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UNAVOIDABLE SIGMA-POROUS SETS

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Abstract

We prove that every separable metric space which admits an ℓ_1-tree as a Lipschitz quotient has a σ-porous subset which contains every Lipschitz curve up to a set of one-dimensional Hausdorff measure zero. This applies to any Banach space containing ℓ_1. We also obtain an infinite-dimensional counterexample to the Fubini theorem for the σ-ideal of σ-porous sets.

1. *Introduction*

1.1. *Overview*

In the present paper, we are concerned with the question whether porous sets (for definition, see Subsection 1.2), in a metric space X, are small on Lipschitz curves.

Our interest stems from the discovery in [**10**] that the problem of describing Banach spaces, where porous sets are null on many (infinite-dimensional) surfaces, is intimately related to the well-known problem of existence of common points of Fréchet differentiability of (countably many) real-valued Lipschitz functions on Banach spaces. This relation has been refined by J. Lindenstrauss, D. Preiss and J. Tišer (book in preparation) who show that the problem, in which spaces porous sets are null on many n-dimensional surfaces, is closely connected to the problem of existence of many common points of Fréchet differentiability of exactly n Lipschitz functions. In particular, for $n = 1$, it is shown in *loc. cit.* that in every Banach space with separable dual porous sets are null on residually many C^1 curves, and this is related to the known result [**9**, **11**] that real-valued Lipschitz functions on such spaces have (many) points of Fréchet differentiability.

However, what happens if the dual X^* is not separable? On every separable Banach space with non-separable dual, there is a nowhere Fréchet differentiable Lipschitz function (moreover, a nowhere Fréchet differentiable equivalent norm; see [**2**, Proposition 4.12]). It is then natural to conjecture that Banach spaces, in which porous sets are negligible on curves, necessarily have separable duals. This conjecture is supported by a result of J. Lindenstrauss, D. Preiss and J. Tišer (book in preparation) who show that the space ℓ_1 contains a σ-porous subset whose complement is null on all curves. This is achieved by using a variant of rather complicated examples [**12**] of badly non-differentiable Lipschitz functions.

In the present paper, we describe a σ-porous subset S, the complement of which is null on all Lipschitz curves, in any Banach space containing ℓ_1. We refer to such sets S as 'unavoidable'. Interestingly, our method of construction of these sets is not related to differentiability. Instead, we recall that every Banach space containing ℓ_1 admits an ℓ_1-tree as a Lipschitz quotient ([**7**]; see Subsections 1.3 and 1.4 for definitions). We notice that, since ℓ_1-trees do not contain many curves, it is relatively easy to construct porous subsets of an ℓ_1-tree that are large on curves. The next idea is to observe that the preimage of a porous set under a Lipschitz quotient

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mapping is porous. However, this pullback by a Lipschitz quotient mapping does not preserve smallness on curves (a curve may be a preimage of one point). Therefore, we have to make suitable modifications to the preimage of a σ-porous subset of an ℓ_1-tree. In spite of this, our method is considerably simpler than that of J. Lindenstrauss, D. Preiss and J. Tišer, and has the advantage that it works even in metric spaces admitting an ℓ_1-tree as a Lipschitz quotient, where the differentiability approach does not seem to be at all possible.

Our construction does not cover separable Banach spaces with non-separable duals which do not admit an ℓ_1-tree as a Lipschitz quotient. This leads to two open problems: whether such spaces exist at all (candidates must be separable spaces with non-separable duals which do not contain ℓ_1), and if yes, do they still contain unavoidable σ-porous sets.

1.2. *σ-porous sets*

Let us recall the notion of porosity. Let (X, d) be a metric space. A set $A \subset X$ is called porous if there is a number $\lambda > 0$ with the following property. For every $x \in A$ and for each $r > 0$, there exist $z \in X$ and $\rho \leqslant r$ such that $\rho > \lambda d(x, z)$ and $B(z, \rho) \cap A = \emptyset$. A countable union of porous sets is called σ-porous.

The notion of σ-porosity was introduced by Dolzhenko [3], and since then it has been studied and used by many authors.

Porous sets are nowhere dense, and hence σ-porous sets are of first category. The Lebesgue density theorem implies that every σ-porous subset of a finite-dimensional space is of Lebesgue measure zero.

Further information on porous sets can be found in survey paper [15].

1.3. *Lipschitz quotient mappings*

A mapping $f : X \to Y$ between metric spaces X, Y is called Lipschitz quotient if there exist constants $0 < c \leqslant L < \infty$ such that

$$B(f(x), cr) \subset fB(x, r) \subset B(f(x), Lr)$$

for all $x \in X$ and $r > 0$. The biggest possible constant c is called the co-Lipschitz constant of f, and the smallest possible L is called the Lipschitz constant of f.

The notions of co-uniform and co-Lipschitz mappings were introduced in several texts (for example, [5, 6, 14]) but were first systematically studied in [1].

1.4. *ℓ_1-trees*

A complete metric space (T, d) is called a metric tree (an $\mathbb{R}$ tree in the terminology of [4]) if for any two points x, y in T the interval $\langle x, y\rangle = \{z \in T : d(x, z) + d(z, y) = d(x, y)\}$ in T is isometric to $[0, d(x, y)]$, and any injective continuous path in T which starts at x and ends at y coincides with $\langle x, y\rangle$. In this case, any continuous path in T between x and y contains $\langle x, y\rangle$.

An example of a metric tree, which is important for us, is an ℓ_1-tree defined in [7].

In order to introduce ℓ_1-trees, we first define the ℓ_1-union of metric spaces. Suppose that (Y, d_Y) and (Z, d_Z) are two metric spaces that intersect at a single point p. The ℓ_1 union $Y \cup_1 Z$ of Y and Z is $(Y \cup Z, d)$, where the metric d agrees with d_Y on $Y \times Y$, d agrees with d_Z on $Z \times Z$, and if $y \in Y$, $z \in Z$, then $d(y, z)$ is defined to be equal to $d_Y(y, p) + d_Z(p, z)$.

We refer to an image I of an isometric embedding $i : [0, +\infty) \to T$ as a closed ray and to $p = i(0)$ as its endpoint.

Now, an ℓ_1-tree is defined by the following construction. Let I_1 be a closed ray and define $T_1 = I_1$. Having defined T_n, let I_{n+1} be a closed ray; the intersection of the ray with T_n is the endpoint p_{n+1} of I_{n+1}; put $T_{n+1} = T_n \cup_1 I_{n+1}$. The completion, T, of $\bigcup_{n=1}^{\infty} T_n$ is called an ℓ_1-tree if the set $\mathcal{P} = \{p_n\}_{n\geqslant 1}$ of all nodal points is dense in T. We say that $(I_n, T_n)_{n\geqslant 1}$

describe an allowed construction of T. (There are always many different allowed constructions of a given ℓ_1-tree.) We denote the distance in T by $\mathrm{dist}_T(\cdot,\cdot)$.

We reiterate that any ℓ_1-tree is a metric tree; this is proved in [7, Proposition 4.1].

1.5. *Summary of main results*

In this paper we prove (Theorem 4.6) that every separable metric space X which admits a Lipschitz quotient mapping onto an ℓ_1-tree, contains a σ-porous set $S \subset X$ such that for every Lipschitz curve $\gamma : [0,1] \to X$, the one-dimensional Hausdorff measure of $\gamma([0,1]) \setminus S$ is zero.

When X is a Cartesian product of a separable metric space Y which admits a Lipschitz quotient mapping onto an ℓ_1-tree and a real line $\mathbb{R}$, the σ-porous set S given by Theorem 4.6 is a new counterexample to the Fubini theorem for σ-porous sets in Banach spaces. We explain this counterexample in Section 6.

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2. *Properties of ℓ_1-trees*

In this section, we discuss some general properties of an ℓ_1-tree which will be used in our construction later.

Assume that $(I_n, T_n)_{n\geqslant 1}$ describe an allowed construction of T. Recall that $\mathcal{P}$ denotes the set of all nodes in T.

For every n and $r > 0$, let us denote by $p_n \oplus r$ the unique point of the ray I_n defined by the condition $\mathrm{dist}_T(p_n, p_n \oplus r) = r$.

For any point $q \in T$, we define the set $\mathcal{A}(q)$ of ancestors of q as follows:

$$\mathcal{A}(q) = \{p_k \in \langle p_1, q\rangle \cap \mathcal{P} \colon \text{there exists } \varepsilon = \varepsilon(p_k) > 0 \text{ such that } p_k \oplus r \in \langle p_1, q\rangle \text{ for all } 0 < r < \varepsilon\}$$

(for the definition of the interval $\langle x, y\rangle$, see Subsection 1.4). If $q \in I_k \setminus \{p_k\}$, we say that p_k is a parent of q. It is an immediate consequence of the definition of an ℓ_1-union that the parent of q is one of its ancestors. Some further properties of the set of ancestors are listed in the following lemma.

Lemma 2.1. *Assume that T is an ℓ_1-tree and $(I_n, T_n)_{n\geqslant 1}$ describes an allowed construction of T.*

(1) *If a point q belongs to one of the rays of T, then the set $\mathcal{A}(q)$ of ancestors of q is finite and is equal to $\{q^{(i)}\}_{i=0,\ldots,N-1}$, where $q^{(i-1)}$ is the parent of $q^{(i)}$ for every $1 \leqslant i \leqslant N$, $N = \#\mathcal{A}(q)$, $q^{(0)} = p_1$ and $q^{(N)} = q$. Moreover,*

$$\langle p_1, q\rangle = \bigcup_{1\leqslant i\leqslant N} \langle q^{(i-1)}, q^{(i)}\rangle. \tag{2.1}$$

(2) *For any $q \in T$ and two distinct ancestors $p_k, p_{k'}$ of q, either $p_k \in \mathcal{A}(p_{k'})$ or $p_{k'} \in \mathcal{A}(p_k)$.*

(3) *For any $q \in T$, the set $\mathcal{A}(q)$ of its ancestors can be represented as $\{p_1 = q^{(0)}, q^{(1)}, q^{(2)}, \ldots\}$, where $q^{(i)}$ are such that $\mathcal{A}(q^{(i)}) = \{q^{(0)}, \ldots, q^{(i-1)}\}$ for all $0 \leqslant i < \#\mathcal{A}(q)$. Moreover, for $q \in T \setminus \bigcup_{k\geqslant 1} I_k$,*

$$\langle p_1, q\rangle = \{q\} \cup \bigcup_{i\geqslant 1} \langle q^{(i-1)}, q^{(i)}\rangle \tag{2.2}$$

and $q^{(i)} \to q$ as $i \to \infty$.

Proof. (1) Assume that $q \in I_k \setminus \{p_k\}$, and let us prove this statement by induction on k. If $k = 1$, then the statement clearly holds; in this case, $N = 1$ and $\mathcal{A}(q) = \{p_1\}$.

Assume that the statement holds for all points in $\bigcup_{j<k} I_j$, in particular, for the parent p_k of q. Since p_k is an ancestor of q and $\langle p_1, q\rangle = \langle p_1, p_k\rangle \cup \langle p_k, q\rangle$, we get by the induction hypothesis that equality (2.1) holds (where the endpoints of the intervals on the right-hand side are the ancestors of p_k, the point $p_k = q^{(N-1)}$ itself and the point $q^{(N)} = q$).

In order to finish the proof of the first part of the lemma, it is enough to show that $\mathcal{A}(q) = \{p_k\} \cup \mathcal{A}(p_k)$. It is obvious that $\mathcal{A}(q)$ contains $\{p_k\} \cup \mathcal{A}(p_k)$. Let us show that every ancestor of q other than p_k is an ancestor of p_k.

Let $p_s \in \mathcal{A}(q) \setminus \{p_k\}$, then by (2.1) the node p_s belongs to $\langle q^{(i-1)}, q^{(i)}\rangle$ for some $i \leqslant N$. If $i < N$, then by the induction hypothesis p_s must be an ancestor of p_k. If $i = N$ and p_s is an internal node in the interval $\langle p_k, q\rangle$, then the point $p_s \oplus \varepsilon$ belongs to the ray I_s for any $\varepsilon > 0$. Since $s > k$, $p_s \oplus \varepsilon \notin \langle p_1, q\rangle$, a contradiction.

(2) Assume that $p_k, p_{k'} \in \mathcal{A}(q)$. Let $\phi : [0, 1] \to T$ be an injective continuous path in T connecting p_1 and q. Assume that $t, t' \in (0, 1)$ are such that $\phi(t) = p_k$ and $\phi(t') = p_{k'}$. Without loss of generality assume that $t < t'$. Then $p_k \in \langle p_1, p_{k'}\rangle$.

Let $\varepsilon > 0$ be such that $p_k \oplus r \in \langle p_1, q\rangle$ for all $0 < r < \varepsilon$. Then for $r < \min\{\varepsilon, \operatorname{dist}_T(p_k, p_{k'})\}$ the point $p_k \oplus r$ cannot belong to $\langle p_{k'}, q\rangle$, and thus belongs to $\langle p_1, p_{k'}\rangle$. This implies that $p_k \in \mathcal{A}(p_{k'})$.

(3) If q belongs to one of the rays of T, then the statement follows from the first two parts of this lemma.

Assume that $q \in T \setminus \bigcup_{n\geqslant 1} I_n$. Note that any ℓ_1-tree may be isometrically embedded into the space ℓ_1. Such an embedding is described in the proof of [**7**, Corollary 2.1]: let $\{e_n\}_{n\geqslant 1}$ be the standard basis of ℓ_1. Let M map I_1 isometrically to a ray $\mathbb{R}^+ e_1$. If the mapping M is defined on T_n, then we extend M to T_{n+1} by mapping the closed ray $\overline{I_{n+1}}$ isometrically to $M(p_{n+1}) + \mathbb{R}^+ e_{n+1}$. Finally, M is uniquely extended from $\bigcup_{n\geqslant 1} T_n$ to $\overline{\bigcup_{n\geqslant 1} T_n} = T$. Assume now that $M(q) = (x_i)_{i\geqslant 1}$ and $\{i_1 < i_2 < \cdots\} = \{i : x_i \neq 0\}$. Let $q^{(k)} = M^{-1}(\sum_{1\leqslant i<i_k} x_i e_i)$. Then for each k, the point $q^{(k)}$ is an ancestor of q, $q^{(k)} \to q$ and (2.2) holds. By part (1), $\#\mathcal{A}(q^{(k)}) = k$.

We now show that the set of ancestors of q does not contain any nodes except $q^{(k)}$. By part (2), there could not be two distinct ancestors p, p' of q such that $\#\mathcal{A}(p) = \#\mathcal{A}(p')$. Therefore if $p \in \mathcal{A}(q)$, then $p = q^{(\#\mathcal{A}(p))}$. □

For any point q which belongs to one of the rays of T, we refer to $\#\mathcal{A}(q)$ as the *level* of q.

LEMMA 2.2. *Let T be an ℓ_1-tree, and let $p \neq q$ be two points in T. Let $p^{(i)}$ be the ancestors of p and let $q^{(i)}$ be the ancestors of q as in Lemma 2.1. Put*

$$N = \max\{0 \leqslant i < \min(\#\mathcal{A}(p), \#\mathcal{A}(q)) \mid p^{(i)} = q^{(i)}\}. \tag{2.3}$$

Then N is a finite number (the set of such indices i is always bounded) and the interval $\langle p, q\rangle$ is the concatenation

$$\langle p, p^{(N+1)}\rangle \cup \langle p^{(N+1)}, q^{(N+1)}\rangle \cup \langle q^{(N+1)}, q\rangle$$

of three (possibly degenerate) intervals which do not intersect except at endpoints.

Proof. Note that the set $\{0 \leqslant i < \min(\#\mathcal{A}(p), \#\mathcal{A}(q)) \mid p^{(i)} = q^{(i)}\}$ is not empty, since $p^{(0)} = q^{(0)} = p_1$. Furthermore, $N < \infty$, as otherwise $p = q$ by Lemma 2.1, part (3).

Assume that $\min(\#\mathcal{A}(p), \#\mathcal{A}(q)) \geqslant N + 2$. Let $\phi_1, \phi_2, \phi_3 : [0, 1] \to T$ be injective continuous paths in T connecting p with $p^{(N+1)}$, $p^{(N+1)}$ with $q^{(N+1)}$ and $q^{(N+1)}$ with q, respectively. Note that by (2.2), $\phi_1(0, 1)$ and $\phi_3(0, 1)$ do not intersect the ray I_k containing ϕ_2, since the rays the union of which contains $\phi_1(0, 1) \cup \phi_3(0, 1)$ were added to the ℓ_1-tree later than I_k. Also, for any point $x \in \phi_1(0,1)$, the set $\mathcal{A}(x)$ contains $p^{(N+1)}$ (Lemma 2.1, part (3)), and for any point $y \in \phi_3(0,1)$, the set $\mathcal{A}(y)$ does not contain $p^{(N+1)}$ (the node from $\mathcal{A}(y)$ with $N+1$ ancestors is $q^{(N+1)} \neq p^{(N+1)}$). Therefore, $\phi_1(0,1) \cap \phi_3(0,1) = \emptyset$. This means that the concatenation of ϕ_1, ϕ_2 and ϕ_3 is injective and continuous, and hence coincides with $\langle p, q \rangle$.

If $\#\mathcal{A}(p) = N+1$ and $\#\mathcal{A}(q) \geqslant N+2$, then we can repeat the previous argument with the trivial path $\{p\}$ as ϕ_1.

Finally, if $\#\mathcal{A}(p) = \#\mathcal{A}(q) = N+1$, then p and q are contained in the same ray of T, and thus ϕ_1 and ϕ_3 are constant paths ($p^{(N+1)} = p$ and $q^{(N+1)} = q$). □

Lemma 2.3. *If T is an ℓ_1-tree and $\gamma : [0,1] \to T$ is a continuous curve which connects two distinct points in T, then $\gamma([0,1])$ contains infinitely many nodal points of T.*

Proof. Let $p = \gamma(0)$ and $q = \gamma(1)$. By the property of metric trees, the image of γ contains the interval $\langle p, q \rangle$, which decomposes, by Lemma 2.2, into $\langle p, p^{(N+1)} \rangle \cup \langle p^{(N+1)}, q^{(N+1)} \rangle \cup \langle q^{(N+1)}, q \rangle$, where N is defined in (2.3).

Assume first that the interval $J = \langle p^{(N+1)}, q^{(N+1)} \rangle$ is non-degenerate. Observe that since $p^{(N+1)}$ and $q^{(N+1)}$ have the same parent $p^{(N)} = q^{(N)}$, this interval lies on a ray in T. If $J \cap \mathcal{P}$ is finite, then there is a non-degenerate interval $J' \subset J$ which does not contain any node. Then the distance between the middle point of J' and any node of T is at least half the length of J'. Thus the nodes are not dense in T, a contradiction. Therefore $J \cap \mathcal{P}$ is infinite.

If the interval J is degenerate, then consider the interval among $\langle p, p^{(N+1)} \rangle$ and $\langle q^{(N+1)}, q \rangle$ which is non-degenerate, say $\langle q^{(N+1)}, q \rangle$. By the definition of ancestors, this interval contains $\langle q^{(N+1)}, q^{(N+1)} \oplus \varepsilon \rangle$ for some $\varepsilon > 0$; the latter is an interval in a ray of T, so the above argument applies. □

Lemma 2.4. *Assume that T is an ℓ_1-tree and p is a node in T. Let $r > \rho$ be two positive numbers and x be any point in the ball $B(p \oplus r, \rho)$. Then $p \in \mathcal{A}(x)$ and $\mathcal{A}(x) \setminus (\{p\} \cup \mathcal{A}(p)) \subset B(p \oplus r, \rho)$.*

Proof. Denote by z the centre $p \oplus r$ of the ball $B(p \oplus r, \rho)$. By Lemma 2.2, the interval $\langle z, x \rangle$ is equal to the concatenation $\langle z, z^{(N+1)} \rangle \cup \langle z^{(N+1)}, x^{(N+1)} \rangle \cup \langle x^{(N+1)}, x \rangle$, where N is defined as in (2.3). If the interval $\langle z, z^{(N+1)} \rangle$ is non-degenerate, then its length is at least $r = \mathrm{dist}_T(z, p)$, since p is a parent of z. This cannot be true, because the length of $\langle z, x \rangle$ is less than ρ, which is less than r. Thus $z = z^{(N+1)}$. Therefore, $z^{(N)}$, as the parent of z, coincides with p. This implies $x^{(N)} = z^{(N)} = p$, so $p \in \mathcal{A}(x)$.

The set $\mathcal{A}(x) \setminus (\{p\} \cup \mathcal{A}(p))$ consists of $x^{(N+i)}$, $i \geqslant 1$. Since each $x^{(N+i)}$ is in the interval $\langle z, x \rangle$, one has $\mathrm{dist}_T(x^{(N+i)}, z) \leqslant \mathrm{dist}_T(x, z) < \rho$. □

3. *An auxiliary construction in ℓ_1-tree*

Before we get to the construction of a large σ-porous set in a metric space X admitting an ℓ_1-tree as a Lipschitz quotient, we need to do preliminary work at the level of ℓ_1-tree itself.

Let T be an ℓ_1-tree. We will use the notation as in Definition 1.4 of an ℓ_1-tree. We will now describe the sets $\mathcal{P}_{n,m,k}$ and the families $\mathcal{F}_k$ of balls in T.

Let $\mathcal{P}_n = \mathcal{P} \cap I_n \setminus \{p_n\}$ be the set of those nodes of the ℓ_1-tree T which belong to the nth open ray (this is the set of nodes with parent p_n). As follows from Lemma 2.3, $\mathcal{P}_n$ is a dense subset of I_n for each $n \geqslant 1$. It is easy to prove that any countable dense subset of a ray can be split into infinitely many disjoint sets, each of which is dense in this ray. By $\mathcal{P}_{n,m,k}$ we denote disjoint subsets of $\mathcal{P}_n$ ($m \geqslant k \geqslant 1$) such that $\mathcal{P}_{n,m,k}$ is dense in I_n for every pair (m,k). By

$\mathcal{P}_{\cdot,m,k}$ we denote the union $\bigcup_{n\geqslant 1}\mathcal{P}_{n,m,k}$ and $\mathcal{P}_{\cdot,\cdot,k}=\bigcup_{m\geqslant k}\mathcal{P}_{\cdot,m,k}$. Recall that for any point p which belongs to one of the rays of T, its level is defined as $\#\mathcal{A}(p)$.

LEMMA 3.1. *Let T be an ℓ_1-tree. For each $k=1,2,\dots$ there exists a family $\mathcal{F}_k$ of balls in T with the following properties.*

(a) *For any node $p\in\mathcal{P}_{\cdot,m,k}$ there exists $r\in[10k/2^m,10(k+1)/2^m]$ such that $B(p\oplus r,1/2^m)\in\mathcal{F}_k$.*

(b) *Any ball in $\mathcal{F}_k$ is of the form $B(p\oplus r,1/2^m)$, where $p\in\mathcal{P}_{\cdot,m,k}$, $m\geqslant k$, and $r\in[10k/2^m,10(k+1)/2^m]$.*

(c) *For any two distinct balls $B=B(q,\rho)$ and $B'=B(q',\rho')$ in $\mathcal{F}_k$ either $B\cap B'=\emptyset$ or $B(q,5\rho)\subset B(q',\rho')$ or $B(q',5\rho')\subset B(q,\rho)$. If q and q' are of the same level in T, then $B\cap B'=\emptyset$.*

Proof. We fix k and construct the family $\mathcal{F}_k$ by induction. At step n we build the part $\mathcal{F}_k^n$ of $\mathcal{F}_k$ which consists of balls with centres of level $n+1$. We will ensure that, after step n, the family $U_k^n=\bigcup_{i=1}^n\mathcal{F}_k^i$ satisfies condition (a) for all nodes p of level at most n, together with conditions (b) and (c).

For $n=1$, let

$$\mathcal{F}_k^1=\left\{B\left(p\oplus\frac{10k}{2^m},\frac{1}{2^m}\right)\,\middle|\,p\in\mathcal{P}_{1,m,k},m\geqslant k\right\}.$$

Note that condition (c) holds for $\mathcal{F}_k^1$, since balls from $\mathcal{F}_k^1$ are disjoint.

Assume that the families $\mathcal{F}_k^i$ of balls are constructed for all $i<n$. Define

$$\tilde{\mathcal{F}_k^n}=\left\{B\left(p\oplus\frac{10k}{2^m},\frac{1}{2^m}\right)\,\middle|\,p\text{ is of level }n,\ p\notin\bigcup_{B\in U_k^{n-1}}B,\ p\in\mathcal{P}_{\cdot,m,k}\text{ for some }m\geqslant k\right\}.$$

It is easy to see that any two balls from $\tilde{\mathcal{F}_k^n}$ are disjoint. That a ball in $\tilde{\mathcal{F}_k^n}$ does not intersect a ball in U_k^{n-1} follows from Lemma 3.2.

LEMMA 3.2. *Let p be a node of level n and q be a node of level at most $n-1$, and $r_1>\rho_1$ be positive numbers. If a ball $B(q\oplus r_1,\rho_1)$ does not contain p, then for any $r_2>\rho_2>0$ the intersection $B(q\oplus r_1,\rho_1)\cap B(p\oplus r_2,\rho_2)$ is empty.*

Proof. Assume that $x\in B(q\oplus r_1,\rho_1)\cap B(p\oplus r_2,\rho_2)$. Then $p,q\in\mathcal{A}(x)$ by Lemma 2.4 and $\mathcal{A}(x)\setminus(\{q\}\cup\mathcal{A}(q))\subset B(q\oplus r_1,\rho_1)$. Since the level of p is greater than the level of q, we have $p\notin\{q\}\cup\mathcal{A}(q)$. Thus $p\in B(q\oplus r_1,\rho_1)$, a contradiction. □

We now return to the proof of Lemma 3.1 and construct a family $\tilde{\tilde{\mathcal{F}}}_k^n$ of balls $B=B(p\oplus r,1/2^m)$ such that p is a node of level n, $p\in\mathcal{P}_{\cdot,m,k}$, and there exists a ball from U_k^{n-1} which contains p. We then put $\mathcal{F}_k^n=\tilde{\mathcal{F}}_k^n\cup\tilde{\tilde{\mathcal{F}}}_k^n$.

Fix any such p of level n, and let m_0 be such that $p\in\mathcal{P}_{\cdot,m_0,k}$. Let $\{B_v\}_{v=1}^N$ be those balls from U_k^{n-1} which contain p. By condition (c), the centres of these balls are of different levels (and hence the number of the balls is finite), and they can be so enumerated that $B_1\subset B_2\subset\dots\subset B_N$. Let $B_v=B(q_v,1/2^{m_v})$.

Consider the set of indices

$$\left\{v:\ \text{both }B\left(p\oplus\frac{10k}{2^{m_0}},\frac{1}{2^{m_0}}\right)\cap B_v\text{ and }B\left(p\oplus\frac{10k}{2^{m_0}},\frac{1}{2^{m_0}}\right)\setminus B_v\text{ are not empty}\right\}.\tag{3.1}$$

If this set is not empty, let v_0 be the maximal index in it (that is, B_{v_0} is the biggest ball among all B_v which non-trivially intersects $B\left(p\oplus 10k/2^{m_0},1/2^{m_0}\right)$). Then $(10k-1)/2^{m_0}\leqslant 2/2^{m_{v_0}}$, because B_{v_0} must contain the interval $[p, p \oplus (10k-1)/2^{m_0}]$ in T. Therefore, m_0 is greater than $m_{v_0}+2$.

If $v_0 = N$, then the ball $B = B(p \oplus (10k+2)/2^{m_0}, 1/2^{m_0})$ does not intersect B_N. Indeed, the point in $\overline{B}$ closest to the centre of B_N (which is of level at most n) is $p \oplus (10k+1)/2^{m_0}$; if B_N contained this point, B_N would contain $B\,(p \oplus (10k)/2^{m_0}, 1/2^{m_0})$. Therefore, B is disjoint from any of the balls $B_1, \dots, B_N$. Also, B does not intersect other balls from U_k^{n-1} by Lemma 3.2. In this case, add $B = B(p \oplus (10k+2)/2^{m_0}, 1/2^{m_0})$ to the family $\tilde{\tilde{\mathcal{F}}}_k^n$.

If $v_0 < N$, then the ball $B = B(p \oplus (10k+2)/2^{m_0}, 1/2^{m_0})$ does not intersect B_{v_0} (proved as in the previous paragraph), and therefore B is disjoint from any of the balls B_v for $v \leqslant v_0$. By condition (c), $B(q_{v_0}, 5/2^{m_{v_0}}) \subset B_{v_0+1}$. Since $1/2^{m_0} < (1/2^{m_{v_0}})/4$, we conclude that

$$B\left(p \oplus \frac{10k+2}{2^{m_0}}, \frac{5}{2^{m_0}}\right) \subset B\left(p \oplus \frac{10k-1}{2^{m_0}}, \frac{8}{2^{m_0}}\right) \subset B\left(q_{v_0}, \frac{3}{2^{m_{v_0}}}\right) \subset B_{v_0+1}.$$

Therefore, we may add the ball $B = B(p \oplus (10k+2)/2^{m_0}, 1/2^{m_0})$ to the family $\tilde{\tilde{\mathcal{F}}}_k^n$: the ball B does not intersect B_v for $v \leqslant v_0$, the ball $B(p \oplus (10k+2)/2^{m_0}, 5/2^{m_0})$ is contained in B_{v_0+1} (and therefore, is contained in all B_v, $v_0+1 \leqslant v \leqslant N$), and B does not intersect other balls from U_k^{n-1}.

Consider the case when the set (3.1) is empty and the ball $B\,(p \oplus 10k/2^{m_0}, 1/2^{m_0})$ is properly contained in some of the balls B_v. In this case, denote by v_1 an index such that m_{v_1} is the largest (the ball B_{v_1} is the smallest containing $B\,(p \oplus 10k/2^{m_0}, 1/2^{m_0})$). If $B(p \oplus 10k/2^{m_0}, 5/2^{m_0})$ is contained in B_{v_1}, we add $B = B\,(p \oplus 10k/2^{m_0}, 1/2^{m_0})$ to the family $\tilde{\tilde{\mathcal{F}}}_k^n$. However, if the ball $B(10k/2^{m_0}, 5/2^{m_0})$ is not contained in B_{v_1}, then the ball $B = B(p \oplus (10k+6)/2^{m_0}, 1/2^{m_0})$ does not intersect B_{v_1} (and thus does not intersect B_v for $v \leqslant v_1$). In this case again $m_0 > m_{v_1}+2$ and therefore

$$B\left(p \oplus \frac{10k+6}{2^{m_0}}, \frac{5}{2^{m_0}}\right) \subset B\left(p \oplus \frac{10k+1}{2^{m_0}}, \frac{10}{2^{m_0}}\right) \subset B\left(q_{v_1}, \frac{3.5}{2^{m_{v_1}}}\right) \subset B_{v_1+1}.$$

This implies that the ball $B(p \oplus 10k+6/2^{m_0}, 5/2^{m_0})$ is contained in all B_v, $v \geqslant v_1$, and B does not intersect with any of the balls B_v, $v \leqslant v_1$. Then add B to the family $\tilde{\tilde{\mathcal{F}}}_k^n$.

If (3.1) is empty and the ball $B = B\,(p \oplus 10k/2^{m_0}, 1/2^{m_0})$ does not intersect any of the balls B_v, then add it to the family $\tilde{\tilde{\mathcal{F}}}_k^n$.

These cases describe all possible situations, since the ball $B\,(p \oplus 10k/2^{m_0}, 1/2^{m_0})$ cannot contain any of the balls B_v ($p \in B_v$ and $p \notin B\,(p \oplus 10k/2^{m_0}, 1/2^{m_0})$).

We carry out the above procedure for all nodes p of level n from $\mathcal{P}_{\cdot,m,k}$, such that there exists a ball from U_k^{n-1} which contains p. The balls in $\mathcal{F}_k^n = \tilde{\mathcal{F}}_k^n \cup \tilde{\tilde{\mathcal{F}}}_k^n$ are disjoint, because any two balls $B(p \oplus r, \rho)$ and $B(p' \oplus r', \rho')$ do not intersect when $p \neq p'$ are nodes of the same level, $\rho < r$ and $\rho' < r'$. This guarantees that condition (c) holds for $U_k^n = U_k^{n-1} \cup \mathcal{F}_k^n$. By construction, conditions (a) and (b) are also satisfied. □

4. *Main result*

We are ready to describe our main construction of the σ-porous set $S \subset X$. Let us start with the following general lemma.

LEMMA 4.1. *Assume that X is a separable metric space, T is an ℓ_1-tree and $f : X \to T$ is Lipschitz quotient. There exists a countable dense $\mathcal{X} \subset X$ such that*

$$f(x) \in T' = \bigcup_{l \geqslant 1} T_l \quad \textit{for every } x \in \mathcal{X}. \tag{4.1}$$

Proof. The preimage of T' under f is dense in X since the mapping f is open: for any $x \in X$ and $r > 0$, let $\rho(= cr)$ be such that $fB(x,r) \supset B(f(x),\rho)$; take any $t' \in T' \cap B(f(x),\rho)$ and find $x' \in B(x,r)$ such that $f(x') = t'$.

Since X is separable, we can find a countable dense $\mathcal{X} \subset f^{-1}(T')$. □

LEMMA 4.2. *Assume that X is a separable metric space, T is an ℓ_1-tree, $f : X \to T$ is a Lipschitz quotient mapping with Lipschitz constant L and co-Lipschitz constant c, families $\mathcal{F}_k$ of balls in T are as in Lemma 3.1 and $\mathcal{X} = \{x_1, x_2, \dots\} \subset X$ is a countable dense subset such that (4.1) holds.*

There exist balls $B_{j,m,k} = B(\tilde{x}_{j,m,k}, 1/(2^m L))$ ($j \geqslant 1$, $m \geqslant k$, $k \geqslant 1$) in X such that

(1) $\operatorname{dist}_X(\tilde{x}_{j,m,k}, x_j) < (10(k+1)+1)/(2^m c)$;
(2) $f(\tilde{x}_{j,m,k})$ *are distinct points in T;*
(3) $B(f(\tilde{x}_{j,m,k}), 1/2^m) \in \mathcal{F}_k$.

Proof. For each j, denote by p_{s_j} the parent of $f(x_j)$.

Assume that $k \geqslant 1$ is fixed. There exist distinct nodes $u_{j,m,k} \in \mathcal{P}_{s_j,m,k}$ ($j \geqslant 1$, $m \geqslant k$) such that $\operatorname{dist}_T(f(x_j), u_{j,m,k}) < 1/2^m$. By property (a) of the family $\mathcal{F}_k$ of balls from Lemma 3.1, there exist balls $B(u_{j,m,k} \oplus r_{j,m,k}, 1/2^m) \in \mathcal{F}_k$ such that $10k/2^m \leqslant r_{j,m,k} \leqslant 10(k+1)/2^m$ for every $j \geqslant 1, m \geqslant k$. Denote $y_{j,m,k} = u_{j,m,k} \oplus r_{j,m,k}$. Then

$$y_{j,m,k} \in B(f(x_j), r_{j,m,k} + 2^{-m}) \subset f\left(B\left(x_j, \tfrac{r_{j,m,k}+2^{-m}}{c}\right)\right).$$

Let $\tilde{x}_{j,m,k}$ be a point in the ball $B(x_j, (r_{j,m,k} + 2^{-m})c \subset B(x_j, (10(k+1)+1)/(2^m c)$ such that $f(\tilde{x}_{j,m,k}) = y_{j,m,k}$. Finally, we put $B_{j,m,k} = B(\tilde{x}_{j,m,k}, 1/(2^m L))$. □

LEMMA 4.3. *Assume that X is a separable metric space, T is an ℓ_1-tree, $f : X \to T$ is a Lipschitz quotient mapping with Lipschitz constant L and co-Lipschitz constant c, $\mathcal{X} = \{x_1, x_2, \dots\} \subset X$ is a countable dense subset such that (4.1) holds, and balls $B_{j,m,k}$ ($j \geqslant 1$, $m \geqslant k$, $k \geqslant 1$) are such as in Lemma 4.2.*

Then for each $k \geqslant 1$, the set

$$S_k = X \setminus \bigcup_{j \geqslant 1, m \geqslant k} B_{j,m,k}. \tag{4.2}$$

is porous.

Proof. Let k be fixed. Assume that x is a point in S_k and let $r > 0$. Let $m \geqslant k$ be such that $\rho = 1/(2^m L) < r$. Find $x_j \in \mathcal{X}$ such that the distance between x and x_j is less than ρ. Then the ball $B_{j,m,k}$ has radius $\rho < r$, and the distance $\operatorname{dist}_X(x, \tilde{x}_{j,m,k})$ from x to the centre of $B_{j,m,k}$ is not greater than $\rho(1 + (L/c)(10(k+1)+1))$.

Since the ball $B_{j,m,k}$ lies in the complement of S_k, this proves that the set S_k is porous with constant $\lambda = 1/(1 + L(10(k+1)+1)/c)$. □

In the next two lemmas, we show that the union of porous sets S_k constructed in Lemma 4.3 contains (up to a set of one-dimensional Hausdorff measure zero) any bi-Lipschitz piece of every Lipschitz curve in X.

LEMMA 4.4. *Assume that X is a separable metric space, T is an ℓ_1-tree, $f : X \to T$ is a Lipschitz quotient mapping with Lipschitz constant L and co-Lipschitz constant c, and sets $S_k \subset X$ are as in (4.2).*

Let $\gamma : [0,1] \to X$ be a Lipschitz mapping. Suppose that $A \subset [0,1]$ is a set of positive outer Lebesgue measure such that the restriction $\gamma|_A$ is a bi-Lipschitz mapping. Then the set $\gamma(A)$ intersects the porous set S_k for k large enough.

Proof. Let the map γ be α-Lipschitz and the map $(\gamma|_A)^{-1}$ be $(1/\beta)$-Lipschitz on $\gamma(A)$, that is, for any $a_1, a_2 \in A$,

$$\beta|a_1 - a_2| \leqslant \operatorname{dist}_X(\gamma(a_1), \gamma(a_2)) \leqslant \alpha|a_1 - a_2|.$$

Assume that $\gamma(A)$ lies in $X \setminus S_k$ for arbitrarily large indices k.

If the composition $f \circ \gamma : [0,1] \to T$ is not constant, then there exist points $0 \leqslant a_1 < a_2 \leqslant 1$ such that $(f \circ \gamma)(a_1) \neq (f \circ \gamma)(a_2)$. Therefore, $(f \circ \gamma)([a_1, a_2])$ is a continuous path connecting $(f \circ \gamma)(a_1)$ and $(f \circ \gamma)(a_2)$, and thus by Lemma 2.3, it contains a nodal point $p \in \mathcal{P}$. Assume that $t \in [a_1, a_2]$ is such that $(f \circ \gamma)(t) = p$. Without loss of generality, we may assume that $A' = A \cap [t, 1]$ has a positive outer Lebesgue measure (otherwise, consider $A \cap [0, t]$).

Let $\mathcal{A}(p)$ be the set of ancestors of p. Since this set is finite, there exists k_0 such that for any $k \geqslant k_0$ none of the ancestors of p lies in the set $\mathcal{P}_{\cdot,\cdot,k}$.

Let us observe that if $z \in B_{j,m,k}$, then $f(z) \in B(f(\tilde{x}_{j,m,k}), 1/2^m) \in \mathcal{F}_k$ (see Lemma 4.2). By property (b) of the family $\mathcal{F}_k$ in Lemma 3.1, we have $f(\tilde{x}_{j,m,k}) = u_{j,m,k} \oplus r_{j,m,k}$ with $u_{j,m,k} \in \mathcal{P}_{\cdot,m,k}$ and $r_{j,m,k} \in [10k/2^m, 10(k+1)/2^m]$.

Note that for $k \geqslant k_0$, any path connecting the node p with a point $q = f(z)$ from $f(B_{j,m,k}) \subset B(u_{j,m,k} \oplus r_{j,m,k}, 2^{-m})$ must pass through the points $u_{j,m,k} \oplus r$ for all $r \in [0, r_{j,m,k} - 1/2^m]$.

Indeed, by Lemma 2.2, any such path is a concatenation of the form $\langle p, p^{(N+1)} \rangle \cup \langle p^{(N+1)}, q^{(N+1)} \rangle \cup \langle q^{(N+1)}, q \rangle$ with N as in (2.3). Lemma 2.4 implies that $u_{j,m,k} \in \mathcal{A}(q)$. Since $u_{j,m,k} \in \mathcal{P}_{\cdot,m,k}$ and none of the ancestors of p lies in the set $\mathcal{P}_{\cdot,\cdot,k}$, we get $u_{j,m,k} \notin \mathcal{A}(p)$. Thus the latter interval $\langle q^{(N+1)}, q \rangle$ contains $u_{j,m,k}$. Let $M \geqslant N+1$ be such that $u_{j,m,k} = q^{(M)}$. By Lemma 2.4, the node $q^{(M+1)}$ is in the ball $B(u_{j,m,k} \oplus r_{j,m,k}, 2^{-m})$, and hence $\langle p, q \rangle \supset \langle q^{(M)}, q^{(M+1)} \rangle \ni u_{j,m,k} \oplus r$ for all $r \in [0, r_{j,m,k} - 1/2^m]$.

Therefore, if the curve $\phi = \gamma|_{[t,1]}$ intersects $B_{j,m,k}$, then the image $(f \circ \phi)([t, 1])$ contains the interval

$$E_{j,m,k} = \langle u_{j,m,k}, u_{j,m,k} \oplus (r_{j,m,k} - 1/2^m) \rangle\,.$$

Fix $k \geqslant k_0$ such that $\gamma(A') \subset X \setminus S_k$. Let (j_i, m_i), $i \geqslant 1$ be pairs of indices such that each of the balls $B_{j_i,m_i,k}$ intersects $\gamma(A')$ and the union $\cup_i B_{j_i,m_i,k}$ contains $\gamma(A')$.

Since the curve ϕ intersects $B_{j_i,m_i,k}$ for each i, we conclude that $(f \circ \phi)([t, 1])$ contains $E_{j_i,m_i,k}$ for each i.

Denote by $A_{j_i,m_i,k}$ the preimage $(f \circ \phi)^{-1}(E_{j_i,m_i,k} \setminus \mathcal{P})$. Note that the sets $A_{j_v,m_v,k}$ and $A_{j_w,m_w,k}$ do not intersect for any $v \neq w$; indeed, the straight closed intervals $E_{j_v,m_v,k}$ and $E_{j_w,m_w,k}$ may intersect only at one of the nodes of T since all nodes $u_{j,m,k}$, constructed in Lemma 4.2, are distinct. It is clear that since $(f \circ \phi)$ is αL-Lipschitz, the set $A_{j_i,m_i,k} \subset [t, 1]$ is Borel measurable and

$$\mathcal{L}^1(A_{j_i,m_i,k}) \geqslant \frac{1}{\alpha L} \frac{10k - 1}{2^{m_i}} \tag{4.3}$$

(this follows from the fact that the one-dimensional Hausdorff measure of $E_{j_i,m_i,k} \setminus \mathcal{P}$ is at least $r_{j_i,m_i,k} - 1/2^{m_i} \geqslant (10k - 1)/2^{m_i}$).

As ϕ is bi-Lipschitz on A', we conclude that the diameter of the intersection $A' \cap \phi^{-1}(B_{j_i,m_i,k})$ does not exceed $(1/\beta)\operatorname{diam}(B_{j_i,m_i,k}) = (2/\beta L)(1/2^{m_i})$. By (4.3), the latter is not greater than $(2/\beta L)(\alpha L/(10k - 1))\mathcal{L}^1(A_{j_i,m_i,k})$.

The outer Lebesgue measure of $A' \cap \phi^{-1}(B_{j_i,m_i,k})$ is bounded from above by its diameter, and thus the outer measure of $A' = \bigcup_i \bigl(A' \cap \phi^{-1}(B_{j_i,m_i,k})\bigr)$ is not greater than $\sum_i (2\alpha/(\beta(10k - 1)))\mathcal{L}^1(A_{j_i,m_i,k})$. Since the sets $A_{j_i,m_i,k}$ are disjoint and measurable, we conclude that the sum $\sum_i \mathcal{L}^1(A_{j_i,m_i,k})$ is not greater than 1, and therefore, the outer measure of A' is not greater than $2\alpha/(\beta(10k - 1))$, for arbitrarily large $k \geqslant k_0$. This contradicts the positivity of the outer measure of A'.

Now we have to treat separately the case when the mapping $(f \circ \gamma) : [0, 1] \to T$ is a constant mapping. Denote the constant value of $(f \circ \gamma)$ by q.

Fix any $k \geqslant 1$: assume that $\gamma(A) \subset X \setminus S_k$. As before, this means that $\gamma(A)$ is covered by some of the balls $B_{j,m,k}$. Note that if $B_{j,m,k} \cap \gamma(A) \neq \emptyset$, then $q \in f(B_{j,m,k}) \subset B(f(\tilde{x}_{j,m,k}), 1/2^m) \in \mathcal{F}_k$.

Assume that (j_i, m_i) are pairs of indices such that each of the balls $B_{j_i,m_i,k}$ intersects $\gamma(A)$ and their union $\bigcup_i B_{j_i,m_i,k}$ contains $\gamma(A)$. Then each of the balls $B(f(\tilde{x}_{j_i,m_i,k}), 1/2^{m_i})$ contains the point q and therefore these balls intersect pairwise. Property (c) in Lemma 3.1 implies that these balls form a decreasing sequence (with respect to inclusion). Note that the centres of these balls belong to the rays of T and are distinct by Lemma 4.2. Therefore, balls themselves are distinct. Property (c) in Lemma 3.1 then implies that their diameters decrease by factor of at least five each time. Therefore, the sum of their diameters is not greater than $(5/4)\max_i \operatorname{diam}(B_{j_i,m_i,k}) \leqslant 2.5/(2^k L)$, since each m_i is at least k. On the other hand, A is covered by $\bigcup_i (A \cap \gamma^{-1}(B_{j_i,m_i,k}))$ and $\operatorname{diam}(A \cap \gamma^{-1}(B_{j_i,m_i,k})) \leqslant (1/\beta)\operatorname{diam}(B_{j_i,m_i,k})$. Thus $\operatorname{diam}(A) \leqslant 2.5/(2^k \beta L)$. Since k is arbitrary, this contradicts the positivity of the outer measure of A. □

LEMMA 4.5. *Assume that X is a separable metric space which admits a Lipschitz quotient mapping onto an ℓ_1-tree, $\gamma : [0,1] \to X$ is Lipschitz and A is a subset of $[0,1]$ such that the restriction $\gamma|_A$ is bi-Lipschitz. Then, for almost every $a \in A$, $\gamma(a)$ lies in the σ-porous set*

$$S = \bigcup_{k \geqslant 1} S_k, \tag{4.4}$$

where sets $S_k \subset X$ are as in (4.2).

Proof. Assume that there exists $A \subset [0,1]$ such that $\gamma|_A$ is bi-Lipschitz and the outer Lebesgue measure of $\widetilde{A} = A \setminus \gamma^{-1}(S)$ is positive. Then $\gamma|_{\widetilde{A}}$ is bi-Lipschitz and $\gamma(\widetilde{A}) = \gamma(A) \setminus S$ is contained in $X \setminus S_k$ for any $k \geqslant 1$, in contradiction to Lemma 4.4. □

THEOREM 4.6. *Assume that X is a separable metric space which admits a Lipschitz quotient mapping onto an ℓ_1-tree. Then there exists a σ-porous set $S \subset X$ such that for every Lipschitz curve $\gamma : [0,1] \to X$, the one-dimensional Hausdorff measure of $\gamma([0,1]) \setminus S$ is zero.*

Proof. First, embed X into the space $C([0,1])$ of continuous functions on the interval, by an isometry $\mathrm{iso} : X \to C([0,1])$. Since γ is Lipschitz, it is a corollary of [**8**, Lemma 4] that there are Borel sets $E_k \subset [0,1]$ on which $(\mathrm{iso} \circ \gamma)$ is bi-Lipschitz and such that $\gamma([0,1]) \setminus \bigcup_{k \geqslant 1} \gamma(E_k)$ is of one-dimensional Hausdorff measure zero.

Then for the set S defined in (4.4) the statement follows from Lemma 4.5. □

5. *Remarks*

1. If X is a separable Banach space, then the construction of porous sets S_k can be simplified and made more transparent.

Instead of the family $\mathcal{F}_k$ of balls constructed in Lemma 3.1, let us consider the following family of balls:

$$\mathcal{G}_k = \left\{ B\left(p \oplus \frac{10k}{2^m}, \frac{1}{2^m}\right) \,\middle|\, p \in \mathcal{P}_{\cdot,m,k}, m \geqslant k \right\}. \tag{5.1}$$

Then the balls $B_{j,m,k}$ (see Lemma 4.2) are so chosen that $B(f(\tilde{x}_{j,m,k}), 1/2^m) \in \mathcal{G}_k$ and have the properties as in Lemma 4.2.

We then define $S_k = X \setminus \bigcup_{j \geqslant 1, m \geqslant k} B_{j,m,k}$. They are porous, which is established in the same way as in Lemma 4.3.

In the proof of Lemma 4.4 for the simplified porous sets S_k, the case $(f \circ \gamma) \neq$ const is treated in the same way as it was in the proof of Lemma 4.4.

The proof in the other case $(f \circ \gamma) \equiv q$ is done as follows. Property (c) of the family $\mathcal{F}_k$ of balls, which was instrumental in the original proof, does not hold for the family $\mathcal{G}_k$ of balls. We simply take $x_0 \in X$ such that $f(x_0) \neq q$ and define

$$\gamma_1(t) = \begin{cases} (1-2t)x_0 + 2t\gamma(0), & \text{if } t \in [0, 1/2] \\ \gamma(2t-1), & \text{if } t \in [1/2, 1]. \end{cases}$$

Since $(f \circ \gamma_1)$ is not constant, we have that $\gamma(A) = \gamma_1(0.5(A+1))$ intersects S_k for k large enough.

2. Of course, the same simplification as described in Remark 1 is possible in all separable metric spaces X for which any two points may be connected by a Lipschitz path. Instead of a straight-line segment, γ_1 is defined using a Lipschitz path connecting x_0 and $\gamma(0)$.

3. Trivially, any ℓ_1-tree is a Lipschitz quotient of itself. Hence, Theorem 4.6 applies to ℓ_1-trees.

4. Since for every separable Banach space X containing a subspace isomorphic to ℓ_1 there exists a Lipschitz quotient mapping f from X onto an ℓ_1-tree [**7**, Theorem 2.1], we conclude that Theorem 4.6 holds for every such space.

5. The proof of [**7**, Theorem 2.1] describes how to construct a Lipschitz quotient mapping from a separable Banach space X containing a subspace isomorphic to ℓ_1 onto an ℓ_1-tree. It would be interesting to combine that construction with ours so as to have a geometric description of an unavoidable σ-porous set in X. Such a geometric description might help to understand whether such σ-porous sets exist in all separable Banach spaces with non-separable duals. Note that currently it is not known whether the existence of a Lipschitz quotient mapping from X on to an ℓ_1-tree implies that X contains a subspace isomorphic to ℓ_1.

6. *Counterexample to the Fubini theorem for σ-porous sets*

Recall that the Fubini theorem for measure spaces implies that if $S \subset X_1 \times X_2$ is null (with respect to the product measure), then for almost all $v \in X_1$, the section $M_v = \{w \in X_2 \mid (v, w) \in S\}$ is null.

One would like to consider the Fubini theorem for the σ-ideal of σ-porous sets (instead of null sets). Preiss and Zajíček [**13**] show that there exists a σ-porous set M in the plane such that for each $x \in \mathbb{R}$ except a first category set, the vertical section M_x is not Lebesgue null (moreover, its complement $\mathbb{R} \setminus M_x$ is Lebesgue null). Therefore, no statement directly analogous to the Fubini theorem can hold for the σ-porous sets.

Theorem 4.6 surprisingly gives us a counterexample to the Fubini theorem of a different nature: a σ-porous set S in every separable metric space $X = Y \times \mathbb{R}$ (where Y admits a Lipschitz quotient mapping onto an ℓ_1-tree) such that all sections $S_y \subset \mathbb{R}$, $y \in Y$, are not σ-porous. Moreover, for all $y \in Y$, the complement $\mathbb{R} \setminus S_y$ is Lebesgue null.

Indeed, let X be a Cartesian product of a separable metric space Y, which admits a Lipschitz quotient mapping onto an ℓ_1-tree, and the real line $\mathbb{R}$. The distance in X is given by $d_X((y_1, t_1), (y_2, t_2)) = d_Y(y_1, y_2) + |t_1 - t_2|$. Then X itself admits a Lipschitz quotient mapping onto an ℓ_1-tree (a composition of the projection $X \to Y$ and a Lipschitz quotient from Y onto an ℓ_1-tree is a Lipschitz quotient mapping). Let $S \subset X$ be as in Theorem 4.6. For every $y \in Y$, the section S_y can be treated as a subset of $\mathbb{R}$. Since S contains every Lipschitz curve up to a set of Hausdorff measure zero, its section S_y contains all straight intervals up to a set of Hausdorff measure zero. Therefore, $\mathbb{R} \setminus S_y$ is Lebesgue null for every $y \in Y$ and in particular S_y is not σ-porous.

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