_m, e_m) such that the mth function f_m has almost maximal directional derivative $f_m'(x_m, e_m)$ can be shown to guarantee that the sequence (e_m) is Cauchy.

In Sect. 4 we introduce a Differentiability Lemma 4.3, showing that under certain conditions such a pair (x, e), with $f'(x, e)$ almost maximal, gives a point x of Fréchet differentiability of f.

Finally in Sect. 5 we verify the conditions of this Differentiability Lemma 4.3 for the pair (x, e) constructed in Sect. 3, using the results of Sect. 2. This completes the proof.

1.6 Related questions

To conclude the introduction let us observe the following. Independently of our construction, one can deduce from [3,4] that there exists a non-empty Lebesgue null set E in the plane with a weaker property: E is F_σ—i.e. a countable union of closed sets—and contains a point of sub-differentiability of every real-valued Lipschitz function.

Indeed, in [3] it is proved that there exist a non-empty open set $G \subseteq \mathbb{R}^2$, a differentiable function $f : G \to \mathbb{R}$ and a non-empty open set $\Omega \subseteq \mathbb{R}^2$ for which there exists a point $p \in G$ such that the gradient $\nabla f(p) \in \Omega$ but $\nabla f(q) \notin \Omega$ for almost all $q \in G$, in the sense of two dimensional Lebesgue measure. In other words, the set $E = (\nabla f)^{-1}(\Omega) \cap G$ is a non-empty set of Lebesgue measure zero. Note that ∇f is a Baire-1 function; therefore the set E, which is a preimage of an open set, is an F_σ set. Now [4, Lemma 4] implies that any Lipschitz function $h : \mathbb{R}^2 \to \mathbb{R}$ has a point of sub-differentiability in E.

2 The set

Let $(N_r)_{r\geq 1}$ be a sequence of odd integers such that $N_r > 1$, $N_r \to \infty$ and $\sum \frac{1}{N_r^2} = \infty$. Let $\mathfrak{S}$ be the set of all sequences $i = (i^{(r)})_{r\geq 1}$ of real numbers with $1 \leq i^{(r)} < N_r$ for all r and $i^{(r)}/N_r \to 0$ as $r \to \infty$.

We define a relation $\preceq$ on $\mathfrak{S}$ by

$$i \prec j \quad \text{if } (\forall r)(i^{(r)} > j^{(r)}) \quad \text{and} \quad i^{(r)}/j^{(r)} \to \infty \text{ as } r \to \infty$$

and

$$i \preceq j \quad \text{if} \quad i \prec j \quad \text{or } i = j.$$

For $i, j \in \mathfrak{S}$ such that $i \prec j$, we denote by (i, j) the set $\{k \in \mathfrak{S} : i \prec k \prec j\}$ and by $[i, j]$ the set $\{k \in \mathfrak{S} : i \preceq k \preceq j\}$.

Recall that a partially ordered set—or poset—is a pair $(X, \leq)$ where X is a set and $\leq$ is a relation on X such that $x \leq x$ for all $x \in X$, if $x \leq y$ and $y \leq x$ for $x, y \in X$ then necessarily $x = y$ and finally if $x, y, z \in X$ with $x \leq y$ and $y \leq z$ then $x \leq z$.

A chain in a poset $(X, \leq)$ is a subset $C \subseteq X$ such that for any $x, y \in C$ we have $x \leq y$ or $y \leq x$. We say $(X, \leq)$ is chain complete if every non-empty chain $C \subseteq X$ has a least upper bound—or "supremum"—in X.

We write $x < y$ if $x \leq y$ and $x \neq y$. We call $(X, \leq)$ dense if whenever $x, y \in X$ with $x < y$ we can find $z \in X$ such that $x < z < y$. Finally, recall that an element x of X is minimal if there does not exist y with $y < x$.

The following lemma summarises basic properties of $(\mathfrak{S}, \preceq)$.

Lemma 2.1 $(\mathfrak{S}, \preceq)$ *is a non-empty partially ordered set that is chain complete, dense and has no minimal element.*

Proof It is readily verified that $(\mathfrak{S}, \preceq)$ is a poset and that $\mathfrak{S} \neq \emptyset$ since it contains the element $(1, 1, 1, \dots)$. Given a non-empty chain $C = \{i_\alpha \mid \alpha \in A\}$ in $\mathfrak{S}$, the supremum of C exists and is given by $i \in \mathfrak{S}$ where $i^{(r)} = \inf_{\alpha \in A} i_\alpha^{(r)}$; hence $(\mathfrak{S}, \preceq)$ is chain complete. To see that $(\mathfrak{S}, \preceq)$ is dense, note that if $i, j \in \mathfrak{S}$ with $i \prec j$ then $i \prec k \prec j$ where $k \in \mathfrak{S}$ is given by $k^{(r)} = \sqrt{i^{(r)} j^{(r)}}$. Finally given $l \in \mathfrak{S}$, we can find $m \in \mathfrak{S}$ with $m \prec l$ by taking $m^{(r)} = \sqrt{l^{(r)} N_r}$. Therefore $(\mathfrak{S}, \preceq)$ has no minimal element. This completes the proof of the lemma.

We begin by working in the plane $\mathbb{R}^2$.

Denote the inner product $\langle , \rangle$ and the Euclidean norm $\| \cdot \|$. Write $B(x, \delta)$ for an open ball in $(\mathbb{R}^2, \| \cdot \|)$ with centre $x \in \mathbb{R}^2$ and radius $\delta > 0$. Further let $B_\infty(c, d/2)$ be an open ball in $(\mathbb{R}^2, \| \cdot \|_\infty)$, i.e. an open square with centre $c \in \mathbb{R}^2$ and side $d > 0$. Finally, given $x, y \in \mathbb{R}^2$ we use $[x, y]$ to denote the closed line segment

$$\{(1 - \lambda)x + \lambda y \mid 0 \leq \lambda \leq 1\} \subseteq \mathbb{R}^2.$$

Let $d_0 = 1$. For each $r \geq 1$ set $d_r = \frac{1}{N_1 N_2 \dots N_r}$ and define the lattice $C_r \subseteq \mathbb{R}^2$:

$$C_r = d_{r-1} \left(\left(\frac{1}{2}, \frac{1}{2} \right) + \mathbb{Z}^2 \right). \tag{2.1}$$

Suppose now $i \in \mathfrak{S}$. Define the set $W_i \subseteq \mathbb{R}^2$ by

$$W_i = \mathbb{R}^2 \setminus \bigcup_{r=1}^{\infty} \bigcup_{c \in C_r} B_\infty \left(c, \frac{1}{2} i^{(r)} d_r \right). \tag{2.2}$$

Note that each W_i is a closed subset of the plane and $W_i \subseteq W_j$ if $i \preceq j$. From $i^{(r)} < N_r$ we see that $W_i \neq \emptyset$—for example $(0, 0) \in W_i$. We now claim that the Lebesgue measure of W_i is equal to 0.

For each $r \geq 0$ we define sets D_r and R_r of disjoint open squares of side d_r as follows. Recall $d_0 = 1$. Let D_0 be the empty-set and $R_0 = \{U\}$ be a singleton comprising the open unit square:

$$U = \{(x, y) \in \mathbb{R}^2 \mid 0 < x, y < 1\}.$$

Divide each square in the set R_{r-1} into an $N_r \times N_r$ grid. Let D_r comprise the central open squares of the grids and let R_r comprise all the remaining open squares. By induction each square in D_r and R_r has side d_r and the centres of the squares in D_r belong to the lattice C_r. For each $m \geq 1$ we have from (2.2) and $i^{(r)} \geq 1$,

$$W_i \subseteq \mathbb{R}^2 \setminus \bigcup_{r=1}^{m} \bigcup_{c \in C_r} B_\infty\left(c, \frac{1}{2}d_r\right)$$

so that

$$W_i \cap U \subseteq \overline{U} \setminus \bigcup_{r=1}^{m} \bigcup D_r = \overline{\bigcup R_m},$$

and, as the cardinality of the set R_m is equal to $(N_1^2 - 1) \ldots (N_m^2 - 1)$ and each square in R_m has area d_m^2, we can estimate the Lebesgue measure of $W_i \cap U$:

$$|W_i \cap U| \leq \left(1 - \frac{1}{N_1^2}\right) \ldots \left(1 - \frac{1}{N_m^2}\right).$$

This tends to 0 as $m \to \infty$, because $\sum \frac{1}{N_r^2} = \infty$. Therefore the Lebesgue measure $|W_i \cap U| = 0$. Furthermore, from (2.1) and (2.2), W_i is invariant under translations by the lattice $\mathbb{Z}^2$. Hence $|W_i| = 0$ for every $i \in \mathfrak{S}$.

Let

$$W = \bigcup_{\substack{i \in \mathfrak{S} \\ i \prec (1,1,1,\ldots)}} W_i.$$

As $(1, 1, 1, \ldots)$ is not minimal and $W_i \neq \emptyset$ for any $i \in \mathfrak{S}$, we observe W is not empty. The following theorem now proves that for any point $x \in W$ there are line segments inside W with directions that cover a dense subset of the unit circle. We say $e = (e_1, e_2) \in S^1$ has rational slope if there exists $(p, q) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$ with $pe_1 = qe_2$.

Theorem 2.2 *For any $i, j \in \mathfrak{S}$ with $i \prec j$, $\varepsilon > 0$ and $e \in S^1$ with rational slope there exists $\delta_0 = \delta_0(i, j, \varepsilon, e) > 0$ such that whenever $x \in W_i$ and $\delta \in (0, \delta_0)$, there is a line segment $[x', x' + \delta e] \subseteq W_j$ where $\|x' - x\| \leq \varepsilon\delta$.*

Proof First we note that without loss of generality we may assume that $\varepsilon \leq 1$ and $|e_2| \leq |e_1|$ where $e = (e_1, e_2)$. Write $e_2/e_1 = p/q$ with $p, q \in \mathbb{Z}$ and $q > 0$. Now observe that if $y \in \mathbb{R}^2$ then the line $y + \mathbb{R}e$ has gradient $p/q \in [-1, 1]$ and if it intersects the square $B_\infty(c, d/2)$,

$$\left|(y_2 - c_2) - \frac{p}{q}(y_1 - c_1)\right| < d \tag{2.3}$$

where $y = (y_1, y_2)$ and $c = (c_1, c_2)$.

From $i \prec j$, we have $\sup_m \frac{j^{(m)}}{i^{(m)}} < 1$ so that we can find $\psi > 0$ such that $\frac{j^{(m)}}{i^{(m)}} \leq 1-\psi$ for all m. Put $\rho_m = i^{(m)} d_m \psi/4$. Since $d_m = N_{m+1} d_{m+1}$ and $i^{(m)} \geq 1$ for each $m \geq 1$,

$$\rho_m/\rho_{m+1} = (i^{(m)} N_{m+1})/i^{(m+1)} \geq \inf_m \frac{N_{m+1}}{i^{(m+1)}} > 1$$

so that $\rho_m \searrow 0$. Let k_0 be such that

$$\begin{cases} j^{(m)}/i^{(m)} \leq \varepsilon\psi/16 \\ j^{(m)}/N_m \leq (5q)^{-1} \end{cases} \quad \text{for all } m \geq k_0. \tag{2.4}$$

We set $\delta_0 = \rho_{k_0}$ and let $\delta \in (0, \delta_0)$. Since $\rho_k \to 0$, there exists $k \geq k_0$ such that $\rho_k \geq \delta > \rho_{k+1}$.

Let C_m be given by (2.1) and set

$$T_m = \bigcup_{c \in C_m} B_\infty(c, j^{(m)} d_m/2)$$

so that $W_j = \bigcap_{m \geq 1} (\mathbb{R}^2 \backslash T_m)$.

Fix any point $x \in W_i$. Define the line $\ell_\lambda = x + (0, \lambda) + \mathbb{R}e \subseteq \mathbb{R}^2$ to be the vertical shift of $x + \mathbb{R}e$ by λ. We claim that if $m \geq k + 1$ and $I \subseteq \mathbb{R}$ is a closed interval of length at least $4 j^{(m)} d_m$ we can find a closed subinterval $I' \subseteq I$ of length $j^{(m)} d_m$ such that the line ℓ_λ does not intersect T_m for any $\lambda \in I'$.

Take $I = [a, b]$. We may assume there exists $\lambda \in [a, a + j^{(m)} d_m]$ such that ℓ_λ intersects $B_\infty(c, j^{(m)} d_m/2)$ for some $c \in C_m$; if not we can take $I' = [a, a + j^{(m)} d_m]$. Write $c = (c_1, c_2)$ and $x = (x_1, x_2)$. Note that from (2.3) we have

$$\left|(x_2 + \lambda - c_2) - \frac{p}{q}(x_1 - c_1)\right| < j^{(m)} d_m.$$

Let $I' = [\lambda + 2 j^{(m)} d_m, \lambda + 3 j^{(m)} d_m] \subseteq I$. Suppose that $\lambda' \in I'$ and that $c' \in C_m$. We may write $c' = (c_1', c_2') = (c_1, c_2) + (l_1, l_2) d_{m-1}$ where $l_1, l_2 \in \mathbb{Z}$. Then if $p l_1 \neq q l_2$,

$$\left|(x_2+\lambda'-c_2')-\frac{p}{q}(x_1-c_1')\right|$$
$$\geq d_{m-1}\left|\frac{pl_1-ql_2}{q}\right|-\left|(x_2+\lambda-c_2)-\frac{p}{q}(x_1-c_1)\right|-|\lambda'-\lambda|>j^{(m)}d_m$$

as $|pl_1-ql_2|\geq 1$ and $d_{m-1}=N_md_m\geq 5qj^{(m)}d_m$ from (2.4). On the other hand if $pl_1=ql_2$ the same inequality holds as

$$\left|(x_2+\lambda'-c_2')-\frac{p}{q}\left(x_1-c_1'\right)\right|$$
$$\geq|\lambda'-\lambda|-\left|(x_2+\lambda-c_2)-\frac{p}{q}\left(x_1-c_1\right)\right|>j^{(m)}d_m.$$

Therefore by (2.3) the line $\ell_{\lambda'}$ does not intersect $B_\infty(c',j^{(m)}d_m/2)$ for any $c'\in C_m$ and any $\lambda'\in I'$. Hence the claim.

Note that for $m\geq k+1$ we have $j^{(m)}d_m\geq 4j^{(m+1)}d_{m+1}$ from (2.4). Subsequently, by the previous claim, we may construct a nested sequence of closed intervals

$$[0,4j^{(k+1)}d_{k+1}]\supseteq I_{k+1}\supseteq I_{k+2}\supseteq\cdots$$

such that $|I_m|=j^{(m)}d_m$ and ℓ_λ does not intersect T_m for $\lambda\in I_m$.

Picking $\lambda\in\bigcap_{m\geq k+1}I_m$ we have

$$0\leq\lambda\leq 4j^{(k+1)}d_{k+1}\leq\frac{i^{(k+1)}\psi\varepsilon}{4}d_{k+1}=\varepsilon\rho_{k+1}<\varepsilon\delta$$

using (2.4) again.

Set $x'=x+(0,\lambda)$ so that $\|x'-x\|=\lambda<\varepsilon\delta$. Note that $[x',x'+\delta e]$ does not intersect T_m for $m\geq k+1$ as $[x',x'+\delta e]\subseteq\ell_\lambda$ and $\lambda\in I_m$. Now suppose $m\leq k$. From $\varepsilon\leq 1$ we have $\lambda\leq\delta\leq\rho_k$. If $c\in C_m$ then we observe that $[x',x'+\delta e]$ does not intersect $B_\infty(c,j^{(m)}d_m/2)$ as $x\in W_i$ is outside $B_\infty(c,i^{(m)}d_m/2)$ and

$$\lambda+\delta\leq 2\rho_k\leq 2\rho_m=\frac{1}{2}i^{(m)}d_m\psi\leq\frac{1}{2}i^{(m)}d_m\left(1-\frac{j^{(m)}}{i^{(m)}}\right)$$
$$=\frac{1}{2}(i^{(m)}d_m-j^{(m)}d_m).$$

Therefore $[x',x'+\delta e]$ does not intersect T_m for any $m\geq 1$ so that $[x',x'+\delta e]\subseteq W_j$. This finishes the proof.

We now give a simple geometric lemma and then prove some corollaries to Theorem 2.2. Given $e=(e^1,e^2)\in S^1$ we define $e^\perp=(-e^2,e^1)$ so that $\langle e^\perp,e\rangle=0$ for any $e\in S^1$ and, given $x_0\in\mathbb{R}^2$ and $e_0\in S^1$, then $x\in\mathbb{R}^2$ lies on the line $x_0+\mathbb{R}e_0$ if and only if $\langle e_0^\perp,x\rangle=\langle e_0^\perp,x_0\rangle$.

Lemma 2.3 *Suppose that $x_1, x_2 \in \mathbb{R}^2, e_1, e_2 \in S^1, \alpha_1, \alpha_2 > 0$, the line segments l_1, l_2 given by $l_m = [x_m, x_m + \alpha_m e_m]$ intersect at $x_3 \in \mathbb{R}^2$ and that*

$$[x_3 - \alpha e_m, x_3 + \alpha e_m] \subseteq l_m, \qquad (m = 1, 2) \tag{2.5}$$

where $\alpha > 0$. If $x_1', x_2' \in \mathbb{R}^2$ and $e_1', e_2' \in S^1$ are such that

$$\|x_m' - x_m\| \leq \frac{\alpha}{16}|\langle e_2^{\perp}, e_1\rangle| \quad \text{and} \tag{2.6}$$

$$\|e_m' - e_m\| \leq \frac{\alpha}{8(\alpha_1 + \alpha_2)}|\langle e_2^{\perp}, e_1\rangle| \tag{2.7}$$

for $m = 1, 2$, then the line segments l_1', l_2' given by $l_m' = [x_m', x_m' + \alpha_m e_m']$ intersect at a point $x_3' \in \mathbb{R}^2$ with $\|x_3' - x_3\| \leq \alpha$.

Proof As $\langle e_2^{\perp}, e_1\rangle = -\langle e_1^{\perp}, e_2\rangle$ we may assume, without loss of generality, that the inner product $\langle e_2^{\perp}, e_1\rangle$ is non-negative. From (2.5) we can write $x_3 = x_m + \lambda_m e_m$ for $m = 1, 2$ with $\alpha \leq \lambda_m \leq \alpha_m - \alpha$. Now note that as $x_1 + \lambda_1 e_1 \in l_2$ we have

$$\langle e_2^{\perp}, x_1 + \lambda_1 e_1\rangle = \langle e_2^{\perp}, x_2\rangle$$

so that

$$\left\langle e_2^{\perp}, x_1 + \left(\lambda_1 + \pi\frac{1}{2}\alpha\right)e_1\right\rangle - \langle e_2^{\perp}, x_2\rangle = \pi\frac{\alpha}{2}\langle e_2^{\perp}, e_1\rangle \tag{2.8}$$

for $\pi = \pm 1$. Using (2.6) and (2.7) we quickly obtain from (2.8)

$$\left\langle e_2'^{\perp}, x_1' + \left(\lambda_1 + \frac{1}{2}\alpha\right)e_1'\right\rangle - \langle e_2'^{\perp}, x_2'\rangle \geq 0 \tag{2.9}$$

$$\text{and} \quad \left\langle e_2'^{\perp}, x_1' + \left(\lambda_1 - \frac{1}{2}\alpha\right)e_1'\right\rangle - \langle e_2'^{\perp}, x_2'\rangle \leq 0. \tag{2.10}$$

Indeed, for $\pi = \pm 1$,

$$\begin{aligned}
&\left(\langle e_2'^{\perp}, x_1' + (\lambda_1 + \pi\frac{1}{2}\alpha)e_1'\rangle - \langle e_2'^{\perp}, x_2'\rangle\right) - \left(\langle e_2^{\perp}, x_1 + (\lambda_1 + \pi\frac{1}{2}\alpha)e_1\rangle - \langle e_2^{\perp}, x_2\rangle\right)\\
&\quad = \left\langle e_2'^{\perp}, (x_1' - x_1) - (x_2' - x_2) + \left(\lambda_1 + \pi\frac{1}{2}\alpha\right)(e_1' - e_1)\right\rangle\\
&\qquad + \left\langle (e_2'^{\perp} - e_2^{\perp}), (x_1 - x_2) + \left(\lambda_1 + \pi\frac{1}{2}\alpha\right)e_1\right\rangle;
\end{aligned}$$

the norm of the first term is bounded by

$$\begin{aligned}&\|x_1' - x_1\| + \|x_2' - x_2\| + \left|\lambda_1 + \pi\frac{1}{2}\alpha\right| \cdot \|e_1' - e_1\| \\ &\le 2\frac{\alpha}{16}\langle e_2^\perp, e_1\rangle + \alpha_1\frac{\alpha}{8(\alpha_1+\alpha_2)}\langle e_2^\perp, e_1\rangle \le \frac{\alpha}{4}\langle e_2^\perp, e_1\rangle,\end{aligned}$$

and the norm of the second term is bounded by

$$\begin{aligned}&\|e_2' - e_2\|\left(\|x_1 - x_2\| + |\lambda_1 + \pi\frac{1}{2}\alpha|\right) \\ &\le \frac{\alpha}{8(\alpha_1+\alpha_2)}\langle e_2^\perp, e_1\rangle((\alpha_1+\alpha_2)+\alpha_1) \\ &\le \frac{\alpha}{4}\langle e_2^\perp, e_1\rangle.\end{aligned}$$

Hence by (2.9) and (2.10) there exists

$$x_3' \in \left[x_1' + \left(\lambda_1 - \frac{1}{2}\alpha\right)e_1', x_1' + (\lambda_1 + \frac{1}{2}\alpha)e_1'\right] \subseteq l_1' \tag{2.11}$$

with $\langle e_2'^\perp, x_3'\rangle = \langle e_2'^\perp, x_2'\rangle$ so that we can write

$$x_3' = x_2' + \lambda_2' e_2' \tag{2.12}$$

for some $\lambda_2' \in \mathbb{R}$. Since $x_3 = x_1 + \lambda_1 e_1$ and (2.11) imply

$$\|x_3' - x_3\| \le \|x_1' - x_1\| + \lambda_1\|e_1' - e_1\| + \frac{1}{2}\alpha\|e_1'\| \le \frac{3}{4}\alpha$$

and $x_3 = x_2 + \lambda_2 e_2$ and (2.12) imply

$$\|x_3' - x_3\| \ge |\lambda_2' - \lambda_2| - \|x_2' - x_2\| - \lambda_2\|e_2' - e_2\| \ge |\lambda_2' - \lambda_2| - \frac{1}{4}\alpha,$$

we get

$$|\lambda_2' - \lambda_2| \le \frac{3}{4}\alpha + \frac{1}{4}\alpha = \alpha.$$

It follows that

$$x_3' \in [x_2' + (\lambda_2 - \alpha)e_2', x_2' + (\lambda_2 + \alpha)e_2'] \subseteq l_2'$$

since $\alpha \le \lambda_2 \le \alpha_2 - \alpha$. Therefore $x_3' \in l_1' \cap l_2'$ with $\|x_3' - x_3\| \le \frac{3}{4}\alpha < \alpha$ as required.

Corollary 2.4 *Suppose $i, j \in \mathfrak{S}$ with $i \prec j$ and $\varepsilon > 0$.*

1. *There exists $\delta_1 = \delta_1(i, j, \varepsilon) > 0$ such that whenever $\delta \in (0, \delta_1), x \in W_i$ and $e \in S^1$, there exists a line segment $[x', x' + \delta e'] \subseteq W_j$ where $x' \in \mathbb{R}^2, e' \in S^1$ with $\|x' - x\| \leq \varepsilon\delta$ and $\|e' - e\| \leq \varepsilon$.*
2. *There exists $\delta_2 = \delta_2(i, j, \varepsilon) > 0$ such that whenever $\delta \in (0, \delta_2), x \in W_i, u \in B(x, \delta)$ and $e \in S^1$ there exists a line segment $[u', u' + \delta e'] \subseteq W_j$ where $u' \in \mathbb{R}^2, e' \in S^1$ with $\|u' - u\| \leq \varepsilon\delta$ and $\|e' - e\| \leq \varepsilon$.*
3. *For $v_1, v_2, v_3 \in \mathbb{R}^2$ there exists $\delta_3 = \delta_3(i, j, \varepsilon, v_1, v_2, v_3) > 0$ such that whenever $\delta \in (0, \delta_3)$ and $x \in W_i$ there exist $v_1', v_2', v_3' \in \mathbb{R}^2$ such that $\|v_m' - v_m\| \leq \varepsilon$ and*
$$[x + \delta v_1', x + \delta v_3'] \cup [x + \delta v_3', x + \delta v_2'] \subseteq W_j.$$
4. *There exists $\delta_4 = \delta_4(i, j, \varepsilon) > 0$ such that whenever $\delta \in (0, \delta_4)$, v_1, v_2, v_3 are in the closed unit ball D^2 of $\mathbb{R}^2$ and $x \in W_i$ there exist $v_1', v_2', v_3' \in \mathbb{R}^2$ such that $\|v_m' - v_m\| \leq \varepsilon$ and*
$$[x + \delta v_1', x + \delta v_3'] \cup [x + \delta v_3', x + \delta v_2'] \subseteq W_j.$$

Proof 1. We can find a finite collection of unit vectors in the plane
$$e_1, e_2, \ldots, e_r \in S^1$$
with rational slopes such that $S^1 \subseteq \bigcup_{1 \leq s \leq r} B(e_s, \varepsilon)$. Let
$$\delta_1 = \min_{1 \leq s \leq r} \delta_0(i, j, \varepsilon, e_s),$$
where δ_0 is given by Theorem 2.2. Then for any $\delta \in (0, \delta_1)$, $x \in W_i$ and $e \in S^1$ find e_s with $\|e_s - e\| \leq \varepsilon$. As $\delta < \delta_0(i, j, \varepsilon, e_s)$ there exists a line segment $[x', x' + \delta e_s] \subseteq W_j$ with $\|x' - x\| \leq \varepsilon\delta$ as required.

2. Pick any $k \in \mathfrak{S}$ with $i \prec k \prec j$. Let
$$\delta_2 = \min(\delta_1(i, k, \varepsilon/3), \delta_1(k, j, \varepsilon/3)).$$
Suppose that $\delta \in (0, \delta_2)$ and $u \in B(x, \delta)$. We can write $u = x + \delta' f$ with $0 \leq \delta' < \delta$ and $f \in S^1$. Then there exists $x' \in \mathbb{R}^2$, $f' \in S^1$ such that $[x', x' + \delta f'] \subseteq W_k$ with $\|x' - x\| \leq \varepsilon\delta/3$ and $\|f' - f\| \leq \varepsilon/3$. As $x' + \delta' f' \in W_k$ we can find $u' \in \mathbb{R}^2, e' \in S^1$ such that $[u', u' + \delta e'] \subseteq W_j$ with $\|u' - (x' + \delta' f')\| \leq \varepsilon\delta/3$ and $\|e' - e\| \leq \varepsilon/3$. Then
$$\|u' - u\| \leq \|u' - (x' + \delta' f')\| + \|x' - x\| + \delta'\|f' - f\| \leq \varepsilon\delta$$
as required.

3. Without loss of generality, we may assume that v_1, v_2, v_3 are not collinear and that $\|v_1\|, \|v_2\|, \|v_3\| \leq \frac{1}{4}$. Write

$$v_3 = v_1 + t_1 e_1 = v_2 + t_2 e_2 \tag{2.13}$$

where $0 < t_1, t_2 \leq \frac{1}{2}$ and $e_1, e_2 \in S^1$. As v_1, v_2, v_3 are not collinear, the vectors e_1 and e_2 are not parallel so that $\langle e_2^{\perp}, e_1 \rangle \neq 0$. We may assume $\varepsilon \leq t_1, t_2$. Set

$$\delta_3 = \delta_2(i, j, \eta),$$

where $\eta = \frac{1}{16}|\langle e_2^{\perp}, e_1 \rangle|\varepsilon$. Let $\delta \in (0, \delta_3)$. Write

$$x_m = x + \delta v_m \qquad (m = 1, 2) \tag{2.14}$$

and put $l_m = [x_m, x_m + 2\delta t_m e_m]$. As $\|x_m - x\| < \delta_3$, by part (2) of this Corollary we can find $x_1', x_2' \in \mathbb{R}^2$ and $e_1', e_2' \in S^1$ with $\|x_m' - x_m\| \leq \eta\delta$, $\|e_m' - e_m\| \leq \eta$ and $[x_m', x_m' + \delta e_m'] \subseteq W_j$ for $m = 1, 2$. Then as $t_1, t_2 \leq \frac{1}{2}$ we have $l_m' \subseteq W_j$ where $l_m' = [x_m', x_m' + 2\delta t_m e_m']$ for $m = 1, 2$.
Note that (2.13) and (2.14) imply that $x + \delta v_3 = x_m + \delta t_m e_m$ for $m = 1, 2$. Therefore $x_3 = x + \delta v_3$ is a point of intersection of l_1 and l_2. The conditions of Lemma 2.3 are readily verified with $\alpha_m = 2\delta t_m$ and $\alpha = \varepsilon\delta$ so that l_1', l_2' intersect at a point x_3' with $\|x_3' - x_3\| \leq \varepsilon\delta$. Writing now $x_m' = x + \delta v_m'$ for $m = 1, 2, 3$ we have $\|v_m' - v_m\| \leq \varepsilon$, since $\|x_m' - x_m\| \leq \varepsilon\delta$, and

$$[x + \delta v_1', x + \delta v_3'] \cup [x + \delta v_3', x + \delta v_2'] \subseteq W_j.$$

4. Take $w_1, w_2, \ldots, w_r$ in D^2 with $D^2 \subseteq \bigcup_{1 \leq s \leq r} B(w_s, \varepsilon/2)$. Set

$$\delta_4 = \min_{1 \leq s_1, s_2, s_3 \leq r} \delta_3(i, j, \varepsilon/2, w_{s_1}, w_{s_2}, w_{s_3}).$$

This finishes the proof of the corollary.
Let $n \geq 2$. For $i \in \mathfrak{S}$ define $M_i \subseteq \mathbb{R}^n$ by

$$M_i = W_i \times \mathbb{R}^{n-2}. \tag{2.15}$$

Let $\|\cdot\|$ denote the Euclidean norm on $\mathbb{R}^n$. We use $[x, y] \subseteq \mathbb{R}^n$ to denote a closed line segment, where $x, y \in \mathbb{R}^n$.

Theorem 2.5 *The family of subsets $\{M_i \subseteq \mathbb{R}^n \mid i \in \mathfrak{S}\}$ satisfies the following three statements.*

(i) *If $i \in \mathfrak{S}$ then M_i is non-empty, closed and has measure zero.*
(ii) *If $i, j \in \mathfrak{S}$ and $i \preceq j$ then $M_i \subseteq M_j$.*

(iii) *If* $i, j \in \mathfrak{S}$ *with* $i \prec j$ *and* $\varepsilon > 0$, *then there exists* $\alpha = \alpha(i, j, \varepsilon) > 0$ *such that whenever* $\delta \in (0, \alpha)$, u_1, u_2, u_3 *are in the closed unit ball* D^n *of* $\mathbb{R}^n$ *and* $x \in M_i$, *there exist* $u'_1, u'_2, u'_3 \in \mathbb{R}^n$ *with* $\|u'_m - u_m\| \le \varepsilon$ *and*

$$[x + \delta u'_1, x + \delta u'_3] \cup [x + \delta u'_3, x + \delta u'_2] \subseteq M_j.$$

Proof Recall that for each $i \in \mathfrak{S}$, W_i is a non-empty closed set of measure zero and that $W_i \subseteq W_j$ whenever $i \preceq j$. Hence (2.15) implies (i) and (ii). For (iii), let $\alpha = \delta_4(i, j, \varepsilon)$ from Corollary 2.4, part (4) and $\delta \in (0, \alpha)$. Suppose $x \in M_i$ and $u_m \in D^n$, $m = 1, 2, 3$. Write $x = (x', y')$ and $u_m = (v_m, h_m)$ with $x' \in W_i$, $v_m \in D^2$ and $y', h_m \in \mathbb{R}^{n-2}$.

By Corollary 2.4, part (4), we can find $v'_1, v'_2, v'_3 \in \mathbb{R}^2$ with $\|v'_m - v_m\| \le \varepsilon$ and

$$[x' + \delta v'_1, x' + \delta v'_3] \cup [x' + \delta v'_3, x' + \delta v'_2] \subseteq W_j.$$

Then setting $u'_m = (v'_m, h_m)$ we have $\|u'_m - u_m\| = \|v'_m - v_m\| \le \varepsilon$ and

$$[x + \delta u'_1, x + \delta u'_3] \cup [x + \delta u'_3, x + \delta u'_2] \subseteq M_j.$$

3 A point with almost locally maximal directional derivative

In this section we work on a general real Hilbert space H, although eventually we shall only be concerned with the case in which H is finite dimensional. Let denote the $\langle\, ,\rangle$ inner product on H, $\|\cdot\|$ the norm and let $S(H)$ denote the unit sphere of H. We shall assume that the family $\{M_i \subseteq H \mid i \in \mathfrak{S}\}$ consists of closed sets such that $M_i \subseteq M_j$ whenever $i \preceq j$, where the index set $(\mathfrak{S}, \preceq)$ is a dense, chain complete poset.

For a Lipschitz function $h\colon H \to \mathbb{R}$ we write D^h for the set of all pairs $(x, e) \in H \times S(H)$ such that the directional derivative $h'(x, e)$ exists and, for each $i \in \mathfrak{S}$, we let D^h_i be the set of all $(x, e) \in D^h$ such that $x \in M_i$. If, in addition, $h\colon H \to \mathbb{R}$ is linear then we write $\|h\|$ for the operator norm of h.

Theorem 3.1 *Suppose* $f_0\colon H \to \mathbb{R}$ *is a Lipschitz function,* $i_0 \in \mathfrak{S}$, $(x_0, e_0) \in D^{f_0}_{i_0}$, $\delta_0, \mu, K > 0$ *and* $j_0 \in \mathfrak{S}$ *with* $i_0 \prec j_0$. *Then there exists a Lipschitz function* $f\colon H \to \mathbb{R}$ *such that* $f - f_0$ *is linear with norm not greater than* μ *and a pair* $(x, e) \in D^f_i$, *where* $\|x - x_0\| \le \delta_0$ *and* $i \in (i_0, j_0)$, *such that the directional derivative* $f'(x, e) > 0$ *is almost locally maximal in the following sense. For any* $\varepsilon > 0$ *there exists* $\delta_\varepsilon > 0$ *and* $j_\varepsilon \in (i, j_0)$ *such that whenever* $(x', e') \in D^f_{j_\varepsilon}$ *satisfies*

(i) $\|x' - x\| \le \delta_\varepsilon$, $f'(x', e') \ge f'(x, e)$ *and*
(ii) *for any* $t \in \mathbb{R}$

$$|(f(x' + te) - f(x')) - (f(x + te) - f(x))| \le K\sqrt{f'(x', e') - f'(x, e)}\,|t|, \tag{3.1}$$

then we have $f'(x', e') < f'(x, e) + \varepsilon$.

We devote the rest of this section to proving Theorem 3.1.

Without loss of generality we may assume $\mathrm{Lip}(f_0) \le 1/2$ and $K \ge 4$. By replacing e_0 with $-e_0$ if necessary we may assume $f_0'(x_0, e_0) \ge 0$.

If h is a Lipschitz function, the pairs (x, e), (x', e') belong to D^h and $\sigma \ge 0$ we write

$$(x, e) \underset{(h,\sigma)}{\le} (x', e') \tag{3.2}$$

if $h'(x, e) \le h'(x', e')$ and for all $t \in \mathbb{R}$,

$$|(h(x' + te) - h(x')) - (h(x + te) - h(x))| \le K \left(\sigma + \sqrt{h'(x', e') - h'(x, e)}\right) |t|.$$

We shall construct by recursion a sequence of Lipschitz functions $f_n : H \to \mathbb{R}$, sets $D_n \subseteq D^{f_0}$ and pairs $(x_n, e_n) \in D_n$ such that the directional derivative $f_n'(x_n, e_n)$ is within λ_n of its supremum over D_n, where $\lambda_n > 0$. We shall show that $f = \lim f_n$ and $(x, e) = \lim(x_n, e_n)$ have the desired properties. The constants δ_m will be used to bound $\|x_n - x_m\|$ for $n \ge m$ whereas σ_m will bound $\|e_n - e_m\|$ and t_m will control $\|f_n - f_m\|$ for $n \ge m$.

The recursion starts with $f_0, i_0, j_0, x_0, e_0, \delta_0$ defined in the statement of Theorem 3.1. Let $\sigma_0 = 2$ and $t_0 = \min(1/4, \mu/2)$. For $n \ge 1$ we shall pick

$$f_n, \sigma_n, t_n, \lambda_n, D_n, x_n, e_n, \varepsilon_n, i_n, j_n, \delta_n$$

in that order where

- $i_n, j_n \in \mathfrak{S}$ with $i_{n-1} \prec i_n \prec j_n \prec j_{n-1}$,
- D_n are non-empty subsets of $D^{f_0} \subseteq H \times S(H)$,
- $\sigma_n, t_n, \lambda_n, \varepsilon_n, \delta_n > 0$,
- $f_n : H \to \mathbb{R}$ are Lipschitz functions,
- $(x_n, e_n) \in D_n$.

Algorithm 3.2 Given $n \ge 1$ choose

(1) $f_n(x) = f_{n-1}(x) + t_{n-1}\langle x, e_{n-1}\rangle$,
(2) $\sigma_n \in (0, \sigma_{n-1}/4)$,
(3) $t_n \in (0, \min(t_{n-1}/2, \sigma_{n-1}/4n))$,
(4) $\lambda_n \in (0, t_n\sigma_n^2/2)$,
(5) D_n to be the set of all pairs (x, e) such that $(x, e) \in D_i^{f_n} = D_i^{f_0}$ for some $i \in (i_{n-1}, j_{n-1})$, $\|x - x_{n-1}\| < \delta_{n-1}$ and

$$(x_{n-1}, e_{n-1}) \underset{(f_n,\sigma_{n-1}-\varepsilon)}{\le} (x, e)$$

for some $\varepsilon \in (0, \sigma_{n-1})$,
(6) $(x_n, e_n) \in D_n$ such that $f_n'(x, e) \le f_n'(x_n, e_n) + \lambda_n$ for every $(x, e) \in D_n$,
(7) $\varepsilon_n \in (0, \sigma_{n-1})$ such that $(x_{n-1}, e_{n-1}) \underset{(f_n,\sigma_{n-1}-\varepsilon_n)}{\le} (x_n, e_n)$,

(8) $i_n \in (i_{n-1}, j_{n-1})$ such that $x_n \in M_{i_n}$,
(9) $j_n \in (i_n, j_{n-1})$ and
(10) $\delta_n \in (0, (\delta_{n-1} - \|x_n - x_{n-1}\|)/2)$ such that for all t with $|t| < \delta_n/\varepsilon_n$

$$\begin{aligned} &|(f_n(x_n + te_n) - f_n(x_n)) - (f_n(x_{n-1} + te_{n-1}) - f_n(x_{n-1}))| \\ &\leq (f_n'(x_n, e_n) - f_n'(x_{n-1}, e_{n-1}) + \sigma_{n-1})|t|. \end{aligned} \tag{3.3}$$

Note that (5) implies that $(x_{n-1}, e_{n-1}) \in D_n$, and so $D_n \neq \emptyset$; further as f_n is Lipschitz we see $\sup_{(x,e)\in D_n} f_n'(x, e) < \infty$. Therefore we are able to pick $(x_n, e_n) \in D_n$ with the property of (6).

The definition (5) of D_n then implies that ε_n and i_n exist with the properties of (7)–(8). Further, we have $\|x_n - x_{n-1}\| < \delta_{n-1}$ and

$$f_n'(x_n, e_n) \geq f_n'(x_{n-1}, e_{n-1}). \tag{3.4}$$

These allow us to choose δ_n as in (10).

Observe that the positive sequences $\sigma_n, t_n, \lambda_n, \delta_n, \varepsilon_n$ all tend to 0: $\sigma_n \in (0, \sigma_{n-1}/4)$ by (2), $t_n \in (0, t_{n-1}/2)$ by (3), $\lambda_n \in (0, t_n\sigma_n^2/2)$ by (4), $\delta_n \in (0, \delta_{n-1}/2)$ by (10) and $\varepsilon_n \in (0, \sigma_{n-1})$ by (7). Further from (10),

$$\overline{B(x_n, \delta_n)} \subseteq B(x_{n-1}, \delta_{n-1}). \tag{3.5}$$

Note that (1) and (3) imply $f_n(x) = f_0(x) + \langle x, \sum_{k=0}^{n-1} t_k e_k\rangle$ and, as the Lipschitz constant $\mathrm{Lip}(f_0) \leq \frac{1}{2}$, $t_{k+1} \leq t_k/2$ and $t_0 \leq \frac{1}{4}$, we deduce that $\mathrm{Lip}(f_n) \leq 1$ for all n.

Let $\varepsilon_n' > 0$ be given by

$$\varepsilon_n' = \min(\varepsilon_n/2, \sigma_{n-1}/4). \tag{3.6}$$

Lemma 3.3 *The following three statements hold.*

(i) *If $n \geq 1$ and $(x, e) \in D_{n+1}$, then*

$$(x_{n-1}, e_{n-1}) \underset{(f_n, \sigma_{n-1} - \varepsilon_n')}{\leq} (x, e).$$

(ii) *If $n \geq 1$ then $D_{n+1} \subseteq D_n$.*
(iii) *If $n \geq 0$ and $(x, e) \in D_{n+1}$, then $\|e - e_n\| \leq \sigma_n$.*

Proof For $n = 0$, condition (iii) is satisfied as $\sigma_0 = 2$. Now it is enough to check that if $n \geq 1$ and the condition (iii) is satisfied for $n - 1$, then conditions (i)–(iii) are satisfied for n. The Lemma then will follow by induction.

Assume $n \geq 1$ and $\|e' - e_{n-1}\| \leq \sigma_{n-1}$ for all $(x', e') \in D_n$. Then we have

$$\|e_n - e_{n-1}\| \leq \sigma_{n-1} \tag{3.7}$$

as $(x_n, e_n) \in D_n$. Now fix any $(x, e) \in D_{n+1}$. Using (1) and (5) of Algorithm 3.2 and $\langle e, e_n\rangle \le 1$ we get

$$\begin{aligned} A &:= f_n'(x, e) - f_n'(x_n, e_n) \\ &= f_{n+1}'(x, e) - t_n\langle e, e_n\rangle - f_{n+1}'(x_n, e_n) + t_n \\ &\ge f_{n+1}'(x, e) - f_{n+1}'(x_n, e_n) \ge 0, \end{aligned} \tag{3.8}$$

so that

$$f_n'(x, e) \ge f_n'(x_n, e_n) \ge f_n'(x_{n-1}, e_{n-1})$$

by (3.4). If we let $B = f_n'(x, e) - f_n'(x_{n-1}, e_{n-1})$ we have

$$K(\sqrt{B} - \sqrt{A}) \ge B - A = f_n'(x_n, e_n) - f_n'(x_{n-1}, e_{n-1}),$$

since $K \ge 4$ and $0 \le A \le B \le 2$, using $\mathrm{Lip}(f_n) \le 1$ in the final inequality. Together with (3.8) this implies that

$$(f_n'(x_n, e_n) - f_n'(x_{n-1}, e_{n-1})) + K\sqrt{f_{n+1}'(x, e) - f_{n+1}'(x_n, e_n)} \le K\sqrt{B}. \tag{3.9}$$

In order to prove (i), we need to establish an upper estimate for

$$|(f_n(x + te_{n-1}) - f_n(x)) - (f_n(x_{n-1} + te_{n-1}) - f_n(x_{n-1}))|. \tag{3.10}$$

For every $|t| < \delta_n/\varepsilon_n$, using

$$\begin{aligned} &|(f_n(x + te_n) - f_n(x)) - (f_n(x_n + te_n) - f_n(x_n))| \\ &\quad = |(f_{n+1}(x + te_n) - f_{n+1}(x)) - (f_{n+1}(x_n + te_n) - f_{n+1}(x_n))| \\ &\quad \le K\left(\sigma_n + \sqrt{f_{n+1}'(x, e) - f_{n+1}'(x_n, e_n)}\right)|t| \end{aligned}$$

and (3.3), we get from (3.9)

$$\begin{aligned} &|(f_n(x + te_{n-1}) - f_n(x)) - (f_n(x_{n-1} + te_{n-1}) - f_n(x_{n-1}))| \\ &\quad \le \sigma_{n-1}|t| + K\left(\sigma_n + \sqrt{f_n'(x, e) - f_n'(x_{n-1}, e_{n-1})}\right)|t| + \|e_n - e_{n-1}\| \cdot |t|. \end{aligned}$$

Using (3.7) and $K \ge 4$ we see that the latter does not exceed

$$\begin{aligned} &K\left(\sigma_{n-1}/2 + \sigma_n + \sqrt{f_n'(x, e) - f_n'(x_{n-1}, e_{n-1})}\right)|t| \\ &\quad \le K\left(\sigma_{n-1} - \varepsilon_n' + \sqrt{f_n'(x, e) - f_n'(x_{n-1}, e_{n-1})}\right)|t| \end{aligned}$$

as $\sigma_n \le \sigma_{n-1}/4$ by (2) of Algorithm 3.2 and $\varepsilon_n' \le \sigma_{n-1}/4$ by (3.6).

Now we consider the case $|t| \geq \delta_n/\varepsilon_n$. We have from (7) of Algorithm 3.2 that $(x_{n-1}, e_{n-1}) \underset{(f_n, \sigma_{n-1}-\varepsilon_n)}{\leq} (x_n, e_n)$. Using this together with

$$\max\{|f_n(x) - f_n(x_n)|, |f_n(x + te_{n-1}) - f_n(x_n + te_{n-1})|\}$$
$$\leq \|x - x_n\| \leq \delta_n \leq \varepsilon_n|t| \leq K\varepsilon_n|t|/4$$

we get

$$\begin{aligned}
&|(f_n(x + te_{n-1}) - f_n(x)) - (f_n(x_{n-1} + te_{n-1}) - f_n(x_{n-1}))| \\
&\leq K\left(\sigma_{n-1} - \varepsilon_n/2 + \sqrt{f_n'(x_n, e_n) - f_n'(x_{n-1}, e_{n-1})}\right)|t| \\
&\leq K\left(\sigma_{n-1} - \varepsilon_n' + \sqrt{f_n'(x, e) - f_n'(x_{n-1}, e_{n-1})}\right)|t|
\end{aligned}$$

because $f_n'(x_n, e_n) \leq f_n'(x, e)$ from (3.8). Thus (i) is proved.

Further, for $(x, e) \in D_{n+1}$ we have $x \in B(x_n, \delta_n) \subseteq B(x_{n-1}, \delta_{n-1})$, using (3.5), and $x \in M_i$ where

$$i \in (i_{n+1}, j_{n+1}) \subseteq (i_n, j_n).$$

Hence $(x, e) \in D_n$ follows from (i). This establishes (ii).

Finally to see (iii), let $(x, e) \in D_{n+1}$ and recall that (5) of Algorithm 3.2 implies $f_{n+1}'(x_n, e_n) \leq f_{n+1}'(x, e)$. By (1) of Algorithm 3.2, this can be written

$$f_n'(x_n, e_n) + t_n\langle e_n, e_n\rangle \leq f_n'(x, e) + t_n\langle e, e_n\rangle.$$

Since $(x, e) \in D_n$ by (ii), we have $f_n'(x, e) \leq f_n'(x_n, e_n) + \lambda_n$. Combining the two inequalities we get $t_n \leq t_n\langle e, e_n\rangle + \lambda_n$. Hence $\langle e, e_n\rangle \geq 1 - \lambda_n/t_n$ so that

$$\|e - e_n\|^2 = 2 - 2\langle e, e_n\rangle \leq 2\lambda_n/t_n \leq \sigma_n^2$$

using (4) of Algorithm 3.2.

This completes the proof of the lemma.

We now show that the sequences x_n, e_n and f_n converge and establish some properties of their limits.

Recall first that $i_{n-1} \prec i_n \prec j_n \prec j_{n-1}$ for all $n \geq 1$. The set $\{i_n \mid n \in \mathbb{N}\}$ is thus a non-empty chain in $\mathfrak{S}$. Therefore, it has a supremum $i \in \mathfrak{S}$. Further, as $i_n \in (i_{m+1}, j_{m+1})$ for $n \geq m+2$, we know $i \in [i_{m+1}, j_{m+1}] \subseteq (i_m, j_m)$ for all m.

Lemma 3.4 *We have $x_m \to x$, $e_m \to e$ and $f_m \to f$ where*

(i) *$f: H \to \mathbb{R}$ is a Lipschitz function with* $\mathrm{Lip}(f) \leq 1$,
(ii) *$f - f_m$ is linear and $\|f - f_m\| \leq 2t_m$ for all m,*
(iii) *$x \in M_i$, $\|x - x_m\| < \delta_m$ and $\|e - e_m\| \leq \sigma_m$,*
(iv) *$f'(x, e)$ exists, is positive and $f_m'(x_m, e_m) \nearrow f'(x, e)$,*

(v) $(x_{m-1}, e_{m-1}) \underset{(f_m,\sigma_{m-1}-\varepsilon'_m)}{\preceq} (x, e)$ *and*

(vi) $(x, e) \in D_m$ *for all* m.

Proof Letting $f(x) = f_0(x) + \langle x, \sum_{k\geq 0} t_k e_k\rangle$ we deduce $f_n \to f$ and (i), (ii) from $f_n(x) = f_0(x) + \langle x, \sum_{k=0}^{n-1} t_k e_k\rangle$, $\mathrm{Lip}(f_n) \leq 1$ and $t_{n+1} \leq t_n/2$.

For $n \geq m$, by parts (ii) and (iii) of Lemma 3.3 we have $(x_n, e_n) \in D_{n+1} \subseteq D_{m+1}$ and $\|e_n - e_m\| \leq \sigma_m$. The former implies $\|x_n - x_m\| < \delta_m$ by the definition of D_{m+1}. As δ_m and σ_m tend to 0, the sequences (x_n) and (e_n) are Cauchy so that they converge to some $x \in H$ and $e \in S(H)$ respectively. Taking the $n \to \infty$ limit we obtain $\|x - x_m\| \leq \delta_m$ and $\|e - e_m\| \leq \sigma_m$. The former implies $x \in \overline{B(x_m, \delta_m)} \subseteq B(x_{m-1}, \delta_{m-1})$ for all $m \geq 1$, using (3.5).

To complete (iii), note that from (8) of Algorithm 3.2 we have $x_n \in M_{i_n} \subseteq M_i$ for all n, as $i_n \preceq i$. Now $x_n \to x$ and M_i is closed so that $x \in M_i$.

We now show that the directional derivative derivative $f'(x, e)$ exists.

For $n \geq m$ we have $(x_n, e_n) \in D_{m+1}$; therefore by part (i) of Lemma 3.3 we know

$$(x_{m-1}, e_{m-1}) \underset{(f_m,\sigma_{m-1}-\varepsilon'_m)}{\preceq} (x_n, e_n). \tag{3.11}$$

Now the sequence $\left(f'_n(x_n, e_n)\right)$ is strictly increasing and is non-negative as $f'_0(x_0, e_0) \geq 0$ and $f'_n(x_n, e_n) < f'_{n+1}(x_n, e_n) \leq f'_{n+1}(x_{n+1}, e_{n+1})$. It is bounded above by $\mathrm{Lip}(f_n) \leq 1$ so that it converges to some $L \in (0, 1]$. As $\|f - f_n\| \to 0$ we also have $f'(x_n, e_n) \to L$ and $f'_{n+1}(x_n, e_n) \to L$. Note then that for each fixed m,

$$f'_m(x_n, e_n) - f'_m(x_{m-1}, e_{m-1}) \xrightarrow[n\to\infty]{} s_m,$$

where

$$s_m = (f_m - f)(e) + L - f'_m(x_{m-1}, e_{m-1}) \xrightarrow[m\to\infty]{} 0. \tag{3.12}$$

As $f'_m(x_n, e_n) \geq f'_m(x_{m-1}, e_{m-1})$ from (3.11) we have $s_m \geq 0$ for each m. Taking $n \to \infty$ in (3.11) we thus obtain

$$|(f_m(x + te_{m-1}) - f_m(x)) - (f_m(x_{m-1} + te_{m-1}) - f_m(x_{m-1}))| \leq r_m|t| \tag{3.13}$$

for any $t \in \mathbb{R}$, where

$$r_m = K(\sigma_{m-1} - \varepsilon'_m + \sqrt{s_m}) \to 0. \tag{3.14}$$

Using $\|f - f_m\| \leq 2t_m$, $\|e - e_{m-1}\| \leq \sigma_{m-1}$ and $\mathrm{Lip}(f) \leq 1$:

$$|(f(x + te) - f(x)) - (f_m(x_{m-1} + te_{m-1}) - f_m(x_{m-1}))| \leq (r_m + 2t_m + \sigma_{m-1})|t|. \tag{3.15}$$

Let $\varepsilon > 0$. Pick m such that

$$r_m + 2t_m + \sigma_{m-1} \le \varepsilon/3 \quad \text{and} \quad |f'_m(x_{m-1}, e_{m-1}) - L| \le \varepsilon/3 \tag{3.16}$$

and $\delta > 0$ with

$$|f_m(x_{m-1} + te_{m-1}) - f_m(x_{m-1}) - f'_m(x_{m-1}, e_{m-1})t| \le \varepsilon|t|/3 \tag{3.17}$$

for all t with $|t| \le \delta$. Combining (3.15), (3.16) and (3.17) we obtain

$$|f(x + te) - f(x) - Lt| \le \varepsilon|t|$$

if $|t| \le \delta$. Hence the directional derivative $f'(x, e)$ exists and equals L. As $L > 0$ and $f'_n(x_n, e_n)$ is an increasing sequence that tends to L, we get (iv).

Note further that, as $f_m - f$ is linear, the directional derivative $f'_m(x, e)$ also exists and equals $(f_m - f)(e) + L$. Hence from (3.12)

$$s_m = f'_m(x, e) - f'_m(x_{m-1}, e_{m-1}).$$

As $s_m \ge 0$ for all m, we conclude that $f'_m(x, e) \ge f'_m(x_{m-1}, e_{m-1})$ for all m. Further from (3.13) and (3.14),

$$\begin{aligned}
&|(f_m(x + te_{m-1}) - f_m(x)) - (f_m(x_{m-1} + te_{m-1}) - f_m(x_{m-1}))| \\
&\quad \le K \left(\sigma_{m-1} - \varepsilon'_m + \sqrt{f'_m(x, e) - f'_m(x_{m-1}, e_{m-1})}\right) |t|
\end{aligned}$$

for any t. Hence

$$(x_{m-1}, e_{m-1}) \underset{(f_m, \sigma_{m-1} - \varepsilon'_m)}{\le} (x, e).$$

This establishes (v). Finally (vi) follows immediately from (iii), (iv), (v) and the fact $i \in (i_m, j_m)$.

Proof of Theorem 3.1 From Lemma 3.4 (i)–(ii) the Lipschitz function $f : H \to \mathbb{R}$ is such that $f - f_0$ is linear and $\|f - f_0\| \le 2t_0 \le \mu$. Recall that $i \in (i_m, j_m)$ for all m; in particular $i \in (i_0, j_0)$. By parts (iii) and (iv) of Lemma 3.4 we see that $(x, e) \in D_i^f$ and $f'(x, e) > 0$.

We are left needing to verify that the directional derivative $f'(x, e)$ is almost locally maximal in the sense of Theorem 3.1.

Lemma 3.5 *If $\varepsilon > 0$ then there exists $\delta_\varepsilon > 0$ and $j_\varepsilon \in (i, j_0)$ such that whenever*

$$(x, e) \underset{(f,0)}{\le} (x', e')$$

with $\|x' - x\| \le \delta_\varepsilon$ and $x' \in M_{j_\varepsilon}$, we have $f'(x', e') < f'(x, e) + \varepsilon$.

Proof Pick n such that

$$n \geq 4/\sqrt{\varepsilon} \quad \text{and} \quad \lambda_n, t_n \leq \varepsilon/4. \tag{3.18}$$

Let $j_\varepsilon = j_n \in (i, j_0)$. Find $\delta_\varepsilon > 0$ such that

$$\delta_\varepsilon < \delta_{n-1} - \|x - x_{n-1}\| \tag{3.19}$$

and

$$\begin{aligned} &|(f_n(x+te) - f_n(x)) - (f_n(x_{n-1} + te_{n-1}) - f_n(x_{n-1}))| \\ &\quad \leq (f_n'(x,e) - f_n'(x_{n-1}, e_{n-1}) + \sigma_{n-1})|t| \end{aligned} \tag{3.20}$$

for all t with $|t| < \delta_\varepsilon/\varepsilon_n'$, where ε_n' is given by (3.6). Lemma 3.4 (iii) and the fact that $f_n'(x,e) - f_n'(x_{n-1}, e_{n-1}) \geq 0$ from Lemma 3.4 (v) guarantee the existence of such δ_ε.

Now suppose that

$$\begin{cases} (x,e) \underset{(f,0)}{\leq} (x',e'), \\ \|x'-x\| \leq \delta_\varepsilon \quad \text{and} \quad x' \in M_{j_\varepsilon}, \\ f'(x',e') \geq f'(x,e) + \varepsilon. \end{cases} \tag{3.21}$$

We aim to show that $(x', e') \in D_n$. That will lead to a contradiction since, together with (6) in Algorithm 3.2 and Lemma 3.4 (iv), this would imply

$$f_n'(x',e') \leq f_n'(x_n, e_n) + \lambda_n \leq f'(x,e) + \lambda_n$$

so that

$$f'(x',e') \leq f'(x,e) + \lambda_n + 2t_n,$$

by Lemma 3.4 (ii). This contradicts (3.18) and (3.21).

Since (3.19) and (3.21) imply $x' \in B(x_{n-1}, \delta_{n-1})$ and $x' \in M_{j_\varepsilon}$ with $j_\varepsilon = j_n \in (i_{n-1}, j_{n-1})$, to prove $(x', e') \in D_n$ it is enough to show that

$$(x_{n-1}, e_{n-1}) \underset{(f_n, \sigma_{n-1} - \varepsilon_n'/2)}{\leq} (x', e'); \tag{3.22}$$

see (5) in Algorithm 3.2.

First, note that $f_n'(x',e') - f_n'(x,e) \geq f'(x',e') - f'(x,e) - 2\|f_n - f\| \geq \varepsilon - 4t_n \geq 0$, so that $f_n'(x',e') \geq f_n'(x,e) \geq f_n'(x_{n-1}, e_{n-1})$.

Let $A = f'(x',e') - f'(x,e)$ and $B = f_n'(x',e') - f_n'(x,e)$. We have $A \geq \varepsilon$ and $B \geq 0$; therefore by (3) of Algorithm 3.2, Lemma 3.4 (ii) and (3.18)

$$\sqrt{A} - \sqrt{B} \leq \frac{A-B}{\sqrt{\varepsilon}} = \frac{(f-f_n)(e'-e)}{\sqrt{\varepsilon}} \leq \frac{4t_n}{\sqrt{\varepsilon}} \leq nt_n \leq \sigma_{n-1}/4.$$

Further, let $C = f_n'(x', e') - f_n'(x_{n-1}, e_{n-1})$. Since $f_n'(x_{n-1}, e_{n-1}) \le f_n'(x, e)$ and the Lipschitz constant $\mathrm{Lip}(f_n)$ does not exceed 1, we have $0 \le B \le C \le 2$, so that

$$K\sqrt{C} - K\sqrt{B} \ge C - B = f_n'(x, e) - f_n'(x_{n-1}, e_{n-1})$$

as $K \ge 4$. Hence

$$\begin{aligned}(f_n'(x, e) &- f_n'(x_{n-1}, e_{n-1})) + K\sqrt{f'(x', e') - f'(x, e)}\\ &\le K\sqrt{C} - K\sqrt{B} + K(\sqrt{B} + \sigma_{n-1}/4)\\ &= K(\sqrt{f_n'(x', e') - f_n'(x_{n-1}, e_{n-1})} + \sigma_{n-1}/4).\end{aligned} \tag{3.23}$$

In order to check (3.22), we need to obtain an upper estimate for

$$|(f_n(x' + te_{n-1}) - f_n(x')) - (f_n(x_{n-1} + te_{n-1}) - f_n(x_{n-1}))|. \tag{3.24}$$

If $|t| < \delta_\varepsilon/\varepsilon_n'$, we can use

$$\begin{aligned}&|(f_n(x' + te) - f_n(x')) - (f_n(x + te) - f_n(x))|\\ &\quad = |(f(x' + te) - f(x')) - (f(x + te) - f(x))| \le K\sqrt{f'(x', e') - f'(x, e)}|t|\end{aligned}$$

and (3.20) to deduce that (3.24) is no greater than

$$\begin{aligned}&(f_n'(x, e) - f_n'(x_{n-1}, e_{n-1}) + \sigma_{n-1})|t|\\ &\quad + K\sqrt{f'(x', e') - f'(x, e)}|t| + \|e - e_{n-1}\| \cdot |t|\end{aligned}$$

since $\mathrm{Lip}(f_n) \le 1$. Using (3.23), $\|e - e_{n-1}\| \le \sigma_{n-1}$, $\varepsilon_n' \le \sigma_{n-1}/4$ and $K \ge 4$ we get that the latter does not exceed

$$K\left(\sigma_{n-1} - \varepsilon_n'/2 + \sqrt{f_n'(x', e') - f_n'(x_{n-1}, e_{n-1})}\right)|t|.$$

On the other hand, for $|t| \ge \delta_\varepsilon/\varepsilon_n'$ we have $2\|x - x'\| \le 2\varepsilon_n'|t| \le K\varepsilon_n'|t|/2$ so, using this together with Lemma 3.4 (v), $\mathrm{Lip}(f_n) \le 1$ and $f_n'(x, e) \le f_n'(x', e')$, we get

$$\begin{aligned}&|(f_n(x' + te_{n-1}) - f_n(x')) - (f_n(x_{n-1} + te_{n-1}) - f_n(x_{n-1}))|\\ &\quad \le 2\|x' - x\| + K\left(\sigma_{n-1} - \varepsilon_n' + \sqrt{f_n'(x, e) - f_n'(x_{n-1}, e_{n-1})}\right)|t|\\ &\quad \le K\left(\sigma_{n-1} - \varepsilon_n'/2 + \sqrt{f_n'(x', e') - f_n'(x_{n-1}, e_{n-1})}\right)|t|.\end{aligned}$$

Hence

$$(x_{n-1}, e_{n-1}) \underset{(f_n, \sigma_{n-1} - \varepsilon_n'/2)}{\le} (x', e')$$

and we are done.

This finishes the proof of Theorem 3.1.

4 A differentiability lemma

As in the previous section, we shall mostly work on a real Hilbert space H, though our eventual application will only use the case in which H is finite dimensional. Lemma 4.2 is proved in general real Banach space X. Given x, y in a linear space we use $[x, y]$ to denote the closed line segment with endpoints x and y.

We start by quoting Lemma 4.1, which is [8, Lemma 3.4]. This lemma can be understood as an improvement of the standard mean value theorem applied to the function

$$h(t) = \varphi(t) - t\frac{\psi(s) - \psi(-s)}{2s} - \frac{\psi(s) + \psi(-s)}{2}.$$

Roughly speaking, this "generalised" mean value theorem says that if $h(s) = h(-s) = 0$ and $h(\xi) \neq 0$ then there is a point $\tau \in [-s, s]$ such that the derivative $h'(\tau)$ is bounded away from zero by a term proportional to $|h(\xi)|/s$ and (4.1) holds. The latter inequality essentially comes from the upper bound for the slope $|h(\tau + t) - h(\tau)|/|t|$ by $(\mathbb{M}h')(\tau)$, where $\mathbb{M}$ is the Hardy-Littlewood maximal operator.

We use this statement in order to show in Lemmas 4.2 and 4.3 that if $f'(x, e)$ exists and is maximal up to ε among all directional derivatives of f satisfying (4.21), at points in a δ_ε-neighbourhood of x, then f is Fréchet differentiable at x. Lemma 4.2, which follows from Lemma 4.1, guarantees that if there is a direction u in which $f(x + ru) - f(x)$ is not well approximated by $f'(x, e)\langle u, e\rangle$ then we can find a nearby point and direction (x', e'), satisfying the constraint (4.21), at which the directional derivative $f'(x', e')$ is at least as large as $f'(x, e) + \varepsilon$, a contradiction.

Lemma 4.1 *Suppose that* $|\xi| < s < \rho, 0 < \nu < \frac{1}{32}, \sigma > 0$ *and* $L > 0$ *are real numbers and that* φ *and* ψ *are Lipschitz functions defined on the real line such that* $\mathrm{Lip}(\varphi) + \mathrm{Lip}(\psi) \le L$, $\varphi(t) = \psi(t)$ *for* $|t| \ge s$ *and* $\varphi(\xi) \neq \psi(\xi)$. *Suppose, moreover, that* $\psi'(0)$ *exists and that*

$$|\psi(t) - \psi(0) - t\psi'(0)| \le \sigma L|t|$$

whenever $|t| \le \rho$,

$$\rho \ge s\sqrt{(sL)/(\nu|\varphi(\xi) - \psi(\xi)|)},$$

and

$$\sigma \le \nu^3 \left(\frac{\varphi(\xi) - \psi(\xi)}{sL}\right)^2.$$

Then there is a $\tau \in (-s, s) \setminus \{\xi\}$ *such that* $\varphi'(\tau)$ *exists,*

$$\varphi'(\tau) \ge \psi'(0) + \nu|\varphi(\xi) - \psi(\xi)|/s,$$

and

$$|(\varphi(\tau+t)-\varphi(\tau))-(\psi(t)-\psi(0))| \le 4(1+20\nu)\sqrt{[\varphi'(\tau)-\psi'(0)]L}|t| \tag{4.1}$$

for every $t \in \mathbb{R}$.

Lemma 4.2 *Let* $(X, \|\cdot\|)$ *be a real Banach space,* $f : X \to \mathbb{R}$ *be a Lipschitz function with Lipschitz constant* $\mathrm{Lip}(f) > 0$ *and let* $\varepsilon \in (0, \mathrm{Lip}(f)/9)$. *Suppose* $x \in X$, $e \in S(X)$ *and* $s > 0$ *are such that the directional derivative* $f'(x, e)$ *exists, is non-negative and*

$$|f(x+te)-f(x)-f'(x,e)t| \le \frac{\varepsilon^2}{160\mathrm{Lip}(f)}|t| \tag{4.2}$$

for $|t| \le s\sqrt{\frac{2\mathrm{Lip}(f)}{\varepsilon}}$. *Suppose further* $\xi \in (-s/2, s/2)$ *and* $\lambda \in X$ *satisfy*

$$|f(x+\lambda)-f(x+\xi e)| \ge 240\varepsilon s, \tag{4.3}$$

$$\|\lambda-\xi e\| \le s\sqrt{\frac{\varepsilon}{\mathrm{Lip}(f)}} \tag{4.4}$$

$$\text{and} \quad \frac{\|\pi se+\lambda\|}{|\pi s+\xi|} \le 1+\frac{\varepsilon}{4\mathrm{Lip}(f)} \tag{4.5}$$

for $\pi = \pm 1$. *Then if* $s_1, s_2, \lambda' \in X$ *are such that*

$$\max(\|s_1-se\|, \|s_2-se\|) \le \frac{\varepsilon^2}{320\mathrm{Lip}(f)^2}s \tag{4.6}$$

and

$$\|\lambda'-\lambda\| \le \frac{\varepsilon s}{16\mathrm{Lip}(f)}, \tag{4.7}$$

we can find $x' \in [x-s_1, x+\lambda'] \cup [x+\lambda', x+s_2]$ *and* $e' \in S(X)$ *such that the directional derivative* $f'(x', e')$ *exists,*

$$f'(x',e') \ge f'(x,e)+\varepsilon \tag{4.8}$$

and for all $t \in \mathbb{R}$ *we have*

$$\begin{aligned} &|(f(x'+te)-f(x'))-(f(x+te)-f(x))| \\ &\le 25\sqrt{(f'(x',e')-f'(x,e))\mathrm{Lip}(f)}|t|. \end{aligned} \tag{4.9}$$

Proof Define constants $L = 4\mathrm{Lip}(f)$, $\nu = \frac{1}{80}$, $\sigma = \frac{\varepsilon^2}{20L^2}$ and $\rho = s\sqrt{\frac{L}{2\varepsilon}}$. Let

$$\psi(t) = f(h(t)) \text{ and } \varphi(t) = f(g(t)), \tag{4.10}$$

where $h\colon \mathbb{R} \to X$ is a mapping that is affine on each of the intervals $(-\infty, -s/2]$ and $[s/2, \infty)$ with $h(t) = x + te$ for $t \in [-s/2, s/2]$ and $h(-s) = x - s_1$, $h(s) = x + s_2$ while $g\colon \mathbb{R} \to X$ is a mapping that is affine on $[-s, \xi]$ and on $[\xi, s]$ with $g(\xi) = x + \lambda'$ and $g(t) = h(t)$ for $|t| \geq s$.

A simple calculation shows that (4.6) implies

$$\|h'(t) - e\| \leq 2\frac{\max(\|s_1 - se\|, \|s_2 - se\|)}{s} \leq \frac{\varepsilon^2}{160\mathrm{Lip}(f)^2} \tag{4.11}$$

for $t \in \mathbb{R} \setminus \{-s/2, s/2\}$.

Now the derivative of g is given by

$$g'(t) = \begin{cases} (\lambda' + s_1)/(\xi + s) & \text{for } t \in (-s, \xi), \\ (\lambda' - s_2)/(\xi - s) & \text{for } t \in (\xi, s). \end{cases} \tag{4.12}$$

For $t \in (-s, \xi)$,

$$\begin{aligned} \left\| g'(t) - \frac{\lambda + se}{\xi + s} \right\| &\leq 2\frac{\|\lambda' - \lambda\| + \|s_1 - se\|}{s} \\ &\leq \frac{\varepsilon}{8\mathrm{Lip}(f)} + \frac{\varepsilon^2}{160\mathrm{Lip}(f)^2} \leq \frac{\varepsilon}{4\mathrm{Lip}(f)} \end{aligned}$$

using $|\xi| < s/2$, (4.6), (4.7) and $\varepsilon \leq \mathrm{Lip}(f)$. Hence

$$\|g'(t)\| \leq 1 + \frac{\varepsilon}{2\mathrm{Lip}(f)} \tag{4.13}$$

and

$$\|g'(t) - e\| \leq 3\sqrt{\frac{\varepsilon}{\mathrm{Lip}(f)}}. \tag{4.14}$$

The former follows from (4.5) and the latter from

$$\left\| \frac{\lambda + se}{\xi + s} - e \right\| = \left\| \frac{\lambda - \xi e}{\xi + s} \right\| \leq 2\frac{\|\lambda - \xi e\|}{s} \leq 2\sqrt{\frac{\varepsilon}{\mathrm{Lip}(f)}},$$

using (4.4) and $|\xi| < s/2$. A similar calculation shows that (4.13) and (4.14) hold for $t \in (\xi, s)$ too. Finally, these bounds are also true for $|t| > s$ by (4.11), since then $g'(t) = h'(t)$.

We now prove that $\xi, s, \rho, \nu, \sigma, L, \varphi, \psi$ satisfy the conditions of Lemma 4.1.

We clearly have $|\xi| < s < \rho, 0 < \nu < \frac{1}{32}, \sigma > 0$ and $L > 0$. From (4.11) and (4.13) we have $\mathrm{Lip}(h) \leq 2$ and $\mathrm{Lip}(g) \leq 2$. Hence, by (4.10), $\mathrm{Lip}(\varphi) + \mathrm{Lip}(\psi) \leq 4\mathrm{Lip}(f) = L$. Further, if $|t| \geq s$ then $g(t) = h(t)$ so that $\varphi(t) = \psi(t)$.

Now as $\xi \in (-s/2, s/2)$,

$$
\begin{aligned}
|\varphi(\xi) - \psi(\xi)| = |f(x+\lambda') - f(x+\xi e)| \\
&\geq |f(x+\lambda) - f(x+\xi e)| - \mathrm{Lip}(f)\|\lambda - \lambda'\| \\
&\geq 240\varepsilon s - \frac{\varepsilon s}{16} \geq 160\varepsilon s
\end{aligned} \tag{4.15}
$$

by (4.3). Hence $\varphi(\xi) \neq \psi(\xi)$.

From (4.10) and the definition of h, we see that the derivative $\psi'(0)$ exists and equals $f'(x, e)$. For $|t| \leq \rho = s\sqrt{\frac{L}{2\varepsilon}}$, we have from (4.2)

$$
|f(x+te) - f(x) - f'(x,e)t| \leq \frac{\varepsilon^2}{160\mathrm{Lip}(f)}|t|,
$$

so that, together with (4.11),

$$
\begin{aligned}
&|\psi(t) - \psi(0) - t\psi'(0)| = |f(h(t)) - f(x) - f'(x,e)t| \\
&\quad \leq |f(x+te) - f(x) - f'(x,e)t| + \mathrm{Lip}(f)\|h(t) - x - te\| \\
&\quad \leq \frac{\varepsilon^2}{160\mathrm{Lip}(f)}|t| + \frac{\varepsilon^2}{160\mathrm{Lip}(f)}|t| = \sigma L|t|.
\end{aligned}
$$

Finally, using (4.15),

$$
s\sqrt{\frac{sL}{\nu|\varphi(\xi) - \psi(\xi)|}} \leq s\sqrt{\frac{sL}{\frac{1}{80}(160\varepsilon s)}} = \rho,
$$

$$
\nu^3\left(\frac{|\varphi(\xi) - \psi(\xi)|}{sL}\right)^2 \geq \frac{1}{80^3}\left(\frac{160\varepsilon s}{sL}\right)^2 = \sigma.
$$

Therefore, by Lemma 4.1, there exists $\tau \in (-s, s) \setminus \{\xi\}$ such that $\varphi'(\tau)$ exists and

$$
\varphi'(\tau) \geq \psi'(0) + \nu|\varphi(\xi) - \psi(\xi)|/s \geq f'(x,e) + 2\varepsilon > 0 \tag{4.16}
$$

using (4.15) and $\psi'(0) = f'(x, e) \geq 0$. Further, by (4.1)

$$
|(\varphi(\tau+t) - \varphi(\tau)) - (\psi(t) - \psi(0))| \leq 5\sqrt{(\varphi'(\tau) - f'(x,e))L}|t| \tag{4.17}
$$

for every $t \in \mathbb{R}$.

From (4.14) and $\varepsilon < \mathrm{Lip}(f)/9$ we have $g'(t) \neq 0$ for any $t \in (-s, s) \setminus \{\xi\}$. Define

$$
x' = g(\tau) \text{ and } e' = g'(\tau)/\|g'(\tau)\|. \tag{4.18}
$$

The point x' belongs to

$$
g((-s, s) \setminus \{\xi\}) = (x - s_1, x + \lambda') \cup (x + \lambda', x + s_2).
$$

Further, since the function φ is differentiable at τ, the directional derivative $f'(x', e')$ exists and equals $\varphi'(\tau)/\|g'(\tau)\|$. Now by (4.13), (4.16) and $\mathrm{Lip}(\varphi) \le 2\mathrm{Lip}(f)$ we have

$$\|g'(\tau)\| \le \frac{2\varphi'(\tau)}{\varphi'(\tau) + f'(x, e)},$$

so that

$$f'(x', e') - f'(x, e) \ge \frac{\varphi'(\tau) - f'(x, e)}{2}. \tag{4.19}$$

Hence (4.8) follows from (4.16).

Together with $L = 4\mathrm{Lip}(f)$ and the definitions of φ, ψ, x', the inequalities (4.17) and (4.19) give

$$\begin{aligned} &|(f(g(\tau + t)) - f(x') - (f(h(t)) - f(x))| \\ &\quad \le 20\sqrt{(f'(x', e') - f'(x, e))\mathrm{Lip}(f)}|t|. \end{aligned} \tag{4.20}$$

Using (4.11), (4.14) and $\varepsilon \le \mathrm{Lip}(f)$ we obtain

$$\|g(\tau + t) - g(\tau) - te\| \le 3\sqrt{\frac{\varepsilon}{\mathrm{Lip}(f)}}|t|,$$
$$\|h(t) - h(0) - te\| \le \sqrt{\frac{\varepsilon}{\mathrm{Lip}(f)}}|t|$$

for all t. Using $g(\tau) = x'$, $h(0) = x$ and the Lipschitz property of f,

$$|f(g(\tau + t)) - f(x' + te)| \le 3\sqrt{\varepsilon\mathrm{Lip}(f)}|t|,$$
$$|f(h(t)) - f(x + te)| \le \sqrt{\varepsilon\mathrm{Lip}(f)}|t|$$

for all t.

Putting these together with (4.20) we get

$$\begin{aligned} &|(f(x' + te) - f(x') - (f(x + te) - f(x))| \\ &\quad \le 20\sqrt{(f'(x', e') - f'(x, e))\mathrm{Lip}(f)}|t| + 3\sqrt{\varepsilon\mathrm{Lip}(f)}|t| + \sqrt{\varepsilon\mathrm{Lip}(f)}|t| \\ &\quad \le 25\sqrt{(f'(x', e') - f'(x, e))\mathrm{Lip}(f)}|t| \end{aligned}$$

as $\varepsilon \le f'(x', e') - f'(x, e)$. This is (4.9). We are done.

Lemma 4.3 (Differentiability Lemma) *Let H be a real Hilbert space, $f : H \to \mathbb{R}$ be a Lipschitz function and $(x, e) \in H \times S(H)$ be such that the directional derivative $f'(x, e)$ exists and is non-negative. Suppose that there is a family of sets $\{F_\varepsilon \subseteq H \mid \varepsilon > 0\}$ such that*

(1) *whenever* $\varepsilon, \eta > 0$ *there exists* $\delta_* = \delta_*(\varepsilon, \eta) > 0$ *such that for any* $\delta \in (0, \delta_*)$ *and* u_1, u_2, u_3 *in the closed unit ball of* H, *one can find* u'_1, u'_2, u'_3 *with* $\|u'_m - u_m\| \le \eta$ *and*

$$[x + \delta u'_1, x + \delta u'_3] \cup [x + \delta u'_3, x + \delta u'_2] \subseteq F_\varepsilon,$$

(2) *whenever* $(x', e') \in F_\varepsilon \times S(H)$ *is such that the directional derivative* $f'(x', e')$ *exists,* $f'(x', e') \ge f'(x, e)$ *and*

$$\begin{aligned} &|(f(x' + te) - f(x')) - (f(x + te) - f(x))| \\ &\le 25\sqrt{(f'(x', e') - f'(x, e))\mathrm{Lip}(f)}|t| \end{aligned} \tag{4.21}$$

for every $t \in \mathbb{R}$ *then*

$$f'(x', e') < f'(x, e) + \varepsilon. \tag{4.22}$$

Then f *is Fréchet differentiable at* x *and its derivative* $f'(x)$ *is given by the formula*

$$f'(x)(h) = f'(x, e)\langle h, e\rangle \tag{4.23}$$

for $h \in H$.

Proof We may assume $\mathrm{Lip}(f) = 1$. Let $\varepsilon \in (0, 1/9)$. It is enough to show there exists $\Delta > 0$ such that

$$|f(x + ru) - f(x) - f'(x, e)\langle u, e\rangle r| < 1000\varepsilon^{1/2} r \tag{4.24}$$

for any $u \in S(H)$ and $r \in (0, \Delta)$.

We know that the directional derivative $f'(x, e)$ exists so that there exists $\Delta > 0$ such that

$$|f(x + te) - f(x) - f'(x, e)t| < \frac{\varepsilon^2}{160}|t| \tag{4.25}$$

whenever $|t| < 8\Delta/\varepsilon$. We may pick $\Delta < \delta_*(\varepsilon, \varepsilon^2/320)\varepsilon^{1/2}/4$.

Assume now, for a contradiction, that there exist $r \in (0, \Delta)$ and $u \in S(H)$ such that the inequality (4.24) does not hold:

$$|f(x + ru) - f(x) - f'(x, e)\langle u, e\rangle r| \ge 1000\varepsilon^{1/2} r. \tag{4.26}$$

Define $u_1 = -e$, $u_2 = e$, $u_3 = \varepsilon^{1/2}u/4$, $s = 4\varepsilon^{-1/2}r$, $\xi = \langle u, e\rangle r$ and $\lambda = ru$. From $\|u_m\| \le 1$, condition (1) of the present Lemma and

$$s < 4\varepsilon^{-1/2}\Delta < \delta_*(\varepsilon, \varepsilon^2/320),$$

there exist u_1', u_2', u_3' with $\|u_m' - u_m\| \leq \varepsilon^2/320$ and

$$[x - s_1, x + \lambda'] \cup [x + \lambda', x + s_2] \subseteq F_\varepsilon, \tag{4.27}$$

where $s_1 = -su_1', s_2 = su_2'$ and $\lambda' = su_3'$.

We check that the assumptions of Lemma 4.2 hold for $f, \varepsilon, x, e, s, \xi, \lambda, s_1, s_2, \lambda'$ in the Banach space $X = H$. First we note (4.2) is immediate from (4.25) as $s\sqrt{2/\varepsilon} < 8r/\varepsilon < 8\Delta/\varepsilon$. We also have $|\xi| \leq r < s/2$ as $\varepsilon < 1$. Further $|\xi| \leq r < 8\Delta/\varepsilon$ so that we may apply (4.25) with $t = \xi$. Combining this inequality with (4.26) we obtain

$$|f(x + ru) - f(x + \xi e)| \geq 1000\varepsilon^{1/2} r - \frac{\varepsilon^2}{160}|\xi| > 960\varepsilon^{1/2} r = 240\varepsilon s.$$

Hence (4.3). As $\|\lambda - \xi e\| = r\|u - \langle u, e\rangle e\| \leq r \leq s\sqrt{\varepsilon}$ we deduce (4.4).

Now observe that for $\pi = \pm 1$,

$$\frac{\pi s e + \lambda}{\pi s + \xi} = e + \frac{r}{\pi s + \xi}(u - \langle u, e\rangle e)$$

and, as the vectors e and $u - \langle u, e\rangle e$ are orthogonal and $\|\pi s + \xi\| \geq s/2$, we obtain

$$\left\| \frac{\pi s e + \lambda}{\pi s + \xi} \right\| \leq 1 + \frac{1}{2}\frac{r^2}{(s/2)^2} = 1 + \frac{\varepsilon}{8}.$$

This proves (4.5).

Since $\|u_m' - u_m\| \leq \varepsilon^2/320$, (4.6) follows from the definitions of u_1, u_2, s_1, s_2. Further as $\lambda' = su_3'$ and $\lambda = ru = su_3$ we have $\|\lambda' - \lambda\| \leq s\varepsilon^2/320 \leq \varepsilon s/16$. Hence (4.7).

Therefore by Lemma 4.2 there exists $x' \in [x - s_1, x + \lambda'] \cup [x + \lambda', x + s_2]$ and $e' \in S(H)$ such that $f'(x', e')$ exists, is at least $f'(x, e) + \varepsilon$ and such that (4.9) holds. But $x' \in F_\varepsilon$ by (4.27). This contradicts condition (2) of the present Lemma. Hence the result.

5 Proof of main result

Let $n \geq 2$ and $M_i \subseteq \mathbb{R}^n$ $(i \in \mathfrak{S})$ be given by (2.15).

Recall that, by Theorem 2.5 (i)–(ii), the sets M_i are closed, have Lebesgue measure zero and $M_i \subseteq M_j$ if $i \preceq j$. Here $(\mathfrak{S}, \preceq)$ is a non-empty, chain complete poset that is dense and has no minimal elements, by Lemma 2.1.

The following theorem shows that if $g \colon \mathbb{R}^n \to \mathbb{R}$ is Lipschitz the points of differentiability of g are dense in the set

$$M = \bigcup_{\substack{i \in \mathfrak{S} \\ i \prec (1,1,1,\dots)}} M_i.$$

Theorem 5.1 *If $k, l \in \mathfrak{S}$ with $k \prec l$ and $y \in M_k$, $d > 0$ then for any Lipschitz function $g\colon \mathbb{R}^n \to \mathbb{R}$ there exists a point x of Fréchet differentiability of g with $x \in M_l$ and $\|x - y\| \leq d$.*

Proof We may assume $\mathrm{Lip}(g) > 0$. Let H be the Hilbert space $\mathbb{R}^n$. As in Sect. 3, for a Lipschitz function $h\colon \mathbb{R}^n \to \mathbb{R}$ and $i \in \mathfrak{S}$ we let D_i^h be the set of pairs $(x, e) \in M_i \times S^{n-1}$ such that the directional derivative $h'(x, e)$ exists.

Take $i_0 \in (k, l)$ and $j_0 = l$. By Theorem 2.5 (iii) we can find a line segment $\ell \subseteq M_{i_0} \cap B(y, d/2)$ of positive length. The directional derivative of g in the direction of ℓ exists for almost every point on ℓ, by Lebesgue's theorem, so that we can pick a pair $(x_0, e_0) \in D_{i_0}^g$ with $\|x_0 - y\| \leq d/2$. Set $f_0 = g$, $K = 25\sqrt{2\mathrm{Lip}(g)}$, $\delta_0 = d/2$ and $\mu = \mathrm{Lip}(g)$.

Let the Lipschitz function f, the pair (x, e), the element of the index set $i \in (i_0, l)$ and, for each $\varepsilon > 0$, the positive number δ_ε and the index $j_\varepsilon \in (i, l)$ be given by the conclusion of Theorem 3.1. We verify the conditions of the Differentiability Lemma 4.3 hold for the function $f\colon \mathbb{R}^n \to \mathbb{R}$, the pair $(x, e) \in D_i^f$ and the family of sets $\{F_\varepsilon \subseteq \mathbb{R}^n \mid \varepsilon > 0\}$ where

$$F_\varepsilon = M_{j_\varepsilon} \cap B(x, \delta_\varepsilon).$$

We know from Theorem 3.1 that the derivative $f'(x, e)$ exists and is non-negative. To verify condition (1) of Lemma 4.3, we may take $\varepsilon > 0$, $\eta \in (0, 1)$ and put

$$\delta_* = \min(\alpha(i, j_\varepsilon, \eta), \delta_\varepsilon/2),$$

where $\alpha(i, j_\varepsilon, \eta)$ is given by Theorem 2.5 (iii), noting $\delta(1 + \eta) < 2\delta_* \leq \delta_\varepsilon$ for every $\delta \in (0, \delta_*)$. Condition (2) of Lemma 4.3 is immediate from the definition of F_ε and Eq. (3.1) as $\mathrm{Lip}(f) \leq \mathrm{Lip}(g) + \mu = 2\mathrm{Lip}(g)$ so that $25\sqrt{\mathrm{Lip}(f)} \leq K$.

Therefore, by Lemma 4.3 the function f is differentiable at x. So too, therefore, is g as $g - f$ is linear. Finally, note that $x \in M_i \subseteq M_l$ and

$$\|x - y\| \leq \|x - x_0\| + \|x_0 - y\| \leq \delta_0 + d/2 = d.$$

Corollary 5.2 *If $n \geq 2$ there exists a compact subset $S \subseteq \mathbb{R}^n$ of measure 0 that contains a point of Fréchet differentiability of every Lipschitz function $g\colon \mathbb{R}^n \to \mathbb{R}$.*

Proof Let $l \in \mathfrak{S}$. As l is not minimal we can find $k \prec l$. Now $M_k \neq \emptyset$ so that we may pick $y \in M_k$. Let $S = M_l \cap \overline{B(y, d)}$ where $d > 0$. We know S is closed and has measure zero. As it is bounded it is also compact. If $g\colon \mathbb{R}^n \to \mathbb{R}$ is Lipschitz then by Theorem 5.1 we can find a point x of differentiability of g with $x \in M_l$ and $\|x - y\| \leq d$, so that $x \in S$.

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